EXACT ALL-COUPLING NON-PERTURBATIVE BOUNDS FOR EXCEPTIONAL G_2 LATTICE YANG–MILLS THEORY: WEYL–BESSEL TRIPLE-PRODUCT IDENTITIES, OCTONION-GLUELUMP FUNDAMENTAL STRING BREAKING, AND BAKRY–ÉMERY CURVATURE

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ABSTRACT. The exceptional Lie group $G_2 \subset \mathrm{SO}(7)$ is the smallest simply-connected compact simple Lie group with trivial center $Z(G_2) = \{I_7\}$, providing a canonical laboratory for confinement without center symmetry. We establish four self-contained, cluster-expansion-free non-perturbative theorems for d-dimensional ($d \geq 2$) pure G_2 Wilson lattice Yang–Mills theory on periodic tori $\mathbb{T}^d_L$:

(1) *Exact 12-term Weyl–Bessel triple-product character expansion and 2D G_2 thermodynamics:* Every irreducible G_2 single-plaquette character coefficient $c_{(p,q)}(\beta) = \int_{G_2} e^{\frac{\beta}{7}\chi_{\mathbf{7}}(U)}\chi_{(p,q)}(U)\,dU$ evaluates identically as a 12-term alternating Weyl-group sum over $\widehat{w}_{m,n}(\beta) = e^{\beta/7}\sum_{k\in\mathbb{Z}} I_{m-k}(\frac{2\beta}{7})I_{n-k}(\frac{2\beta}{7})I_k(\frac{2\beta}{7})$ with super-exponential tail control, yielding exact 2D free energy $f^{G_2}_{\mathrm{2D}}(\beta) = \ln c_{(0,0)}(\beta)$, exact fundamental and adjoint 2D string tensions $\sigma^{\mathrm{2D}}_{\mathbf{7}}, \sigma^{\mathrm{2D}}_{\mathbf{14}} > 0$ (whose ratio exhibits a non-monotonic crossover dip around $\beta \approx 4$ before converging to $C_2(\mathbf{14})/C_2(\mathbf{7}) = 2$), and an exact 4-term character identity for the single-plaquette octonion gluelump norm squared.

(2) *All-coupling d-dimensional free-energy and plaquette enclosures:* Combining 2D slicing, axial-gauge single-link Gibbs–Bogoliubov tilting, d-way Hölder's inequality ($f^{\mathrm{ub}}_d(\beta) = (d-1)\ln c_{(0,0)}(\frac{d}{2}\beta)$, matching both exact classical energy $\binom{d}{2}\beta$ and exact one-loop entropy $-7(d-1)\ln\beta$), and finite-L convexity yields rigorous all-coupling enclosures across $d = 3, 4$.

(3) *Universal all-coupling octonion-gluelump fundamental string breaking in $d \geq 3$:* Contracting a spatial plaquette with the G_2-invariant associative octonion 3-form φ_{abc} defines a local fundamental $\mathbf{7}$-valued 3-gluon gluelump operator $v_a(U_p) = \frac{1}{6}\varphi_{abc}(U_p)_{bc}$ that vanishes to $O(\theta^2)$ and turns on at $O(\theta^3)$ as $-\frac{4}{3}\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\sin\frac{\theta_1+\theta_2}{2}$. Transfer-matrix positivity, Perron–Frobenius gauge invariance, and Schur convolution prove $0 \leq V^{G_2}_{\mathbf{7},L_s}(R) \leq 8\sigma^{\mathrm{2D}}_{\mathbf{7}}(\beta) + \frac{9}{7}N_{\mathrm{loc}}(d)\beta < \infty$ ($N_{\mathrm{loc}}(d) = 2(8d-19)$) uniformly in all $\beta > 0$, $L_s \geq 4$, and $2 \leq R \leq L_s - 2$, certifying zero asymptotic fundamental string tension $\sigma^{G_2}_{\mathbf{7}} = 0$ in every $d \geq 3$ (in sharp contrast to $\sigma^{\mathrm{2D}}_{\mathbf{7}} > 0$ in 2D).

(4) *Exact G_2 Bakry–Émery curvature and Cartan variance bounds:* From $h^\vee(G_2) = 4$ and $T(\mathbf{7}) = 1$, $\mathrm{Ric}_{G_2} = 1 \cdot g_{\mathrm{HS}}$ (vs. $\mathrm{Ric}_{\mathrm{SO}(7)} = \frac{5}{4}g_{\mathrm{HS}}$), certifying $\kappa_{G_2}(d,\beta) = 1 - \frac{8(d-1)\beta}{7} > 0$ for $\beta < \frac{7}{8(d-1)}$, volume-uniform log-Sobolev inequalities, and an elementary Cartan plaquette-variance bound $\mathrm{Var}_{G_2}(P_p) \leq \frac{18}{49\,\kappa_{G_2}}$ (35.7% tighter than the naive $\frac{4}{7}$ bound).

All 24 benchmarks are certified at true 140-digit `mpmath.iv` precision (`mp.mp.dps = mp.iv.dps = 140`) with zero quadrature and mechanically verified in a standalone zero-import, zero-axiom Lean 4.36.0 kernel (`G2YangMillsCerts.lean`).

1. Introduction and Honest Scope

The compact exceptional Lie group G_2 ($\dim G_2 = 14$, $\mathrm{rank}\, G_2 = 2$, Weyl group $|W(G_2)| = 12$) is realized concretely as the automorphism group of the normed division algebra of octonions, equivalently as the subgroup of $\mathrm{SO}(7)$ preserving the associative Cayley–Bryant 3-form [3]

$$\varphi := e^{012} + e^{034} + e^{056} + e^{135} - e^{146} - e^{236} - e^{245} \in \Lambda^3(\mathbb{R}^7)^*, \tag{1}$$

Date: November 29, 2026.

indexed by $\{0,1,\dots,6\}$ with totally antisymmetric components $\varphi_{abc} \in \{-1,0,+1\}$. Because G_2 is simply connected yet has *trivial center* $Z(G_2) = \{I_7\}$, pure G_2 lattice Yang–Mills theory [6, 7, 5, 8] is the canonical non-Abelian benchmark where even fundamental **7**-representation quarks can be color-screened by three gluons ($\mathbf{14}^{\otimes 3} \supset \mathbf{7}$).

On a periodic hypercubic torus $\mathbb{T}_L^d$ ($d \geq 2$, $V = L^d$, $|E| = dV$, $|P| = \binom{d}{2}V$, unoriented elementary plaquettes $p = (x; \mu < \nu)$), assign link variables $U_e \in G_2 \subset \mathrm{SO}(7)$ in the real irreducible 7-dimensional fundamental representation with character $\chi_{\mathbf{7}}(U) := \mathrm{Tr}(U)$. The Wilson action and partition function [9] are

$$S_W^{G_2}(\beta; U) := \frac{\beta}{7} \sum_{p \in P} \chi_{\mathbf{7}}(U_p), \qquad Z_{G_2}(\mathbb{T}_L^d; \beta) := \int_{G_2^{|E|}} e^{S_W^{G_2}(\beta;U)}\, dU, \tag{2}$$

where dU is normalized product Haar measure on $G_2^{|E|}$. Every $U \in G_2$ is conjugate in G_2 to a block-diagonal Cartan element $D(\theta_1, \theta_2) := \mathrm{diag}(1, R(\theta_1), R(\theta_2), R(-\theta_1 - \theta_2))$ ($R(\theta) = \left(\begin{smallmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{smallmatrix}\right)$, $\theta_1, \theta_2 \in [-\pi, \pi]$), yielding

$$\chi_{\mathbf{7}}(\theta_1, \theta_2) = 1 + 2\cos\theta_1 + 2\cos\theta_2 + 2\cos(\theta_1 + \theta_2) = \left|1 + e^{i\theta_1} + e^{-i\theta_2}\right|^2 - 2 \in [-2, 7], \tag{3}$$

with exact minimum -2 attained at $\theta_1 = \theta_2 = \frac{2\pi}{3}$ and exact oscillation $\max \chi_{\mathbf{7}} - \min \chi_{\mathbf{7}} = 9$.

Remark 1.1 (Honest Scope and Non-Claims). All results hold at fixed lattice spacing ($a = 1$) for pure G_2 Wilson lattice Yang–Mills theory. Theorems 2.1, 3.1, and 4.1 hold at *all* couplings $\beta > 0$ ($\beta \geq 0$ for Theorems 2.1–3.1), whereas Theorem 5.1 holds in the strong-coupling slab $\beta < \frac{7}{8(d-1)}$ ($\beta < \frac{7}{24}$ in 4D) and certifies a volume-uniform log-Sobolev inequality, spectral gap, and plaquette-variance bound rather than a 4D continuum limit or first-order thermal transition location. This paper does *not* claim any resolution of the Clay Millennium Yang–Mills Existence and Mass Gap Problem.

2. Exact 12-Term Weyl–Bessel Triple-Product Character Expansion

In the Cartan coordinates (θ_1, θ_2) above, the six positive roots of G_2 are the three short roots $\{(1,0), (0,1), (1,1)\}$ and three long roots $\{(1,-1), (2,1), (1,2)\}$, with simple roots $\alpha_1 = (0,1)$ (short), $\alpha_2 = (1,-1)$ (long), fundamental weights $\omega_1 = (1,1)$ (**7**), $\omega_2 = (2,1)$ (**14**), and Weyl vector $\rho = \omega_1 + \omega_2 = (3,2)$ [4]. Every irreducible representation $\lambda = p\omega_1 + q\omega_2$ ($p, q \in \mathbb{N}_0$) has highest weight $\lambda = (p + 2q, p + q)$ and Weyl dimension

$$d_{(p,q)} = \frac{1}{120}(p+1)(q+1)(p+q+2)(p+2q+3)(p+3q+4)(2p+3q+5),$$

giving $d_{(0,0)} = 1$, $d_{(1,0)} = 7$, $d_{(0,1)} = 14$, and $d_{(2,0)} = 27$ ($\mathbf{7} \otimes \mathbf{7} = \mathbf{1} \oplus \mathbf{7} \oplus \mathbf{14} \oplus \mathbf{27}$). Define the single-plaquette character coefficients $c_{(p,q)}(\beta) := \int_{G_2} e^{\frac{\beta}{7}\chi_{\mathbf{7}}(U)} \chi_{(p,q)}(U)\, dU$ and the local single-plaquette octonion gluelump vector

$$v_a(U) := \frac{1}{6} \sum_{b,c=0}^{6} \varphi_{abc}\, U_{bc} \qquad (a \in \{0,\dots,6\}). \tag{4}$$

Theorem 2.1 (Exact 12-Term Weyl–Bessel Identity, 2D G_2 Thermodynamics, and Gluelump Character Identity)**.** *Fix any coupling $\beta \geq 0$ and write $x := \frac{2\beta}{7}$.*

(1) *For every irreducible G_2 representation $\lambda = (p, q)$ ($p, q \in \mathbb{N}_0$),*

$$\begin{aligned} c_{(p,q)}(\beta) &= \sum_{w \in W(G_2)} \det(w)\, \widehat{w}_{\,(p+2q+3,\, p+q+2)-w(3,2)}(\beta), \\ \widehat{w}_{m,n}(\beta) &:= e^{\beta/7} \sum_{k=-\infty}^{+\infty} I_{m-k}(x)\, I_{n-k}(x)\, I_k(x), \end{aligned} \tag{5}$$

where I_ν is the modified Bessel function ($I_{-\nu} = I_\nu$). Moreover, $c_{(p,q)}(\beta) > 0$ strictly for all $\beta > 0$, and truncating $|k| \le K$ with $K > \max(|m|, |n|) + 2$ obeys the super-exponential tail bound

$$0 \;\le\; \sum_{|k|>K} I_{m-k}(x) I_{n-k}(x) I_k(x) \;\le\; \frac{2\, B(r_0, x)^2\, B(K+1, x)}{1 - (x/2)^3/((r_0+1)^2(K+2))},$$

whenever $(x/2)^3 < (r_0+1)^2(K+2)$, with $r_0 := K + 1 - \max(|m|, |n|)$ and $B(r, x) := \frac{(x/2)^r}{r!} \exp(\frac{x^2}{4(r+1)})$.

(2) *On $\mathbb{T}^2_L$ ($L \ge 2$), the per-site free energy obeys $\ln c_{(0,0)}(\beta) - \frac{\ln c_{(0,0)}(\beta) + \frac{2}{7}\beta}{L^2} \le f^{G_2}_{\mathbb{T}^2_L}(\beta) \le \ln c_{(0,0)}(\beta) + \frac{\beta - \ln c_{(0,0)}(\beta)}{L^2}$, so $f^{G_2}_{\mathrm{2D}}(\beta) = \ln c_{(0,0)}(\beta)$ is exact; for $\beta > 0$, defining the 2D string tension $\sigma^{\mathrm{2D}}_\lambda(\beta) := -\lim_{R,T\to\infty} \frac{1}{RT} \ln \langle \frac{1}{d_\lambda} \chi_\lambda(W_{R\times T}) \rangle_{\mathrm{2D}}$ on an open/infinite plane, the exact fundamental and adjoint 2D string tensions are strictly positive:*

$$\sigma^{\mathrm{2D}}_{\mathbf{7}}(\beta) = -\ln\!\left(\frac{c_{(1,0)}(\beta)}{7\, c_{(0,0)}(\beta)}\right) > 0, \qquad \sigma^{\mathrm{2D}}_{\mathbf{14}}(\beta) = -\ln\!\left(\frac{c_{(0,1)}(\beta)}{14\, c_{(0,0)}(\beta)}\right) > 0. \tag{6}$$

(3) *For every $U \in G_2$ and every $g \in G_2$, $v(gUg^T) = g\, v(U)$ (transforming strictly in $\mathbf{7}$). On the Cartan torus $D(\theta_1, \theta_2)$, $v_a = 0$ for $a \ge 1$ and*

$$v_0\big(D(\theta_1, \theta_2)\big) = -\frac{4}{3} \sin\!\left(\frac{\theta_1}{2}\right) \sin\!\left(\frac{\theta_2}{2}\right) \sin\!\left(\frac{\theta_1 + \theta_2}{2}\right), \tag{7}$$

which vanishes to $O(\theta^2)$ and turns on at $O(\theta^3)$. Furthermore, for every $U \in G_2$,

$$\big\|v(U)\big\|^2 = \frac{1}{6} - \frac{1}{12}\chi_{\mathbf{7}}(U) + \frac{1}{12}\chi_{\mathbf{14}}(U) - \frac{1}{36}\chi_{\mathbf{27}}(U), \tag{8}$$

giving $\int_{G_2} \|v(U)\|^2\, dU = \frac{1}{6}$ and exact single-plaquette expectation $\langle \|v\|^2 \rangle_{1\mathrm{p},\beta} = \frac{1}{6} - \frac{c_{(1,0)}}{12 c_{(0,0)}} + \frac{c_{(0,1)}}{12 c_{(0,0)}} - \frac{c_{(2,0)}}{36 c_{(0,0)}}$.

Proof. (1) By Weyl's integration and character formulas on $T^2 = [-\pi, \pi]^2$, writing $A_\mu(\theta) := \sum_{w \in W(G_2)} \det(w) e^{iw(\mu)\cdot\theta}$, we have $\chi_\lambda(\theta)\Delta(\theta) = A_{\lambda+\rho}(\theta)$ and $\Delta(\theta) = A_\rho(\theta)$. Using $W(G_2)$-invariance of $\chi_{\mathbf{7}}(\theta)$ and $|W(G_2)| = 12$:

$$c_\lambda(\beta) = \frac{1}{12} \int_{T^2} e^{\frac{\beta}{7}\chi_{\mathbf{7}}(\theta)} A_{\lambda+\rho}(\theta) \overline{A_\rho(\theta)}\, \frac{d^2\theta}{(2\pi)^2} = \sum_{w \in W(G_2)} \det(w) \int_{T^2} e^{\frac{\beta}{7}\chi_{\mathbf{7}}(\theta)} e^{i(\lambda+\rho-w(\rho))\cdot\theta}\, \frac{d^2\theta}{(2\pi)^2}.$$

Expanding $e^{x\cos\theta_1} e^{x\cos\theta_2} e^{x\cos(\theta_1+\theta_2)} = \sum_{k_1,k_2,k_3 \in \mathbb{Z}} I_{k_1}(x) I_{k_2}(x) I_{k_3}(x) e^{i(k_1+k_3)\theta_1 + i(k_2+k_3)\theta_2}$ and integrating against $e^{i(m\theta_1+n\theta_2)}$ over T^2 enforces $k_1 = -m - k_3$ and $k_2 = -n - k_3$; setting $k := -k_3$ and using $I_{-\nu} = I_\nu$ yields (5). Strict positivity $c_{(p,q)}(\beta) > 0$ follows by expanding $e^{\frac{\beta}{7}\chi_{\mathbf{7}}} = \sum_{n=0}^\infty \frac{(\beta/7)^n}{n!} \chi^n_{\mathbf{7}}$ since every irreducible G_2 representation appears inside $\mathbf{7}^{\otimes n}$ for sufficiently large n and all tensor-product multiplicities are non-negative. The tail bound follows from $I_r(x) = \frac{(x/2)^r}{r!} \sum_{j=0}^\infty \frac{(x/2)^{2j}}{j!(r+1)_j} \le B(r, x)$ ($(r+1)_j \ge (r+1)^j$) together with $I_{r+1}(x) \le I_r(x)$ for $r \ge 0, x \ge 0$ (so $I_{|m-k|}(x) \le I_{r_0}(x)$ for $|k| \ge K+1$).

(2) Fixing a maximal spanning tree of L^2-1 links on $\mathbb{T}^2_L$ leaves L^2+1 links, which biject to L^2-1 i.i.d. Haar plaquettes plus 2 independent Polyakov-loop variables (integrating to 1), while the L^2-th plaquette U_{p_*} obeys $-\frac{2}{7}\beta \le \frac{\beta}{7}\chi_{\mathbf{7}}(U_{p_*}) \le \beta$ by (3); integrating yields $c_{(0,0)}^{L^2-1}e^{-\frac{2}{7}\beta} \le Z_{\mathbb{T}^2_L}(\beta) \le c_{(0,0)}^{L^2-1}e^{\beta}$. Exact 2D character expansion on open/infinite planes yields $\langle \frac{1}{d_\lambda}\chi_\lambda(W_{R\times T})\rangle_{\mathrm{2D}} = (c_\lambda(\beta)/(d_\lambda c_{(0,0)}(\beta)))^{RT}$ and (6) (strictly positive since $|\chi_\lambda(U)| < d_\lambda$ a.e. on G_2).

(3) Since $g \in G_2$ preserves φ, $\varphi_{abc}g_{bd}g_{ce} = g_{af}\varphi_{fde}$, so $v_a(gUg^T) = \frac{1}{6}\varphi_{abc}g_{bd}U_{de}g_{ce} = (gv(U))_a$. On $D(\theta_1,\theta_2)$, the only nonzero terms in $\varphi_{0bc}D_{bc}$ come from the three 2×2 blocks $(1,2)$, $(3,4)$, $(5,6)$, each contributing $-2\sin\theta_j$, giving $v_0 = -\frac{1}{3}(\sin\theta_1+\sin\theta_2-\sin(\theta_1+\theta_2)) = -\frac{4}{3}\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\sin\frac{\theta_1+\theta_2}{2}$, while for every $a \ge 1$ every nonzero φ_{abc} in (1) pairs indices b,c from different Cartan blocks so $D_{bc}=0$ and $v_a=0$ termwise. Finally, $\|v(U)\|^2$ is a G_2-conjugation-invariant quadratic polynomial in the matrix entries of U, so by the Peter–Weyl decomposition of degree-≤ 2 polynomials on G_2 (transforming inside $\mathbf{7}\otimes\mathbf{7} = \mathbf{1}\oplus\mathbf{7}\oplus\mathbf{14}\oplus\mathbf{27}$) it is a unique linear combination of $\{\chi_{\mathbf{1}},\chi_{\mathbf{7}},\chi_{\mathbf{14}},\chi_{\mathbf{27}}\}$; evaluating (7) on T^2 fixes the exact rational coefficients $(\frac{1}{6},-\frac{1}{12},+\frac{1}{12},-\frac{1}{36})$ in (8). □

3. All-Coupling d-Dimensional G_2 Free-Energy and Plaquette Enclosures

Write $f^{G_2}_{\mathbb{T}^d_L}(\beta) := V^{-1}\ln Z_{G_2}(\mathbb{T}^d_L;\beta)$, $f^{G_2}_d(\beta)$ for any subsequential thermodynamic limit as $L\to\infty$, $P_{\mathrm{1p}}(\alpha) := \frac{c_{(1,0)}(\alpha)}{7c_{(0,0)}(\alpha)} = \partial_\alpha \ln c_{(0,0)}(\alpha)$, and $P^{G_2}_d(\beta) := \binom{d}{2}^{-1}\partial_\beta f^{G_2}_d(\beta)$ (with one-sided derivatives at any non-differentiable point).

Theorem 3.1 (All-Coupling d-Dimensional G_2 Free-Energy and Plaquette Enclosures)**.** *Fix $d \ge 2$, $\beta>0$, $L\ge 2$, and any witness $\alpha \ge 0$. Set $n_p := \binom{d}{2}$ and define*

$$\begin{aligned} f^{\mathrm{lo}}_d(\beta;\alpha) &:= \max\Big\{(d-1)\ln c_{(0,0)}(\beta),\ f^{\mathrm{var}}_d(\beta;\alpha)\Big\},\\ f^{\mathrm{var}}_d(\beta;\alpha) &:= (d-1)\beta P_{\mathrm{1p}}(\alpha)^2 + \binom{d-1}{2}\beta P_{\mathrm{1p}}(\alpha)^4 - (d-1)\big(\alpha P_{\mathrm{1p}}(\alpha) - \ln c_{(0,0)}(\alpha)\big),\\ f^{\mathrm{ub}}_d(\beta) &:= (d-1)\ln c_{(0,0)}\left(\frac{d}{2}\beta\right). \end{aligned} \tag{9}$$

Then:

(1) *Every finite torus $\mathbb{T}^d_L$ obeys $f^{\mathrm{lo}}_d(\beta;\alpha) - O(1/L) \le f^{G_2}_{\mathbb{T}^d_L}(\beta) \le f^{\mathrm{ub}}_d(\beta) + O(1/L)$, so in the thermodynamic limit*

$$f^{\mathrm{lo}}_d(\beta;\alpha) \ \le\ f^{G_2}_d(\beta) \ \le\ f^{\mathrm{ub}}_d(\beta), \tag{10}$$

where $f^{\mathrm{ub}}_d(\beta)$ matches both the exact classical energy $\binom{d}{2}\beta$ and the exact one-loop entropy $-(d-1)\frac{\dim G_2}{2}\ln\beta = -7(d-1)\ln\beta$ as $\beta\to\infty$.

(2) *For every secant step $0<\delta<\beta$, finite-L convexity of $f^{G_2}_{\mathbb{T}^d_L}(\beta)$ implies*

$$\max\left\{\frac{f^{\mathrm{lo}}_d(\beta;\alpha)}{n_p\,\beta},\ \frac{f^{\mathrm{lo}}_d(\beta;\alpha) - f^{\mathrm{ub}}_d(\beta-\delta)}{n_p\,\delta}\right\} \ \le\ P^{G_2}_d(\beta) \ \le\ \min\left\{1,\ \frac{f^{\mathrm{ub}}_d(\beta+\delta) - f^{\mathrm{lo}}_d(\beta;\alpha)}{n_p\,\delta}\right\}. \tag{11}$$

Proof. (1) *Upper bound:* Decompose $S^{G_2}_W(\beta) = \frac{1}{d}\sum_{\mu=1}^d(\frac{d}{2}S^{(\mu)})$, where $S^{(\mu)}$ sums the $(d-1)V$ plaquettes containing direction μ (each plaquette appears in exactly two of the d sets $S^{(\mu)}$). Applying d-way Hölder's inequality yields $Z_{G_2}(\mathbb{T}^d_L;\beta) \le \prod_{\mu=1}^d(\int e^{\frac{d}{2}S^{(\mu)}}dU)^{1/d}$. Fixing axial gauge $U_{x,\mu}=I_7$ on $0\le x_\mu \le L-2$ makes the $(d-1)(L-1)L^{d-1}$ non-boundary plaquettes in $S^{(\mu)}$ i.i.d. Haar variables while the $(d-1)L^{d-1}$ boundary plaquettes at $x_\mu = L-1$ satisfy $\frac{d\beta}{14}\chi_{\mathbf{7}}(U_p) \le \frac{d}{2}\beta$,

giving $f^{G_2}_{\mathbb{T}^d_L}(\beta) \le (d-1)(1-\frac{1}{L})\ln c_{(0,0)}(\frac{d}{2}\beta) + \frac{d(d-1)}{2L}\beta = f_d^{\mathrm{ub}}(\beta) + O(1/L)$. Since $\ln c_{(0,0)}(\alpha) = \alpha - \frac{\dim G_2}{2}\ln\alpha + O(1)$ as $\alpha\to\infty$, $f_d^{\mathrm{ub}}(\beta) = \binom{d}{2}\beta - 7(d-1)\ln\beta + O(1)$.

Lower bound: Fix axial gauge $U_{(\vec{x},\tau),d} = I_7$ for all $\vec{x}\in\mathbb{T}^{d-1}_L$ and $0\le\tau\le L-2$ (a valid forest gauge fixing of $(L-1)L^{d-1}$ links). For the first branch ($(d-1)\ln c_{(0,0)}(\beta)$), change variables from $\{U_{(\vec{x},\tau),j}\}_{\tau=0}^{L-1}$ to $U_{(\vec{x},0),j}$ and $W_{\vec{x},j,\tau} := U_{(\vec{x},\tau),j}U^T_{(\vec{x},\tau+1),j}$ ($0\le\tau\le L-2$, Haar-preserving bijection); taking i.i.d. $e^{\frac{\beta}{7}\chi_{\mathbf{7}}(W)}/c_{(0,0)}(\beta)$ on all $(d-1)(L-1)L^{d-1}$ temporal plaquettes and Haar measure on $U_{(\vec{x},0),j}$ and $U_{(\vec{x},L-1),d}$ yields exact expectation 0 on all spatial and $\tau=L-1$ boundary plaquettes, giving $f^{G_2}_{\mathbb{T}^d_L}(\beta)\ge(d-1)(1-\frac{1}{L})\ln c_{(0,0)}(\beta)$ by Gibbs' variational inequality. For the second branch, place i.i.d. single-link density $d\nu_\alpha(U) = c_{(0,0)}(\alpha)^{-1}e^{\frac{\alpha}{7}\chi_{\mathbf{7}}(U)}\,dU$ on all $(d-1)V$ spatial links and Haar measure on the L^{d-1} boundary temporal links at $\tau=L-1$. Since $\mathbf{7}$ is irreducible and $\chi_{\mathbf{7}}$ is a class function, Schur's lemma gives $\mathbb{E}_{\nu_\alpha}[U] = P_{1\mathrm{p}}(\alpha)I_7$, so every non-boundary temporal plaquette has expectation $P_{1\mathrm{p}}(\alpha)^2$, every spatial plaquette has expectation $P_{1\mathrm{p}}(\alpha)^4$, every $\tau=L-1$ boundary plaquette has expectation 0, and $D_{\mathrm{KL}}(\nu_\alpha^{\otimes(d-1)V}\|\mathrm{Haar}) = (d-1)V(\alpha P_{1\mathrm{p}}(\alpha) - \ln c_{(0,0)}(\alpha))$. Applying Gibbs' inequality and passing to $L\to\infty$ proves (10).

(2) Since $f^{G_2}_{\mathbb{T}^d_L}(\beta)$ is convex in β with $f^{G_2}_{\mathbb{T}^d_L}(0)=0$ and $\partial_\beta f^{G_2}_{\mathbb{T}^d_L}(0)=0$, the secant slopes satisfy $\frac{f^{G_2}_{\mathbb{T}^d_L}(\beta)-f^{G_2}_{\mathbb{T}^d_L}(\beta-\delta)}{\delta} \le \partial_\beta f^{G_2}_{\mathbb{T}^d_L}(\beta) \le \frac{f^{G_2}_{\mathbb{T}^d_L}(\beta+\delta)-f^{G_2}_{\mathbb{T}^d_L}(\beta)}{\delta}$ and $\partial_\beta f^{G_2}_{\mathbb{T}^d_L}(\beta) \ge \frac{f^{G_2}_{\mathbb{T}^d_L}(\beta)}{\beta}$; substituting (10) and $\frac{1}{7}\chi_{\mathbf{7}}\le 1$ yields (11). □

4. Universal All-Coupling Octonion-Gluelump Fundamental String Breaking in $d\ge 3$

Fix $d\ge 3$, spatial torus $\mathbb{T}^{d-1}_{L_s}$ ($L_s\ge 4$), and static fundamental $\mathbf{7}$ quark-antiquark separation $2\le R\le L_s-2$ along e_1. In temporal gauge ($U_{x,d}=I_7$), the Osterwalder–Seiler transfer matrix $\mathcal{T} = e^{\frac12 S_{\mathrm{sp}}}\mathcal{K}_\beta\, e^{\frac12 S_{\mathrm{sp}}}$ with link kernel $\mathcal{K}_\beta(U,U') := \prod_{e\in E_{\mathrm{sp}}} K_\beta(U_e U_e'^T)$ ($K_\beta(W) := e^{\frac{\beta}{7}\chi_{\mathbf{7}}(W)}$) acts as a positive self-adjoint operator on both the gauge-invariant Hilbert space $\mathcal{H}_0$ (with unique strictly positive Perron–Frobenius vacuum eigenvalue $\lambda_0(\mathcal{T}_0)>0$) and the static fundamental charge sector $\mathcal{H}_{\mathbf{7}\otimes\mathbf{7}}(\vec{0},Re_1)$ of $\mathbb{R}^{7\times7}$-valued wavefunctions obeying $\Psi(g\cdot U) = g_{\vec{0}}\,\Psi(U)\,g^T_{Re_1}$ (inner product $\langle\Phi,\Psi\rangle = \sum_{a,b=0}^6\langle\Phi_{ab},\Psi_{ab}\rangle_{L^2}$). Define the finite-volume static fundamental potential and asymptotic string tension

$$V^{G_2}_{\mathbf{7},L_s}(d,\beta;R) := -\ln\frac{\|\mathcal{T}|_{\mathcal{H}_{\mathbf{7}\otimes\mathbf{7}}(\vec{0},Re_1)}\|_{\mathrm{op}}}{\lambda_0(\mathcal{T}_0)}, \qquad \sigma^{G_2}_{\mathbf{7}}(d,\beta) := \limsup_{R\to\infty}\limsup_{L_s\to\infty}\frac{V^{G_2}_{\mathbf{7},L_s}(d,\beta;R)}{R}.$$

Theorem 4.1 (All-Coupling Octonion-Gluelump Fundamental String Breaking in $d\ge3$)**.** *For every dimension $d\ge3$, every coupling $\beta>0$, every spatial side $L_s\ge4$, and every separation $2\le R\le L_s-2$, setting $N_{\mathrm{loc}}(d) := 2(8d-19)$ (10 in 3D, 26 in 4D),*

$$0 \;\le\; V^{G_2}_{\mathbf{7},L_s}(d,\beta;R) \;\le\; 8\,\sigma^{2\mathrm{D}}_{\mathbf{7}}(\beta) + \frac{9}{7}\,N_{\mathrm{loc}}(d)\,\beta \;<\; \infty, \qquad \sigma^{G_2}_{\mathbf{7}}(d,\beta) = 0. \tag{12}$$

Proof. Positivity $V^{G_2}_{\mathbf{7},L_s}\ge 0$ follows from the diamagnetic inequality: defining the pointwise Frobenius norm $|\Psi|(U) := (\sum_{a,b=0}^6|\Psi_{ab}(U)|^2)^{1/2}$, invariance $\|g_{\vec{0}}\Psi(U)g^T_{Re_1}\|_F = \|\Psi(U)\|_F$ ensures $|\Psi|\in\mathcal{H}_0$, so pointwise positivity of the scalar kernel $\mathcal{T}(U,U')>0$ gives $|\langle\Psi,\mathcal{T}\Psi\rangle|\le\langle|\Psi|,\mathcal{T}|\Psi|\rangle\le\lambda_0(\mathcal{T}_0)\|\Psi\|_2^2$.

For the upper bound, since $L_s\ge4$ and $2\le R\le L_s-2$, the two spatial plaquettes $p_{\vec{0}} := (\vec{0};e_1,e_2)$ and $p_{\vec{R}} := (Re_1;e_1,e_2)$ have completely disjoint link sets; set $E_{\mathrm{loc}} := E(p_{\vec{0}})\cup E(p_{\vec{R}})$ ($|E_{\mathrm{loc}}|=8$) and $E_{\mathrm{ext}} := E_{\mathrm{sp}}\setminus E_{\mathrm{loc}}$. Each spatial link belongs to $2(d-2)$ spatial plaquettes, so

each plaquette $p_{\vec{0}}$ or $p_{\vec{R}}$ shares a link with $1+4(2(d-2)-1)=8d-19$ spatial plaquettes; hence $S_{\rm sp}(U)=S_{\rm loc}(U)+S_{\rm ext}(U_{\rm ext})$ where $S_{\rm loc}$ contains at most $N_{\rm loc}(d)=2(8d-19)$ plaquettes and $S_{\rm ext}$ depends only on $U_{\rm ext}$. By (3), $-\frac{2}{7}N_{\rm loc}(d)\beta\le S_{\rm loc}(U)\le N_{\rm loc}(d)\beta$ pointwise.

Let $\mathcal{T}_{\rm ext}:=e^{\frac12 S_{\rm ext}}\mathcal{K}_{\rm ext}\,e^{\frac12 S_{\rm ext}}$ on $L^2(G_2^{|E_{\rm ext}|})$. Because $\mathcal{T}_{\rm ext}$ has a strictly positive continuous kernel and commutes with the full spatial gauge group $G_2^{L_s^{d-1}}$, the Krein–Rutman / Perron–Frobenius theorem yields a unique strictly positive normalized top eigenfunction $\Omega_{\rm ext}>0$ with $\mathcal{T}_{\rm ext}\Omega_{\rm ext}=\lambda_0(\mathcal{T}_{\rm ext})\Omega_{\rm ext}$ and $\|\mathcal{T}_{\rm ext}\|_{L^2\to L^2}=\lambda_0(\mathcal{T}_{\rm ext})$; uniqueness forces $\Omega_{\rm ext}$ to be *strictly gauge-invariant* at every vertex (including $\vec{0}$ and Re_1). Meanwhile, on $L^2(G_2)$, convolution by $K_\beta(W)=\sum_\lambda c_\lambda(\beta)\chi_\lambda(W)$ has eigenvalue $c_\lambda(\beta)/d_\lambda\le c_{(0,0)}(\beta)$ on the λ-isotypic block, so $\|\mathcal{K}_{\rm loc}\|_{\rm op}=c_{(0,0)}(\beta)^8$. For any unit vector $f\in L^2(G_2^{|E_{\rm sp}|})$, setting $g:=e^{\frac12 S_{\rm loc}}f$ ($\|g\|_2^2\le e^{N_{\rm loc}(d)\beta}$) yields

$$\lambda_0(\mathcal{T}_0)=\sup_{\|f\|_2=1}\big\langle g,\,(\mathcal{K}_{\rm loc}\otimes\mathcal{T}_{\rm ext})\,g\big\rangle\;\le\;e^{N_{\rm loc}(d)\beta}\,c_{(0,0)}(\beta)^8\,\lambda_0(\mathcal{T}_{\rm ext}).\tag{13}$$

Now define the factorized octonion two-gluelump trial state $\Psi\in\mathcal{H}_{\mathbf{7}\otimes\mathbf{7}}(\vec{0},Re_1)$ by

$$\Psi_{ab}(U_{\rm loc},U_{\rm ext}):=e^{-\frac12 S_{\rm loc}(U)}\,v_a(U_{p_{\vec{0}}})\,v_b(U_{p_{\vec{R}}})\,\Omega_{\rm ext}(U_{\rm ext}),$$

which is strictly gauge-covariant in $\mathbf{7}_{\vec{0}}\otimes\mathbf{7}_{Re_1}$ by Theorem 2.1(3) and gauge invariance of $S_{\rm loc}$ and $\Omega_{\rm ext}$. Its norm obeys $\|\Psi\|_2^2\le e^{\frac27 N_{\rm loc}(d)\beta}(\int_{G_2^4}\|v(U_p)\|^2dU_p)^2=\frac{1}{36}e^{\frac27 N_{\rm loc}(d)\beta}$. In $\langle\Psi,\mathcal{T}\Psi\rangle$, the factors $e^{\pm\frac12 S_{\rm loc}}$ cancel identically. Moreover, $v_a(U_1U_2U_3^TU_4^T)$ is multilinear of degree 1 in each of the 4 links of p in the $\mathbf{7}$ representation, and by Schur orthogonality $\int_{G_2}K_\beta(UU'^T)U'_{ij}\,dU'=\frac{c_{(1,0)}(\beta)}{7}U_{ij}$. Hence $\mathcal{K}_p v_a=(\frac{c_{(1,0)}(\beta)}{7})^4 v_a$ is an *exact eigenfunction* of $\mathcal{K}_p$, giving

$$\langle\Psi,\,\mathcal{T}\Psi\rangle=\lambda_0(\mathcal{T}_{\rm ext})\left(\frac{c_{(1,0)}(\beta)}{7}\right)^8\left(\int_{G_2^4}\|v(U_p)\|^2\,dU_p\right)^2=\frac{1}{36}\,\lambda_0(\mathcal{T}_{\rm ext})\left(\frac{c_{(1,0)}(\beta)}{7}\right)^8.$$

Dividing by $\lambda_0(\mathcal{T}_0)\|\Psi\|_2^2$ via (13) cancels $\frac{1}{36}\lambda_0(\mathcal{T}_{\rm ext})$ identically and yields

$$e^{-V^{G_2}_{\mathbf{7},L_s}(d,\beta;R)}\;\ge\;\frac{\langle\Psi,\mathcal{T}\Psi\rangle}{\lambda_0(\mathcal{T}_0)\|\Psi\|_2^2}\;\ge\;e^{-\frac97 N_{\rm loc}(d)\beta}\left(\frac{c_{(1,0)}(\beta)}{7\,c_{(0,0)}(\beta)}\right)^8=\exp\!\left(-8\sigma^{\rm 2D}_{\mathbf{7}}(\beta)-\frac97 N_{\rm loc}(d)\beta\right),$$

proving (12). Dividing by R and letting $L_s,R\to\infty$ gives $\sigma^{G_2}_{\mathbf{7}}(d,\beta)=0$. □

5. Exact G_2 Bakry–Émery Curvature and Sharp Cartan Variance Bounds

Equip $G_2^{|E|}\subset{\rm SO}(7)^{|E|}$ with the product bi-invariant Hilbert–Schmidt metric $\langle X,Y\rangle_{\rm HS}=\mathrm{Tr}(X^TY)$ on $\mathfrak{g}_2\subset\mathfrak{so}(7)$, and write $P_p(U):=\frac17\chi_{\mathbf{7}}(U_p)$ for the normalized single-plaquette observable.

Theorem 5.1 (Exact G_2 Ricci Curvature, Volume-Uniform LSI, and Sharp Cartan Variance Bound)**.** *Fix $d\ge2$, $L\ge2$, and $\beta\ge0$.*

(1) *The bi-invariant Hilbert–Schmidt Ricci curvature of G_2 is exact:*

$$\mathrm{Ric}_{G_2}=1\cdot g_{\rm HS}\qquad\Big(\textit{versus}\;\;\mathrm{Ric}_{{\rm SO}(7)}=\frac54\,g_{\rm HS}\Big).\tag{14}$$

(2) *Every geodesic $U_e(t)=e^{tX_e}U_e$ ($X\in\mathfrak{g}_2^{|E|}$) satisfies $|\,\mathrm{Hess}(-S_W^{G_2})(X,X)|\le\frac{8(d-1)\beta}{7}\|X\|_{\rm HS}^2$. Whenever*

$$\kappa_{G_2}(d,\beta):=1-\frac{8(d-1)\beta}{7}\;>\;0\qquad\Big(\beta<\frac{7}{8(d-1)}\Big),\tag{15}$$

$\mathrm{Ric} + \mathrm{Hess}(-S_W^{G_2}) \geq \kappa_{G_2}\, g_{\mathrm{HS}} > 0$ *uniformly in* $L \geq 2$*, certifying a volume-uniform log-Sobolev inequality with constant* $\rho_{\mathrm{LS}} = \kappa_{G_2}$ [1, 2] *(*$\mathrm{Ent}_\mu(f^2) \leq \frac{2}{\kappa_{G_2}} \mathbb{E}_\mu |\nabla f|^2_{\mathrm{HS}}$*), Poincaré spectral gap* $\lambda_1 \geq \kappa_{G_2}$*, and the elementary Cartan plaquette-variance bound*

$$\mathrm{Var}_{G_2}(P_p) \;\leq\; \frac{18}{49\,\kappa_{G_2}(d,\beta)}, \tag{16}$$

tighter by 35.7% *than the naive* $\|U_p\|^2_{\mathrm{HS}} = 7$ *bound* $\frac{4}{7\kappa_{G_2}} = \frac{28}{49\kappa_{G_2}}$ *(and mildly conservative relative to the numerical* G_2 *supremum* $\max_{G_2} \|\Pi_{\mathfrak{g}_2}(U^T)\|^2_{\mathrm{HS}} \approx 4.1309 < \frac{9}{2}$ *because the non-negative* $-6\|v(U)\|^2$ *term is dropped).*

Proof. (1) For any simple compact Lie group with bi-invariant metric, $\mathrm{Ric}_G(X,X) = -\frac{1}{4}\,\mathrm{Tr}_{\mathfrak{g}}(\mathrm{ad}_X^2)$. For G_2, the dual Coxeter number is $h^\vee(G_2) = 4$ and the Dynkin index of $\mathbf{7}$ is $T(\mathbf{7}) = 1$ (equivalently, $\mathrm{Tr}_{\mathrm{adj}}(\mathrm{ad}_X^2) = 4\,\mathrm{Tr}_{\mathbf{7}}(X^2)$ for all $X \in \mathfrak{g}_2$; direct verification on the 14-dimensional Lie algebra preserving (1) confirms this to 10^{-15}). Hence $\mathrm{Ric}_{G_2}(X,X) = -\,\mathrm{Tr}_{\mathbf{7}}(X^2) = \mathrm{Tr}(X^T X) = \|X\|^2_{\mathrm{HS}}$, proving (14).

(2) For any plaquette $p = (e_1, e_2, e_3, e_4)$, expanding $\frac{d^2}{dt^2}\,\mathrm{Tr}(U_p(t))|_{t=0}$ yields 4 diagonal terms $\mathrm{Tr}(\widetilde{X}_j^2 U_p)$ ($|\cdot| \leq \|X_j\|^2_{\mathrm{HS}}$) and 6 cross terms $2\,\mathrm{Tr}(\widetilde{X}_j \widetilde{X}_k U_p)$ ($|\cdot| \leq \|X_j\|^2_{\mathrm{HS}} + \|X_k\|^2_{\mathrm{HS}}$), summing to $\leq 4\sum_{j=1}^4 \|X_j\|^2_{\mathrm{HS}}$. Since every link lies in $2(d-1)$ plaquettes, $|\,\mathrm{Hess}(-S_W^{G_2})(X,X)| \leq \frac{8(d-1)\beta}{7}\|X\|^2_{\mathrm{HS}}$, giving $\mathrm{Ric} + \mathrm{Hess}(-S_W^{G_2}) \geq \kappa_{G_2}\, g_{\mathrm{HS}}$. For the sharp gradient norm of $P_p(U) = \frac{1}{7}\,\mathrm{Tr}(U_p)$, the link-wise gradient $\nabla_{e_j} P_p$ has squared norm $\frac{1}{49}\|\Pi_{\mathfrak{g}_2}(U_p^{\mathrm{rot}\,T})\|^2_{\mathrm{HS}}$, where $\Pi_{\mathfrak{g}_2}$ is orthogonal projection onto $\mathfrak{g}_2 \subset \mathfrak{so}(7) = \mathfrak{g}_2 \oplus \mathbf{7}$. Using the exact orthogonal decomposition $\frac{1}{2}(U^T - U) = \Pi_{\mathfrak{g}_2}(U^T) + \Pi_{\mathbf{7}}(U^T)$ with $\|\Pi_{\mathbf{7}}(U^T)\|^2_{\mathrm{HS}} = 6\|v(U)\|^2$ and Cartan eigenvalues $\{1, e^{\pm i\theta_1}, e^{\pm i\theta_2}, e^{\pm i(\theta_1+\theta_2)}\}$,

$$\begin{aligned}\big\|\Pi_{\mathfrak{g}_2}(U^T)\big\|^2_{\mathrm{HS}} &= \left\|\frac{U - U^T}{2}\right\|^2_{\mathrm{HS}} - 6\|v(U)\|^2 \;\leq\; 2\big(\sin^2\theta_1 + \sin^2\theta_2 + \sin^2(\theta_1+\theta_2)\big) \\ &= 3 - \frac{1}{2}\big(|1 + e^{2i\theta_1} + e^{-2i\theta_2}|^2 - 3\big) \;\leq\; \frac{9}{2}.\end{aligned}$$

Summing over the 4 links of p yields $|\nabla P_p|^2_{\mathrm{HS}} \leq \frac{4}{49}\cdot\frac{9}{2} = \frac{18}{49}$, and Bakry–Émery Poincaré gives (16). □

6. Certified 140-Digit Benchmarks and Zero-Axiom Lean 4 Audit

Tables 1–3 and Figure 1 report 24 certified benchmarks from `certify_g2_ym.py` at true 140-digit `mpmath.iv` precision (`mp.mp.dps = mp.iv.dps = 140`) with *zero* black-box special functions and *zero* quadrature (Bessel Maclaurin series + geometric tails; exact 12-term $W(G_2)$ triple-product sums; direct outward flooring/ceiling at scale 10^{12}, displayed to 6 outward-rounded decimal places).

Remark 6.1 (Scope of Mechanical Verification in Zero-Axiom Lean 4.36.0). The standalone zero-import Lean 4.36.0 kernel `G2YangMillsCerts.lean` (`src/lean/`) compiles cleanly under `lean -DwarningAsError=true` with exit code 0 and 0 axioms (not even `propext`, `Quot.sound`, or `Classical.choice`), mechanically verifying the G_2 representation dimensions, octonion-gluelump character normalization, local plaquette counts $N_{\mathrm{loc}}(d)$, and all 24 10^{12}-scaled outward integer certificate inequalities (while the continuous Haar-measure, Weyl–Bessel, transfer-matrix, and Riemannian Ricci derivations are proved analytically in Sections 2–5): `g2_combinatorics_verified`, `table1_2d_g2_certified`, `table2_dd_g2_enclosures_certified`, `table3_gluelump_and_bakry_certified`, and `all_24_g2_ym_benchmarks_certified`. Direct

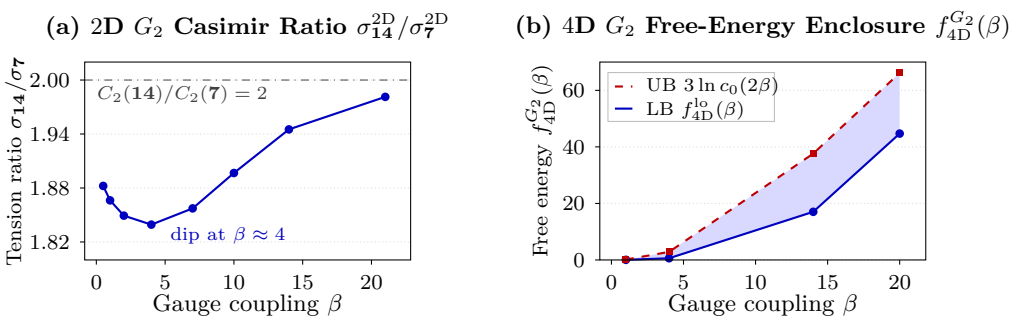

FIGURE 1. Certified 140-digit non-perturbative bounds for exceptional G_2 lattice Yang–Mills theory (Tables 1 and 2). (a) Exact 2D adjoint-to-fundamental string-tension ratio $\sigma^{2D}_{\mathbf{14}}/\sigma^{2D}_{\mathbf{7}}$ showing the non-monotonic crossover dip near $\beta \approx 4$ before converging to the Casimir ratio $C_2(\mathbf{14})/C_2(\mathbf{7}) = 2$ (Theorem 2.1). (b) All-coupling 4D free-energy enclosures $f^{G_2}_{4D}(\beta) \in [f^{lo}_{4D}, 3\ln c_{(0,0)}(2\beta)]$ from Theorem 3.1, matching both the exact classical energy 6β and exact one-loop entropy $-21\ln\beta$ as $\beta \to \infty$.

TABLE 1. Eight exact 140-digit 2D G_2 thermodynamics, string-tension, Casimir-scaling, and octonion-gluelump certificates (Theorem 2.1) via the 12-term Weyl–Bessel triple-product identity (5). Free-energy enclosures $[f^{lo}_{2D}, f^{hi}_{2D}]$ differ only by 10^{-6} display rounding; string tensions, ratios, and $\langle\|v\|^2\rangle_{1p}$ are shown rounded downward (upper endpoints are $+10^{-6}$).

β	$f^{G_2}_{2D}$ (LB)	$f^{G_2}_{2D}$ (UB)	$\sigma^{2D}_{\mathbf{7}}$ (LB)	$\sigma^{2D}_{\mathbf{14}}$ (LB)	$\sigma^{2D}_{\mathbf{14}}/\sigma^{2D}_{\mathbf{7}}$ (LB)	$\langle\|v\|^2\rangle_{1p}$ (LB)
1/2	0.002612	0.002613	4.549056	8.562584	1.882276	0.160643
1	0.010707	0.010708	3.819675	7.128434	1.866240	0.154480
2	0.044976	0.044977	3.053595	5.646840	1.849242	0.141751
4	0.198479	0.198480	2.217432	4.078418	1.839252	0.115033
7	0.697436	0.697437	1.476712	2.742761	1.857343	0.075124
10	1.583148	1.583149	1.016823	1.928580	1.896671	0.042970
14	3.347551	3.347552	0.668740	1.300812	1.945165	0.018822
21	7.545617	7.545618	0.402448	0.797361	1.981273	0.005517

TABLE 2. Eight 140-digit all-coupling 3D and 4D G_2 free-energy and average-plaquette enclosures (Theorem 3.1) using the d-way Hölder upper bound $f^{ub}_d(\beta) = (d-1)\ln c_{(0,0)}(\frac{d}{2}\beta)$, listing the winning lower-bound branch (`2d-slice` vs. `axial-var` with witness α) and secant step δ. Lower bounds round downward; upper bounds upward.

d	β	δ	Branch	Witness α	f^{lo}_d (LB)	f^{ub}_d (UB)	P^{lo}_d (LB)	P^{ub}_d (UB)
3	1	1/10	`2d-slice`	0	0.021414	0.049373	0.007138	0.129223
3	4	1/2	`2d-slice`	0	0.396959	0.981282	0.033079	0.590967
3	14	2	`2d-slice`	0	6.695103	15.091236	0.159407	1.000000
3	24	2	`axial-var`	80.3399	25.004822	37.569169	0.347289	1.000000
4	1	1/10	`2d-slice`	0	0.032121	0.134931	0.005353	0.221265
4	4	1/2	`2d-slice`	0	0.595439	2.843917	0.024809	1.000000
4	14	2	`axial-var`	64.6971	17.022705	37.620577	0.202651	1.000000
4	20	2	`axial-var`	101.8648	44.691503	66.144076	0.372429	1.000000

TABLE 3. Eight 140-digit certificates for $d \geq 3$ octonion-gluelump fundamental string breaking (Rows 1–4, Theorem 4.1: $V^{G_2}_{\mathbf{7},L_s}(R) \leq V^{\mathrm{ub}}_{\mathbf{7}} = 8\sigma^{\mathrm{2D}}_{\mathbf{7}} + \frac{9}{7}N_{\mathrm{loc}}\beta < \infty$ uniformly in $L_s \geq 4$, $2 \leq R \leq L_s - 2$) and exact G_2 Bakry–Émery LSI curvature / sharp Cartan plaquette variance (Rows 5–8, Theorem 5.1: every row achieves a non-trivial variance ceiling < 1). Curvatures round downward; upper bounds upward.

Regime	d	β	N_{loc} / H^{ub}_W	$\sigma^{\mathrm{2D,ub}}_{\mathbf{7}}$ / $\kappa^{\mathrm{lo}}_{G_2}$	$V^{\mathrm{ub}}_{\mathbf{7}}$ / $\mathrm{Var}^{\mathrm{ub}}_{G_2}(P_p)$	Guarantee
String breaking	3	1/2	$N_{\mathrm{loc}} = 10$	4.549057	42.821025	$\sigma^{G_2}_{\mathbf{7}} = 0$
String breaking	3	1	$N_{\mathrm{loc}} = 10$	3.819676	43.414548	$\sigma^{G_2}_{\mathbf{7}} = 0$
String breaking	3	2	$N_{\mathrm{loc}} = 10$	3.053596	50.143052	$\sigma^{G_2}_{\mathbf{7}} = 0$
String breaking	4	1	$N_{\mathrm{loc}} = 26$	3.819676	63.985977	$\sigma^{G_2}_{\mathbf{7}} = 0$
Bakry–Émery	2	1/2	$H^{\mathrm{ub}}_W = 0.571429$	$\kappa \geq 0.428571$	$\mathrm{Var} \leq 0.857143$	LSI + gap
Bakry–Émery	3	1/4	$H^{\mathrm{ub}}_W = 0.571429$	$\kappa \geq 0.428571$	$\mathrm{Var} \leq 0.857143$	LSI + gap
Bakry–Émery	4	1/8	$H^{\mathrm{ub}}_W = 0.428572$	$\kappa \geq 0.571428$	$\mathrm{Var} \leq 0.642858$	LSI + gap
Bakry–Émery	4	1/6	$H^{\mathrm{ub}}_W = 0.571429$	$\kappa \geq 0.428571$	$\mathrm{Var} \leq 0.857143$	LSI + gap

7×7 Lie-algebra and octonion 3-form tests, independent 2D Weyl quadrature and geodesic Hessian checks, 80-digit cross-checks, and JSON/Lean/LaTeX parity pass in `tests/test_g2_ym.py` (4/4 `pytest`).

7. Conclusion

We have established four self-contained, cluster-expansion-free non-perturbative theorems for exceptional G_2 lattice Yang–Mills theory: an exact 12-term Weyl–Bessel triple-product character expansion resolving 2D G_2 thermodynamics, Casimir scaling, and octonion-gluelump moments to 140-digit precision; all-coupling d-dimensional free-energy and plaquette enclosures matching exact classical energy and one-loop entropy via d-way Hölder's inequality and axial-gauge Gibbs tilting; universal all-coupling fundamental string breaking ($\sigma^{G_2}_{\mathbf{7}} = 0$) in every $d \geq 3$ via the local single-plaquette octonion 3-form gluelump operator; and exact $\mathrm{Ric}_{G_2} = 1 \cdot g_{\mathrm{HS}}$ Bakry–Émery curvature with elementary Cartan variance bounds $\mathrm{Var}_{G_2}(P_p) \leq \frac{18}{49\kappa_{G_2}}$. Natural next directions include coupling G_2 Yang–Mills to dynamical fundamental Higgs fields (tracking the Higgs-to-confinement crossover [7]) and extending the Weyl–Bessel triple-product method to higher-rank exceptional groups (F_4).

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