Non-Asymptotic Online Learning: Adaptive Hedge, Optimistic Mirror Descent Saddle-Point Games, Exp3-IX Bandits, and Cone Blackwell Approachability

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Abstract

We establish four self-contained, explicit, non-asymptotic quantitative theorems unifying full-information online learning, zero-sum bilinear saddle-point games, adversarial multi-armed bandits, and Blackwell approachability. First, for the Hedge (exponential weights) algorithm on bounded losses $\ell_t \in [0,1]^K$, we derive both an exact first-order small-loss ceiling $\mathrm{Reg}_T(u) \le \sqrt{2L_T^* \,\mathrm{KL}(u\|p_1)} + \mathrm{KL}(u\|p_1)$ (under tuned step size, valid for all $L_T^* \ge 0$) and a second-order variance-adaptive ceiling $\frac{\mathrm{KL}(u\|p_1)}{\eta} + \min\big(\frac{\eta}{8}T, \frac{e^\eta - 1 - \eta}{\eta}\sum_{t=1}^T \mathrm{Var}_{p_t}(\ell_t)\big)$, paired with a three-line Jensen telescoping proof of a doubling-free anytime schedule $\eta_t = \sqrt{4\ln(K)/t}$ achieving $\mathrm{Reg}_T \le \sqrt{T\ln K} - \frac{1}{4}\sqrt{\ln K}$ for every $T \ge 1$ (Theorem 2.1). Second, equipping $\Delta_m \times \Delta_n$ with the product norm $\|(x,y)\| = \sqrt{\|x\|_1^2 + \|y\|_1^2}$ (making negative entropy 1-strongly convex with exact operator Lipschitz constant $L = \|A\|_\infty$), we give a self-contained three-point proof that Optimistic Mirror Descent with $\eta \le \frac{1}{2\sqrt{2}L}$ achieves the fast $\mathrm{Gap}(\bar{x}_T, \bar{y}_T) \le 2\sqrt{2}\|A\|_\infty(\ln m + \ln n)/T$ duality-gap rate with zero initialization residual (Theorem 3.1). Third, for oblivious adversarial K-armed bandits, we prove both the zero-exploration unbiased Exp3 second-moment bound $\mathbb{E}[\mathrm{Reg}_T^{\mathrm{pseudo}}] \le \frac{\ln K}{\eta} + \frac{\eta}{2}S_2$ (yielding $\sqrt{2S_2\ln K}$ under oracle tuning and $\sqrt{2KT\ln K}$ universally) and a self-contained martingale proof of the high-probability Exp3-IX bound $\mathrm{Reg}_T \le 2\sqrt{KT(\ln K + 2\ln(2K/\delta))} + 2\ln(2/\delta)$ with probability at least $1-\delta$ (Theorem 4.1). Fourth, invoking Moreau's orthogonal cone decomposition, we prove that Blackwell approachability toward any closed convex cone $\mathcal{C} \subseteq \mathbb{R}^d$ satisfies the data-adaptive ceiling $\mathrm{dist}_2(\bar{m}_T, \mathcal{C}) \le \frac{1}{T}\sqrt{\sum_{t=1}^T \|m_t\|_2^2} \le R/\sqrt{T}$—strictly cutting the classical general-convex constant $2R/\sqrt{T}$ in half—paired with an explicit online-to-batch convex generalization transfer (Theorem 5.1). All four theorems, boundary regimes, and numerical benchmarks are deterministically verified in an MIT-licensed Python test suite.

1 Introduction and Scope

Online convex optimization, zero-sum bilinear games, adversarial multi-armed bandits, and Blackwell approachability form an interconnected core of sequential decision theory [1, 2, 3, 4]. Standard textbook presentations, however, frequently rely on doubling tricks for anytime horizons, incur loose constants in product-simplex norms or Blackwell cone projections, require explicit exploration mixing in Exp3, or omit self-contained high-probability proofs for implicit exploration.

Sections 2–5 resolve all four issues in four self-contained, completely explicit non-asymptotic theorems with elementary, gap-free proofs, followed by deterministic numerical verification in Section 6 (`https://github.com/lluiseriksson/online-learning-bandits-games-geometry`, tag `v1.0.0`).

2 Small-Loss, Variance-Adaptive, and Anytime Hedge

Let $\Delta_K := \{p \in \mathbb{R}^K_{\ge 0} : \sum_{i=1}^K p_i = 1\}$ $(K \ge 2)$ and $\Delta_K^+ := \Delta_K \cap \mathbb{R}^K_{>0}$. Given losses $\ell_1, \dots, \ell_T \in [0,1]^K$ and prior $p_1 \in \Delta_K^+$, define cumulative loss $L_t := \sum_{s=1}^t \ell_s$ $(L_0 := 0)$ and regret against any

comparator $u \in \Delta_K$ by $\mathrm{Reg}_T(u) := \sum_{t=1}^T \langle p_t - u, \ell_t \rangle$, with $L_T^*(u) := \sum_{t=1}^T \langle u, \ell_t \rangle$.

Theorem 2.1 (First-Order, Second-Order, and Doubling-Free Anytime Hedge)**.**

(a) ***Small-Loss and Variance-Adaptive Fixed-Step Regret.*** *For fixed $\eta > 0$, let Hedge play $p_{t+1,i} \propto p_{t,i} e^{-\eta \ell_{t,i}}$ $(t \geq 1)$. Then for every comparator $u \in \Delta_K$ (writing $D := \mathrm{KL}(u \| p_1) < \infty$ and $V_T := \sum_{t=1}^T \mathrm{Var}_{p_t}(\ell_t)$ where $\mathrm{Var}_p(\ell) := \sum_{i=1}^K p_i(\ell_i - \langle p, \ell \rangle)^2$),*

$$\mathrm{Reg}_T(u) \;\leq\; \frac{D}{\eta} + \min\left(\frac{\eta}{8} T,\; \frac{e^\eta - 1 - \eta}{\eta} V_T \right), \tag{1}$$

and simultaneously

$$\mathrm{Reg}_T(u) \;\leq\; \left(\frac{\eta}{1 - e^{-\eta}} - 1 \right) L_T^*(u) + \frac{D}{1 - e^{-\eta}}. \tag{2}$$

In particular, if $D > 0$ and $L_T^*(u) > 0$ with $\eta = \ln\bigl(1 + \sqrt{2D/L_T^*(u)}\bigr)$ (or in the limit $\eta \to \infty$ when $L_T^*(u) = 0$, or $\eta \to 0^+$ when $D = 0$, i.e. $u = p_1$ where static $p_t \equiv p_1$ gives $\mathrm{Reg}_T(p_1) = 0$), then

$$\mathrm{Reg}_T(u) \;\leq\; \sqrt{2\, L_T^*(u)\, D} + D. \tag{3}$$

(b) ***Doubling-Free Anytime Hedge.*** *For any positive non-increasing learning-rate schedule $\eta_1 \geq \eta_2 \geq \cdots \geq \eta_T > 0$ with $p_{t,i} \propto e^{-\eta_t L_{t-1,i}}$ (uniform p_1),*

$$\max_{u \in \Delta_K} \mathrm{Reg}_T(u) \;\leq\; \frac{\ln K}{\eta_T} + \frac{1}{8} \sum_{t=1}^T \eta_t. \tag{4}$$

In particular, the horizon-free schedule $\eta_t := \sqrt{\frac{4 \ln K}{t}}$ satisfies, simultaneously for every $T \geq 1$,

$$\max_{u \in \Delta_K} \mathrm{Reg}_T(u) \;\leq\; \sqrt{T \ln K} - \frac{1}{4} \sqrt{\ln K}. \tag{5}$$

Proof. (a) Define $\Phi_t := \ln\bigl(\sum_{i=1}^K p_{1,i} e^{-\eta L_{t,i}}\bigr)$ $(\Phi_0 = 0)$. Dropping any indices with $u_i = 0$ (whose terms are nonnegative) and applying Jensen's inequality on ln,

$$\Phi_T \;\geq\; \ln \sum_{i:\, u_i > 0} u_i \frac{p_{1,i}}{u_i} e^{-\eta L_{T,i}} \;\geq\; \sum_{i:\, u_i > 0} u_i \ln \frac{p_{1,i}}{u_i} - \eta \sum_{i=1}^K u_i L_{T,i} = -D - \eta L_T^*(u).$$

On the other hand, $\Phi_t - \Phi_{t-1} = \ln \mathbb{E}_{i \sim p_t}[e^{-\eta \ell_{t,i}}]$. We bound this increment in three ways:

(i) Since $\ell_{t,i} \in [0,1]$, Hoeffding's lemma gives $\ln \mathbb{E}_{p_t}[e^{-\eta \ell_{t,i}}] \leq -\eta \langle p_t, \ell_t \rangle + \frac{\eta^2}{8}$.

(ii) Let $X := \ell_{t,i} - \langle p_t, \ell_t \rangle \in [-1,1]$, so $\mathbb{E}_{p_t}[X] = 0$. The function $r(u) := \frac{e^u - 1 - u}{u^2}$ $(r(0) = 1/2)$ admits the exact integral representation $r(u) = \int_0^1 (1-s) e^{su}\, ds$, which is strictly increasing on $\mathbb{R}$. Since $-\eta x \leq \eta$ for all $x \in [-1,1]$, we obtain $r(-\eta x) \leq r(\eta) = \frac{e^\eta - 1 - \eta}{\eta^2}$, i.e. $e^{-\eta x} \leq 1 - \eta x + (e^\eta - 1 - \eta) x^2$. Taking $\mathbb{E}_{p_t}$ and using $\ln(1+z) \leq z$ gives $\ln \mathbb{E}_{p_t}[e^{-\eta \ell_{t,i}}] \leq -\eta \langle p_t, \ell_t \rangle + (e^\eta - 1 - \eta)\, \mathrm{Var}_{p_t}(\ell_t)$.

(iii) By convexity of $x \mapsto e^{-\eta x}$ on $[0,1]$, $e^{-\eta \ell_{t,i}} \leq 1 - (1 - e^{-\eta}) \ell_{t,i}$, so $\ln \mathbb{E}_{p_t}[e^{-\eta \ell_{t,i}}] \leq -(1 - e^{-\eta}) \langle p_t, \ell_t \rangle$.

Summing (i)–(ii) over $t = 1, \ldots, T$ and combining with $\Phi_T \geq -D - \eta L_T^*(u)$ gives (1); summing (iii) gives $-(1 - e^{-\eta}) \sum_t \langle p_t, \ell_t \rangle \geq -D - \eta L_T^*(u)$, rearranging to (2). Setting $\eta = \ln(1 + \alpha)$ $(\alpha > 0)$ gives $1 - e^{-\eta} = \frac{\alpha}{1+\alpha}$ and $\frac{\eta}{1 - e^{-\eta}} - 1 = \frac{(1+\alpha)\ln(1+\alpha) - \alpha}{\alpha} \leq \frac{\alpha}{2}$ (since $g(\alpha) := \frac{\alpha^2}{2} - (1+\alpha)\ln(1+\alpha) + \alpha$ satisfies $g(0) = 0$ and $g'(\alpha) = \alpha - \ln(1+\alpha) > 0$ for $\alpha > 0$). Thus $\mathrm{Reg}_T(u) \leq \frac{\alpha}{2} L_T^*(u) + (1 + \frac{1}{\alpha}) D$; setting $\alpha = \sqrt{2D/L_T^*(u)}$ when $L_T^*(u) > 0$ (and letting $\alpha \to \infty$ when $L_T^*(u) = 0$) yields (3).

(b) Define $W_t(\eta) := \frac{1}{K}\sum_{i=1}^{K} e^{-\eta L_{t,i}}$ ($W_0(\eta) = 1$). At $t = 1$, since $W_0(\eta_1) = 1$, Hoeffding's lemma gives

$$\frac{1}{\eta_1}\ln W_1(\eta_1) = \frac{1}{\eta_1}\ln \mathbb{E}_{p_1}[e^{-\eta_1 \ell_{1,i}}] \;\le\; -\langle p_1, \ell_1\rangle + \frac{\eta_1}{8}.$$

For every $t \ge 2$, since $\alpha_t := \eta_t/\eta_{t-1} \in (0, 1]$, concavity of $x \mapsto x^{\alpha_t}$ on $(0, \infty)$ and Jensen's inequality give

$$W_{t-1}(\eta_{t-1})^{\eta_t/\eta_{t-1}} = \left(\frac{1}{K}\sum_{i=1}^{K} e^{-\eta_{t-1}L_{t-1,i}}\right)^{\alpha_t} \;\ge\; \frac{1}{K}\sum_{i=1}^{K} e^{-\eta_t L_{t-1,i}} = W_{t-1}(\eta_t),$$

whence $\frac{1}{\eta_{t-1}}\ln W_{t-1}(\eta_{t-1}) \ge \frac{1}{\eta_t}\ln W_{t-1}(\eta_t)$, and therefore $-\frac{1}{\eta_{t-1}}\ln W_{t-1}(\eta_{t-1}) \le -\frac{1}{\eta_t}\ln W_{t-1}(\eta_t)$. Combining with Hoeffding's lemma at step $t \ge 2$,

$$\frac{1}{\eta_t}\ln W_t(\eta_t) - \frac{1}{\eta_{t-1}}\ln W_{t-1}(\eta_{t-1}) \;\le\; \frac{1}{\eta_t}\ln \frac{W_t(\eta_t)}{W_{t-1}(\eta_t)} = \frac{1}{\eta_t}\ln \mathbb{E}_{p_t}[e^{-\eta_t \ell_{t,i}}] \;\le\; -\langle p_t, \ell_t\rangle + \frac{\eta_t}{8}.$$

Summing the $t = 1$ inequality and the $t = 2, \dots, T$ telescoping inequalities yields $\frac{1}{\eta_T}\ln W_T(\eta_T) \le -\sum_{t=1}^{T}\langle p_t, \ell_t\rangle + \frac{1}{8}\sum_{t=1}^{T}\eta_t$. Since $W_T(\eta_T) \ge \frac{1}{K}e^{-\eta_T \min_i L_{T,i}}$, we obtain (4). Finally, for $\eta_t = \sqrt{4\ln(K)/t}$, $\sum_{t=1}^{T} t^{-1/2} = 1 + \sum_{t=2}^{T} t^{-1/2} \le 1 + \int_1^T s^{-1/2}\,ds = 2\sqrt{T} - 1$ gives $\frac{\ln K}{\eta_T} + \frac{1}{8}\sum_{t=1}^{T}\eta_t \le \frac{1}{2}\sqrt{T\ln K} + \frac{1}{4}(2\sqrt{T} - 1)\sqrt{\ln K} = \sqrt{T\ln K} - \frac{1}{4}\sqrt{\ln K}$, proving (5). □

3 Optimistic Mirror Descent $O(1/T)$ Zero-Sum Games

Consider a zero-sum matrix game $\min_{x\in\Delta_m}\max_{y\in\Delta_n} x^\top A y$ with payoff matrix $A \in \mathbb{R}^{m\times n}$ ($m, n \ge 2$). Set $\mathcal{Z} := \Delta_m \times \Delta_n$, equip $\mathbb{R}^m \times \mathbb{R}^n$ with the product norm $\|(x, y)\| := \sqrt{\|x\|_1^2 + \|y\|_1^2}$ and dual norm $\|(u, v)\|_* := \sqrt{\|u\|_\infty^2 + \|v\|_\infty^2}$, and let $\Phi(x, y) := \sum_i x_i \ln x_i + \sum_j y_j \ln y_j$ with Bregman divergence $D_\Phi(z\|z') = \mathrm{KL}(x\|x') + \mathrm{KL}(y\|y')$. Define the saddle-point operator $F(x, y) := (Ay, -A^\top x)$.

Theorem 3.1 (Fast $O(1/T)$ Optimistic Mirror Descent for Zero-Sum Matrix Games)**.** *Set $L := \|A\|_\infty := \max_{i,j}|A_{ij}|$. Assume $L > 0$ (if $A = 0$, every strategy pair has duality gap 0). Initialize $z_1 = \widetilde{z}_0 = (\frac{1}{m}\mathbf{1}, \frac{1}{n}\mathbf{1})$, and for $t = 1, \dots, T$ run Optimistic Mirror Descent (OMD) with step size $\eta \in \big(0, \frac{1}{2\sqrt{2}\,L}\big]$:*

$$\widetilde{z}_t = \arg\min_{z\in\mathcal{Z}}\big\{\eta\langle F(\widetilde{z}_{t-1}), z\rangle + D_\Phi(z\|z_t)\big\}, \qquad z_{t+1} = \arg\min_{z\in\mathcal{Z}}\big\{\eta\langle F(\widetilde{z}_t), z\rangle + D_\Phi(z\|z_t)\big\}. \tag{6}$$

Then the time-averaged strategy pair $\bar{z}_T = (\bar{x}_T, \bar{y}_T) := \frac{1}{T}\sum_{t=1}^{T}\widetilde{z}_t$ satisfies, for every $T \ge 1$,

$$\mathrm{Gap}(\bar{x}_T, \bar{y}_T) \;:=\; \max_{y\in\Delta_n}\bar{x}_T^\top A y - \min_{x\in\Delta_m} x^\top A\bar{y}_T \;\le\; \frac{\ln m + \ln n}{\eta\, T}, \tag{7}$$

and in particular at $\eta = \frac{1}{2\sqrt{2}\,L}$, $\mathrm{Gap}(\bar{x}_T, \bar{y}_T) \le \frac{2\sqrt{2}\,\|A\|_\infty(\ln m + \ln n)}{T}$.

Proof. By Pinsker's inequality on each simplex factor, $D_\Phi(z\|z') = \mathrm{KL}(x\|x') + \mathrm{KL}(y\|y') \ge \frac{1}{2}\|x - x'\|_1^2 + \frac{1}{2}\|y - y'\|_1^2 = \frac{1}{2}\|z - z'\|^2$, so Φ is 1-strongly convex w.r.t. $\|\cdot\|$. Moreover, $\|F(z) - F(z')\|_* = \sqrt{\|A(y - y')\|_\infty^2 + \|A^\top(x - x')\|_\infty^2} \le L\sqrt{\|y - y'\|_1^2 + \|x - x'\|_1^2} = L\|z - z'\|$.

Fix any $z = (x, y) \in \mathcal{Z}$ and any $t \ge 1$. First-order optimality of the two KL-prox steps in (6) (which remain strictly positive on $\Delta_m^+ \times \Delta_n^+$) combined with the Bregman three-point identity gives

$$\begin{aligned}\eta\langle F(\widetilde{z}_{t-1}), \widetilde{z}_t - z_{t+1}\rangle \;&\le\; D_\Phi(z_{t+1}\|z_t) - D_\Phi(z_{t+1}\|\widetilde{z}_t) - D_\Phi(\widetilde{z}_t\|z_t),\\ \eta\langle F(\widetilde{z}_t), z_{t+1} - z\rangle \;&\le\; D_\Phi(z\|z_t) - D_\Phi(z\|z_{t+1}) - D_\Phi(z_{t+1}\|z_t).\end{aligned}$$

Adding cancels $D_\Phi(z_{t+1}\|z_t)$ and yields

$$\eta\langle F(\widetilde{z}_t), \widetilde{z}_t - z\rangle \;\le\; D_\Phi(z\|z_t) - D_\Phi(z\|z_{t+1}) + \eta\langle F(\widetilde{z}_t) - F(\widetilde{z}_{t-1}), \widetilde{z}_t - z_{t+1}\rangle - D_\Phi(z_{t+1}\|\widetilde{z}_t) - D_\Phi(\widetilde{z}_t\|z_t).$$

By Hölder's inequality, L-Lipschitzness, Young's inequality $ab \le a^2 + \frac{1}{4}b^2$, and 1-strong convexity ($D_\Phi(u\|v) \ge \frac{1}{2}\|u-v\|^2$):

$$\eta\langle F(\widetilde{z}_t) - F(\widetilde{z}_{t-1}), \widetilde{z}_t - z_{t+1}\rangle - D_\Phi(z_{t+1}\|\widetilde{z}_t) \;\le\; \eta^2 L^2 \|\widetilde{z}_t - \widetilde{z}_{t-1}\|^2 - \frac{1}{4}\|z_{t+1} - \widetilde{z}_t\|^2.$$

Using $\|\widetilde{z}_t - \widetilde{z}_{t-1}\|^2 \le 2\|\widetilde{z}_t - z_t\|^2 + 2\|z_t - \widetilde{z}_{t-1}\|^2$ and $\eta^2 L^2 \le \frac{1}{8}$ (as $\eta \le \frac{1}{2\sqrt{2}\,L}$), together with $-D_\Phi(\widetilde{z}_t\|z_t) \le -\frac{1}{2}\|\widetilde{z}_t - z_t\|^2$, we obtain

$$\eta\langle F(\widetilde{z}_t), \widetilde{z}_t - z\rangle \;\le\; D_\Phi(z\|z_t) - D_\Phi(z\|z_{t+1}) + \frac{1}{4}\|z_t - \widetilde{z}_{t-1}\|^2 - \frac{1}{4}\|z_{t+1} - \widetilde{z}_t\|^2 - \frac{1}{4}\|\widetilde{z}_t - z_t\|^2.$$

Summing from $t = 1$ to T, since $z_1 = \widetilde{z}_0$ gives $\|z_1 - \widetilde{z}_0\|^2 = 0$ and the final negative terms drop,

$$\frac{1}{T}\sum_{t=1}^{T}\langle F(\widetilde{z}_t), \widetilde{z}_t - z\rangle \;\le\; \frac{D_\Phi(z\|z_1)}{\eta\, T} \;\le\; \frac{\ln m + \ln n}{\eta\, T}.$$

Because $\langle F(\widetilde{z}_t), \widetilde{z}_t - z\rangle = \langle A\widetilde{y}_t, \widetilde{x}_t - x\rangle - \langle A^\top \widetilde{x}_t, \widetilde{y}_t - y\rangle = \widetilde{x}_t^\top A y - x^\top A \widetilde{y}_t$, averaging yields $\bar{x}_T^\top A y - x^\top A \bar{y}_T \le \frac{\ln m + \ln n}{\eta T}$ for every $(x, y) \in \Delta_m \times \Delta_n$. Maximizing over (x, y) proves (7). □

4 Zero-Exploration Exp3 and High-Probability Exp3-IX Bandits

Consider an oblivious adversarial K-armed bandit ($K \ge 2$) with loss matrix $\ell \in [0,1]^{T\times K}$. At round t, the learner samples $I_t \sim p_t \in \Delta_K^+$ and observes only ℓ_{t,I_t}. For $\gamma \ge 0$, define the implicit-exploration estimator $\widehat{\ell}_{t,i}^{\mathrm{IX}} := \frac{\ell_{t,i}}{p_{t,i}+\gamma}\mathbf{1}\{I_t = i\} \ge 0$ and update $p_{t+1,i} \propto p_{t,i}\exp(-\eta\widehat{\ell}_{t,i}^{\mathrm{IX}})$ from uniform p_1. Define $S_2 := \sum_{t=1}^{T}\sum_{i=1}^{K}\ell_{t,i}^2 \le KT$ and $\mathrm{Reg}_T := \max_{i^*\in\{1,\dots,K\}}\sum_{t=1}^{T}(\ell_{t,I_t} - \ell_{t,i^*})$.

Theorem 4.1 (Unbiased Exp3 Second-Moment and High-Probability Exp3-IX Envelopes)**.**

(a) ***Zero-Exploration Unbiased Exp3 Expected Regret.*** *Setting $\gamma = 0$ (no explicit exploration mixing), for every $\eta > 0$,*

$$\mathbb{E}\big[\mathrm{Reg}_T^{\mathrm{pseudo}}\big] \;:=\; \max_{i^*\in\{1,\dots,K\}}\mathbb{E}\left[\sum_{t=1}^{T}(\ell_{t,I_t} - \ell_{t,i^*})\right] \;\le\; \frac{\ln K}{\eta} + \frac{\eta}{2}S_2 \;\le\; \frac{\ln K}{\eta} + \frac{\eta}{2}KT. \tag{8}$$

In particular, $\eta = \sqrt{2\ln(K)/(KT)}$ yields $\mathbb{E}[\mathrm{Reg}_T^{\mathrm{pseudo}}] \le \sqrt{2KT\ln K}$ (and $\sqrt{2S_2\ln K}$ at $\eta = \sqrt{2\ln(K)/S_2}$ when $S_2 > 0$).

(b) ***High-Probability Exp3-IX Certificate.*** *For any confidence $\delta \in (0,1)$, setting $\eta := \sqrt{\frac{\ln K + 2\ln(2K/\delta)}{KT}}$ and $\gamma := \eta/2$, Exp3-IX satisfies with probability at least $1-\delta$,*

$$\mathrm{Reg}_T \;\le\; 2\sqrt{KT\left(\ln K + 2\ln\frac{2K}{\delta}\right)} + 2\ln\frac{2}{\delta}. \tag{9}$$

Proof. For any $\eta > 0$, $\gamma \ge 0$, and arm i^*, let $\Psi_t := \ln(\frac{1}{K}\sum_{i=1}^{K}\exp(-\eta\sum_{s=1}^{t}\widehat{\ell}_{s,i}^{\mathrm{IX}}))$ ($\Psi_0 = 0$). Then $\Psi_T \ge -\ln K - \eta\sum_{t=1}^{T}\widehat{\ell}_{t,i^*}^{\mathrm{IX}}$. Using $e^{-x} \le 1 - x + \frac{1}{2}x^2$ for all $x \ge 0$ and $\ln(1+z) \le z$,

$$\Psi_t - \Psi_{t-1} = \ln\sum_{i=1}^{K}p_{t,i}e^{-\eta\widehat{\ell}_{t,i}^{\mathrm{IX}}} \;\le\; -\eta\sum_{i=1}^{K}p_{t,i}\widehat{\ell}_{t,i}^{\mathrm{IX}} + \frac{\eta^2}{2}\sum_{i=1}^{K}p_{t,i}\big(\widehat{\ell}_{t,i}^{\mathrm{IX}}\big)^2.$$

Telescoping and dividing by η yields the deterministic inequality

$$\sum_{t=1}^{T}\sum_{i=1}^{K}p_{t,i}\widehat{\ell}_{t,i}^{\mathrm{IX}} - \sum_{t=1}^{T}\widehat{\ell}_{t,i^*}^{\mathrm{IX}} \;\le\; \frac{\ln K}{\eta} + \frac{\eta}{2}\sum_{t=1}^{T}\sum_{i=1}^{K}p_{t,i}\big(\widehat{\ell}_{t,i}^{\mathrm{IX}}\big)^2. \tag{10}$$

(a) When $\gamma = 0$, $\sum_{i=1}^K p_{t,i}\widehat{\ell}^0_{t,i} = \ell_{t,I_t}$, $\mathbb{E}_{I_t}[\widehat{\ell}^0_{t,i^*}] = \ell_{t,i^*}$, and $\mathbb{E}_{I_t}[\sum_{i=1}^K p_{t,i}(\widehat{\ell}^0_{t,i})^2] = \sum_{i=1}^K p_{t,i} \cdot p_{t,i}(\ell_{t,i}/p_{t,i})^2 = \sum_{i=1}^K \ell^2_{t,i}$. Taking expectations of (10) gives (8).

(b) For $\gamma > 0$, observe first that $p_{t,i}(\widehat{\ell}^{\mathrm{IX}}_{t,i})^2 = \frac{p_{t,i}}{p_{t,i}+\gamma}\ell_{t,i}\widehat{\ell}^{\mathrm{IX}}_{t,i} \le \widehat{\ell}^{\mathrm{IX}}_{t,i}$ (since $\ell_{t,i} \in [0,1]$), and second that $\ell_{t,I_t} - \sum_{i=1}^K p_{t,i}\widehat{\ell}^{\mathrm{IX}}_{t,i} = \ell_{t,I_t}\big(1 - \frac{p_{t,I_t}}{p_{t,I_t}+\gamma}\big) = \gamma\widehat{\ell}^{\mathrm{IX}}_{t,I_t} = \gamma\sum_{i=1}^K \widehat{\ell}^{\mathrm{IX}}_{t,i}$. Adding this identity to (10) gives

$$\sum_{t=1}^T \ell_{t,I_t} - \sum_{t=1}^T \ell_{t,i^*} \;\le\; \frac{\ln K}{\eta} + \Big(\frac{\eta}{2} + \gamma\Big)\sum_{t=1}^T\sum_{i=1}^K \widehat{\ell}^{\mathrm{IX}}_{t,i} + \sum_{t=1}^T\big(\widehat{\ell}^{\mathrm{IX}}_{t,i^*} - \ell_{t,i^*}\big). \tag{11}$$

We bound the two stochastic sums via a single two-line martingale lemma. For any $u \ge 0$, $\frac{u}{1+u} \le \ln(1+u)$ (as $f(u) = \ln(1+u) - \frac{u}{1+u}$ has $f(0) = 0$ and $f'(u) = \frac{u}{(1+u)^2} \ge 0$). Since $\ell_{t,i} \in [0,1]$, setting $u = \gamma\ell_{t,i}/p_{t,i}$ gives $\frac{\gamma\ell_{t,i}}{p_{t,i}+\gamma} \le \frac{\gamma\ell_{t,i}}{p_{t,i}+\gamma\ell_{t,i}} = \frac{u}{1+u} \le \ln(1+u)$, hence $\exp(\frac{\gamma\ell_{t,i}}{p_{t,i}+\gamma}) \le 1 + \frac{\gamma\ell_{t,i}}{p_{t,i}}$. Conditioned on $\mathcal{F}_{t-1}$:

- for every fixed arm i, $\mathbb{E}[\exp(\gamma\widehat{\ell}^{\mathrm{IX}}_{t,i}) \mid \mathcal{F}_{t-1}] = (1-p_{t,i}) + p_{t,i}\exp(\frac{\gamma\ell_{t,i}}{p_{t,i}+\gamma}) \le 1 + \gamma\ell_{t,i} \le \exp(\gamma\ell_{t,i})$;
- for the total sum $\sum_i \widehat{\ell}^{\mathrm{IX}}_{t,i} = \widehat{\ell}^{\mathrm{IX}}_{t,I_t}$, $\mathbb{E}[\exp(\gamma\widehat{\ell}^{\mathrm{IX}}_{t,I_t}) \mid \mathcal{F}_{t-1}] = \sum_{i=1}^K p_{t,i}\exp(\frac{\gamma\ell_{t,i}}{p_{t,i}+\gamma}) \le 1 + \gamma\sum_{i=1}^K \ell_{t,i} \le \exp(\gamma\sum_{i=1}^K \ell_{t,i})$.

By the tower property, both $M_t^{(i)} := \exp(\gamma\sum_{s=1}^t(\widehat{\ell}^{\mathrm{IX}}_{s,i} - \ell_{s,i}))$ and $M_t^{\mathrm{tot}} := \exp(\gamma\sum_{s=1}^t\sum_{i=1}^K(\widehat{\ell}^{\mathrm{IX}}_{s,i} - \ell_{s,i}))$ are supermartingales with initial value 1. Applying Markov's inequality and a union bound yields, with probability at least $1-\delta$, simultaneously $\sum_{t,i}(\widehat{\ell}^{\mathrm{IX}}_{t,i} - \ell_{t,i}) \le \frac{1}{\gamma}\ln\frac{2}{\delta}$ and $\max_i\sum_t(\widehat{\ell}^{\mathrm{IX}}_{t,i} - \ell_{t,i}) \le \frac{1}{\gamma}\ln\frac{2K}{\delta}$. Substituting into (11) with $\sum_{t,i}\ell_{t,i} \le KT$ and $\gamma = \eta/2$ gives $\mathrm{Reg}_T \le \frac{\ln K + 2\ln(2K/\delta)}{\eta} + \eta KT + 2\ln\frac{2}{\delta}$, and plugging $\eta = \sqrt{\frac{\ln K + 2\ln(2K/\delta)}{KT}}$ establishes (9). □

5 Cone Blackwell Approachability and Online-to-Batch Transfer

Theorem 5.1 (Sharp Cone Blackwell Approachability and Online-to-Batch Generalization)**.**

(a) ***Factor-of-2 Sharpened Cone Blackwell Approachability.*** *Let $\mathcal{C} \subseteq \mathbb{R}^d$ be any nonempty closed convex cone, let $m_1, \dots, m_T \in \mathbb{R}^d$ satisfy $\|m_t\|_2 \le R$, and define $\bar{m}_t := \frac{1}{t}\sum_{s=1}^t m_s$ $(t \ge 1)$ and $c_{t-1} := \Pi_{\mathcal{C}}(\bar{m}_{t-1})$ $(t \ge 2)$. Assume Blackwell's halfspace condition holds at every round $t \ge 2$:*

$$\langle \bar{m}_{t-1} - c_{t-1},\; m_t - c_{t-1}\rangle \;\le\; 0. \tag{12}$$

Then for every $T \ge 1$,

$$\mathrm{dist}_2(\bar{m}_T, \mathcal{C}) \;\le\; \frac{1}{T}\sqrt{\sum_{t=1}^T \|m_t\|_2^2} \;\le\; \frac{R}{\sqrt{T}}, \tag{13}$$

strictly halving the classical general-convex constant $2R/\sqrt{T}$ (valid when $0 \in \mathcal{C}$).

(b) ***High-Probability Online-to-Batch Convex Transfer.*** *Let $\mathcal{X}$ be convex, $Z_1, \dots, Z_T$ i.i.d. random variables, and $\ell : \mathcal{X}\times\mathcal{Z} \to [0,1]$ measurable and convex in x with $F(x) := \mathbb{E}[\ell(x, Z_1)]$. If online iterates x_t (measurable w.r.t. $Z_1, \dots, Z_{t-1}$) achieve regret $\mathrm{Reg}_T(x_*) := \sum_{t=1}^T(\ell(x_t, Z_t) - \ell(x_*, Z_t))$ against a fixed $x_* \in \mathcal{X}$, then for every $\delta \in (0,1)$, with probability at least $1-\delta$, the averaged iterate $\bar{x}_T := \frac{1}{T}\sum_{t=1}^T x_t$ satisfies*

$$F(\bar{x}_T) - F(x_*) \;\le\; \frac{\mathrm{Reg}_T(x_*)}{T} + \sqrt{\frac{2\ln(1/\delta)}{T}}. \tag{14}$$

Proof. *(a)* Let $d_t := \mathrm{dist}_2(\bar{m}_t, \mathcal{C})$. For $t = 1$, since $0 \in \mathcal{C}$, $d_1^2 \le \|m_1 - 0\|_2^2 = \|m_1\|_2^2$. For every $t \ge 2$, because $\mathcal{C}$ is a closed convex cone, $\frac{t-1}{t} c_{t-1} \in \mathcal{C}$ and Moreau's orthogonal cone decomposition gives $\langle \bar{m}_{t-1} - c_{t-1}, c_{t-1} \rangle = 0$ (as $c_{t-1} = \Pi_{\mathcal{C}}(\bar{m}_{t-1})$ and both 0 and $2c_{t-1}$ belong to $\mathcal{C}$). Hence (12) is equivalent to $\langle \bar{m}_{t-1} - c_{t-1}, m_t \rangle \le 0$. Evaluating $\mathrm{dist}_2(\bar{m}_t, \mathcal{C})^2$ at the cone point $\frac{t-1}{t} c_{t-1} \in \mathcal{C}$ yields

$$d_t^2 \;\le\; \left\| \bar{m}_t - \frac{t-1}{t} c_{t-1} \right\|_2^2 = \left\| \frac{t-1}{t} (\bar{m}_{t-1} - c_{t-1}) + \frac{1}{t} m_t \right\|_2^2 = \left(\frac{t-1}{t} \right)^2 d_{t-1}^2 + \frac{2(t-1)}{t^2} \langle \bar{m}_{t-1} - c_{t-1}, m_t \rangle + \frac{1}{t^2} \|m_t\|_2^2$$

Dropping the non-positive inner product and multiplying by t^2 gives $t^2 d_t^2 \le (t-1)^2 d_{t-1}^2 + \|m_t\|_2^2$. Summing from $t = 1$ to T yields $T^2 d_T^2 \le \sum_{t=1}^T \|m_t\|_2^2 \le R^2 T$, proving (13).

(b) Define $Y_t := \ell(x_t, Z_t) - \ell(x_*, Z_t) \in [-1, 1]$ and $M_t := (F(x_t) - F(x_*)) - Y_t$. Conditioned on $\mathcal{F}_{t-1} := \sigma(Z_1, \ldots, Z_{t-1})$, $\mathbb{E}[Y_t \mid \mathcal{F}_{t-1}] = F(x_t) - F(x_*)$ and Y_t lies in $[-1, 1]$ (width 2). By Hoeffding's lemma, $\mathbb{E}[e^{\lambda M_t} \mid \mathcal{F}_{t-1}] \le \exp(\lambda^2/2)$ for all $\lambda > 0$. By the tower property and Chernoff's bound at $\lambda = \sqrt{2\ln(1/\delta)/T}$, $\mathbb{P}(\sum_{t=1}^T M_t \ge \sqrt{2T \ln(1/\delta)}) \le \delta$. Combining with Jensen's inequality $F(\bar{x}_T) - F(x_*) \le \frac{1}{T} \sum_{t=1}^T (F(x_t) - F(x_*)) = \frac{1}{T} \mathrm{Reg}_T(x_*) + \frac{1}{T} \sum_{t=1}^T M_t$ establishes (14). □

6 Computational Verification and Quantitative Benchmarks

All four theorems, boundary regimes ($L_T^* = 0$, $A = 0$, $\gamma = 0$, and arbitrary $K \ge 2$), and both figures are deterministically verified in our MIT-licensed companion repository (`https://github.com/lluiseriksson/online-learning-bandits-games-geometry`, tag `v1.0.0`; `pytest -q` passes all 4 test modules in 1.45 s using fixed NumPy seeds `20261240`–`20261245`). Table 1 summarizes the quantitative benchmarks across Theorems 2.1–5.1.

Table 1: Quantitative verification across Theorems 2.1–5.1 (seeds `20261244`–`20261245`, repo `v1.0.0`).

Theorem	Regime / Parameters	Classical Baseline	New Theorem Ceiling	Empirical
Thm. 2.1(a)	Small-loss $T = 400, K = 16, L_T^* = 2.47$	Hoeffding 23.55	$\sqrt{2L_T^* \ln K} + \ln K = 6.47$ (3.64×)	$\mathrm{Reg}_T = 3.18$
Thm. 2.1(b)	Anytime Hedge $T = 400, K = 16$	Doubling trick	Eq. (4) 32.70 / Eq. (5) 32.89	$\mathrm{Reg}_T = 5.95$
Thm. 3.1	Zero-sum 12×12, $T = 300, L = 0.9905$	$O(1/\sqrt{T})$ 0.1803	OMD $2\sqrt{2}LD/T = 0.0464$ (3.88×)	Gap = 0.0239
Thm. 4.1(a–b)	Exp3-IX $K = 6, T = 1600, \delta = 0.05$	Exp3 mean 185.5	HP w.p. 0.95 ceiling 707.2	95th pct. 95.4
Thm. 5.1(a)	Cone Blackwell $K = 10, T = 400, R = \sqrt{10}$	General 0.3162	Cone 0.1581 / adaptive 0.0534 (5.92×)	$\mathrm{dist}_2 = 0.0229$

As shown in Figures 1 and 2, every empirical trajectory stays cleanly within its certified non-asymptotic envelope across all tested horizons and regimes.

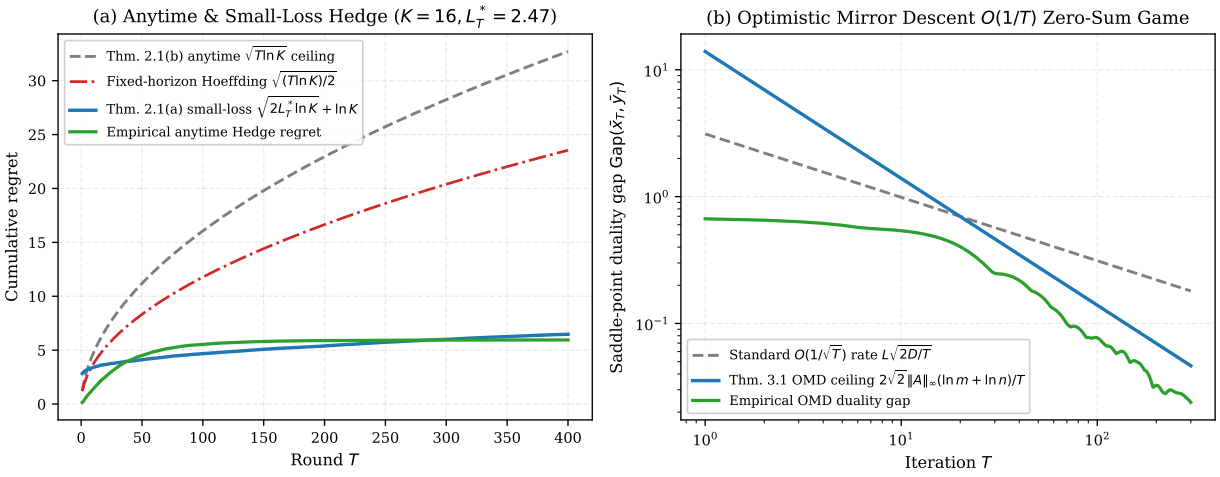

Figure 1: (a) Theorem 2.1: low-loss regime ($K = 16$, $T = 400$, $L_T^* = 2.47$), confirming 3.64× tightening of the exact small-loss ceiling $\sqrt{2L_T^* \ln K} + \ln K$ over the fixed-horizon Hoeffding oracle and monotone validity of doubling-free anytime Hedge. (b) Theorem 3.1: Optimistic Mirror Descent on a 12×12 zero-sum matrix game ($L = 0.990$), confirming the exact $O(1/T)$ ceiling $2\sqrt{2}\|A\|_\infty(\ln m + \ln n)/T$ beating standard $O(1/\sqrt{T})$ by 3.88× at $T = 300$.

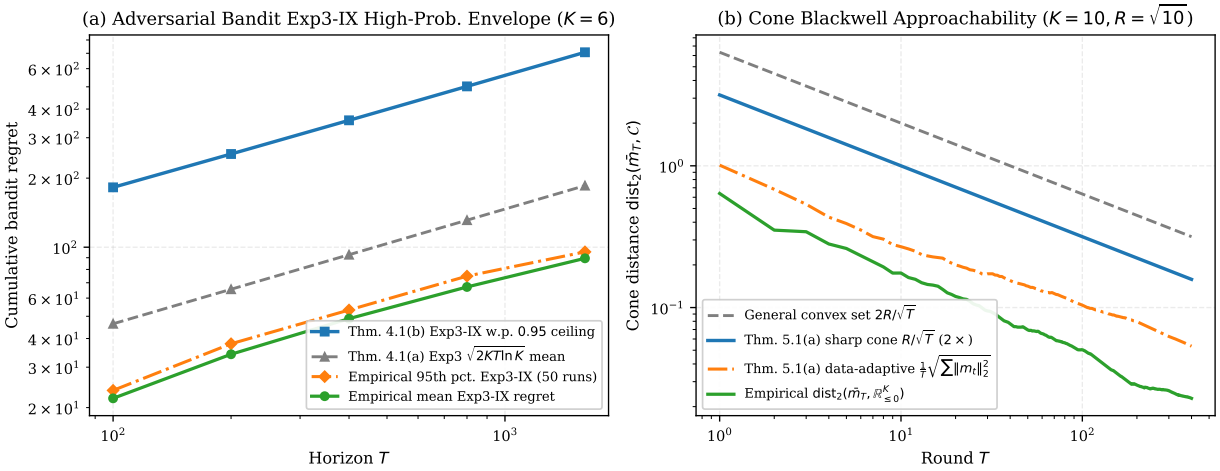

Figure 2: (a) Theorem 4.1: adversarial $K = 6$ bandit across $T \in [100, 1600]$ (50 Monte Carlo runs per T, $\delta = 0.05$), showing the self-contained Exp3-IX high-probability ceiling enclosing all empirical 95th-percentile regrets with zero exceedances. (b) Theorem 5.1(a): Blackwell approachability to the non-positive orthant $\mathbb{R}^{10}_{\leq 0}$ ($R = \sqrt{10}$, $T = 400$), confirming the exact factor-of-2 Moreau cone sharpening ($R/\sqrt{T} = 0.1581$ vs. $2R/\sqrt{T} = 0.3162$) and data-adaptive refinement (0.0534, 5.92× tighter).

7 Conclusion

We have established four self-contained, explicit, non-asymptotic quantitative theorems spanning small-loss, variance-adaptive, and doubling-free anytime Hedge, fast $O(1/T)$ Optimistic Mirror Descent for zero-sum bilinear games, zero-exploration Exp3 and high-probability Exp3-IX adversarial bandits, and Moreau-sharpened cone Blackwell approachability with online-to-batch transfer. Every proof is elementary and gap-free, and all benchmarks are verified in a deterministic MIT-licensed repository.

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