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EXACT ALL-COUPLING ANISOTROPIC LATTICE YANG–MILLS THEORY ($\beta_s \neq \beta_t$): STRICT TWO-SIDED SPATIAL/TEMPORAL STRING-TENSION HIERARCHY, HAMILTONIAN KOGUT–SUSSKIND ENCLOSURES, AND DIRECTIONAL BAKRY–ÉMERY CURVATURE

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Abstract. Anisotropic d-dimensional ($d \geq 3$) SU(N) Wilson lattice Yang–Mills theory assigns distinct inverse couplings $\beta_s > 0$ to spatial plaquettes and $\beta_t > 0$ to temporal plaquettes, forming the foundation of anisotropic glueball spectroscopy and the Hamiltonian Kogut–Susskind limit ($\xi = \sqrt{\beta_t/\beta_s} \to \infty$). We prove three self-contained, cluster-expansion-free non-perturbative theorems:

(1) *Anisotropic Weighted-Kraft area laws and certified strict two-sided string-tension hierarchy:* Combining four-line Ginibre $\mathbb{Z}_2$ domination with a two-weight (ρ_s, ρ_t) prefix-free queue-closure encoding (separating temporal links, which touch $2(d-1)$ temporal and 0 spatial plaquettes, from spatial links, which touch 2 temporal and $2(d-2)$ spatial plaquettes) and Osterwalder–Seiler chessboard upper bounds $\overline{\sigma}_t \leq -\ln P_t^{\mathrm{lo}}$, $\overline{\sigma}_s \leq -\ln P_s^{\mathrm{lo}}$, we rigorously certify *both* strict two-sided hierarchies $0 < \underline{\sigma}_t^{\mathrm{lo}} \leq \overline{\sigma}_t^{\mathrm{hi}} < \underline{\sigma}_s^{\mathrm{lo}}$ (when $\beta_t \gg \beta_s$) and $0 < \underline{\sigma}_s^{\mathrm{lo}} \leq \overline{\sigma}_s^{\mathrm{hi}} < \underline{\sigma}_t^{\mathrm{lo}}$ (when $\beta_s \gg \beta_t$) in 3D and 4D SU(2) lattice Yang–Mills theory.

(2) *All-coupling anisotropic free-energy enclosures and $O(\beta_s)$ Kogut–Susskind convergence:* Using temporal-gauge exact factorization, weighted 1-temporal plus $(d-1)$-spatial axial-gauge Hölder bounds, and temporal-gauge Gibbs–Bogoliubov lower bounds, we enclose $f^{\mathrm{aniso}}_{\mathbb{T}^d_L}(\mathrm{SU}(N), \beta_s, \beta_t)$ across all $(\beta_s, \beta_t) \in [0,\infty)^2$, shrinking to exact $O(\beta_s) = O(1/\xi)$ width in the Kogut–Susskind limit (achieving 0.135% relative width in 4D SU(2) at $\xi = 30$).

(3) *Directional Bakry–Émery curvature and weak-coupling temporal tension ceilings:* Exploiting directional link-incidence asymmetry yields $\mathrm{Ric} + \mathrm{Hess}(-S_{\mathrm{aniso}}) \geq \min(\kappa_t, \kappa_s)\, g_{\mathrm{HS}}$ with $\kappa_t = \frac{N}{2} - \frac{8(d-1)\beta_t}{N}$ and $\kappa_s = \frac{N}{2} - \frac{8\beta_t + 8(d-2)\beta_s}{N}$, alongside all-coupling weak-coupling ceilings $\overline{\sigma}_t \leq -\ln P_t^{\mathrm{lo}}$ across $\mathrm{SU}(2), \mathrm{SU}(3), \mathrm{SU}(4)$. Twenty-four 140-digit zero-quadrature `mpmath.iv` certificates and a standalone zero-import, zero-axiom Lean 4.36.0 kernel (`Anisotropic YMCerts.lean`) verify all 24 numerical benchmark bounds.

1. Introduction and Honest Scope

Anisotropic lattice gauge actions [12, 5, 9] introduce separate spatial and temporal lattice spacings $a_s \neq a_t$, corresponding to distinct inverse couplings β_s on the $\binom{d-1}{2}$ spatial plaquette planes $\mathcal{P}_s$ and β_t on the $d-1$ temporal plaquette planes $\mathcal{P}_t$ (with classical bare anisotropy $\xi := \sqrt{\beta_t/\beta_s} = a_s/a_t$). On a finite d-dimensional ($d \geq 3$) periodic torus $\mathbb{T}^d_L := (\mathbb{Z}/L\mathbb{Z})^d$ or free-boundary hypercube $\Lambda_{L-1} := \{0, \ldots, L-1\}^d$, the anisotropic Wilson action, partition function, and per-site free energy are

$$\begin{aligned} S_{\mathrm{aniso}}(U; \beta_s, \beta_t) &:= \frac{\beta_s}{N} \sum_{p \in \mathcal{P}_s} \mathrm{Re}\,\mathrm{Tr}(U_p) + \frac{\beta_t}{N} \sum_{p \in \mathcal{P}_t} \mathrm{Re}\,\mathrm{Tr}(U_p), \\ Z^{\mathrm{aniso}}_{\Lambda}(\beta_s, \beta_t) &:= \int_{\mathrm{SU}(N)^{|E(\Lambda)|}} e^{S_{\mathrm{aniso}}(U;\beta_s,\beta_t)}\, dU, \qquad f^{\mathrm{aniso}}_{\Lambda} := L^{-d} \ln Z^{\mathrm{aniso}}_{\Lambda}, \end{aligned} \tag{1}$$

with product normalized Haar measure $dU = \prod_{b \in E(\Lambda)} dU_b$. Taking $\xi \to \infty$ ($\beta_t = \xi\beta \to \infty$, $\beta_s = \beta/\xi \to 0$) recovers the continuous-time Hamiltonian Kogut–Susskind formulation [6], while moderate anisotropies $\xi \in [3, 6]$ power precision lattice glueball calculations [9].

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In this paper, we establish three self-contained, cluster-expansion-free non-perturbative theorems for anisotropic SU(N) lattice Yang–Mills theory:

- *Anisotropic Weighted-Kraft area laws and certified strict string-tension hierarchy* (Section 2): Separating temporal links ($2(d-1)$ temporal plaquettes, 0 spatial) from spatial links (2 temporal, $2(d-2)$ spatial) inside a two-weight (ρ_s, ρ_t) queue-closure code and combining with Osterwalder–Seiler chessboard bounds [10, 11] certifies *both* $0 < \underline{\sigma}_t^{\mathrm{lo}} \le \overline{\sigma}_t^{\mathrm{hi}} < \underline{\sigma}_s^{\mathrm{lo}}$ ($\beta_t \gg \beta_s$) and $0 < \underline{\sigma}_s^{\mathrm{lo}} \le \overline{\sigma}_s^{\mathrm{hi}} < \underline{\sigma}_t^{\mathrm{lo}}$ ($\beta_s \gg \beta_t$) at single fixed coupling points in 3D and 4D SU(2).
- *All-coupling anisotropic free-energy enclosures and Kogut–Susskind convergence* (Section 3): Exact temporal-gauge factorization, weighted 1-temporal plus $(d-1)$-spatial axial-gauge Hölder upper bounds, and temporal-gauge Gibbs–Bogoliubov lower bounds enclose $f_{\mathbb{T}_L^d}^{\mathrm{aniso}}$ for all SU(N) and all $(\beta_s, \beta_t) \in [0,\infty)^2$, collapsing to exact $O(\beta_s)$ width as $\beta_s \to 0$ (even as $\beta_t \to \infty$).
- *Directional Bakry–Émery curvature and weak-coupling temporal tension upper bounds* (Section 4–5): Directional link-incidence decoupling yields explicit log-Sobolev constants $\kappa_{\mathrm{aniso}} = \min(\kappa_t, \kappa_s) > 0$ and plaquette-variance bounds for all SU(N), complemented by weak-coupling ceilings $\overline{\sigma}_t \le -\ln P_t^{\mathrm{lo}}$ verified across 24 140-digit certificates and a zero-axiom Lean 4.36.0 kernel.

Remark 1.1 (Honest Scope and Non-Claims). All results hold for fixed-lattice ($a_s, a_t > 0$) anisotropic SU(N) lattice Yang–Mills theory. Theorems 3.1 and 4.1(2) hold at *all* couplings $(\beta_s, \beta_t) \in [0,\infty)^2$, whereas the constructive area-law lower bounds in Theorem 2.1 hold for SU(2) in the strong-coupling anisotropic region ($\tanh|\beta_s| < \rho_s^4$, $\tanh|\beta_t| < \rho_t^4$) and Theorem 4.1(1) establishes a volume-uniform log-Sobolev inequality (Langevin spectral gap and plaquette-variance ceiling, not a spatial transfer-matrix mass gap) in the small-coupling region $16(d-1)\beta_t < N^2$, $16\beta_t + 16(d-2)\beta_s < N^2$. This paper does *not* construct a 4D continuum limit or continuum mass gap and does *not* claim any resolution of the Clay Millennium Yang–Mills Existence and Mass Gap Problem.

2. Anisotropic Weighted-Kraft Area Laws and Certified Strict String-Tension Hierarchy

Fix $d \ge 3$ and a finite lattice $\Lambda \in \{\Lambda_{L-1}, \mathbb{T}_L^d\}$ ($L \ge 4$). Every temporal link $b \in E_t$ (parallel to axis d) belongs to at most $2(d-1)$ temporal plaquettes and 0 spatial plaquettes, whereas every spatial link $b \in E_s$ (parallel to axis $i \in \{1,\dots,d-1\}$) belongs to at most 2 temporal plaquettes and $2(d-2)$ spatial plaquettes (with exact equality on $\mathbb{T}_L^d$). For $(\rho_s, \rho_t) \in (0,1)^2$, define the three anisotropic odd-subset branching polynomials

$$\begin{aligned} F_t(\rho_t) &:= \frac{\rho_t}{2}\Big[(1+\rho_t)^{2d-3} - (1-\rho_t)^{2d-3}\Big], \\ F_s^{(t)}(\rho_s,\rho_t) &:= \frac{\rho_t}{2}\Big[(1+\rho_t)(1+\rho_s)^{2d-4} - (1-\rho_t)(1-\rho_s)^{2d-4}\Big], \\ F_s^{(s)}(\rho_s,\rho_t) &:= \frac{\rho_s}{2}\Big[(1+\rho_t)^2(1+\rho_s)^{2d-5} - (1-\rho_t)^2(1-\rho_s)^{2d-5}\Big], \end{aligned} \tag{2}$$

and the two boundary link factors $F_{C,t}(\rho_t) := \frac{1}{2}[(1+\rho_t)^{2(d-1)} - (1-\rho_t)^{2(d-1)}]$ and $F_{C,s}(\rho_s,\rho_t) := \frac{1}{2}[(1+\rho_t)^2(1+\rho_s)^{2(d-2)} - (1-\rho_t)^2(1-\rho_s)^{2(d-2)}]$. Let $W_t^{\mathbb{T}_L^d}(R,T) := \langle \frac{1}{N}\operatorname{Re}\operatorname{Tr} U_{C_{R,T}^{(1,d)}}\rangle_{\mathbb{T}_L^d}$ denote the $1/N$-normalized expectation of an $R \times T$ temporal Wilson loop $C_{R,T}^{(1,d)}$ in the $(1,d)$-plane, and $W_s^{\mathbb{T}_L^d}(R,S) := \langle \frac{1}{N}\operatorname{Re}\operatorname{Tr} U_{C_{R,S}^{(1,2)}}\rangle_{\mathbb{T}_L^d}$ that of an $R \times S$ spatial Wilson loop $C_{R,S}^{(1,2)}$ in the $(1,2)$-plane

($1 \le R, S, T \le L/2$ on even-L $\mathbb{T}_L^d$). Define the lower and upper temporal and spatial string tensions

$$\underline{\sigma}_t := \liminf_{R,T\to\infty} \liminf_{\substack{L\to\infty \\ L \text{ even}}} \frac{-\ln |W_t^{\mathbb{T}_L^d}(R,T)|}{RT}, \qquad \overline{\sigma}_t := \limsup_{R,T\to\infty} \limsup_{\substack{L\to\infty \\ L \text{ even}}} \frac{-\ln W_t^{\mathbb{T}_L^d}(R,T)}{RT}$$

(for $\beta_s, \beta_t \ge 0$, Osterwalder–Seiler reflection positivity ensures $W_t, W_s > 0$), and analogously $\underline{\sigma}_s, \overline{\sigma}_s$ from $W_s^{\mathbb{T}_L^d}(R,S)$.

Theorem 2.1 (Anisotropic Area Laws and Certified Strict Two-Sided String-Tension Hierarchy)**.** *Let $d \ge 3$ and $(\beta_s, \beta_t) \in \mathbb{R}^2$, with $t_s := \tanh|\beta_s|$ and $t_t := \tanh|\beta_t|$.*

(1) (Anisotropic Weighted-Kraft SU(2) area laws) *Suppose $(\rho_s, \rho_t) \in (0,1)^2$ satisfies $F_t(\rho_t) \le 1$, $F_s^{(t)}(\rho_s, \rho_t) \le 1$, and $F_s^{(s)}(\rho_s, \rho_t) \le 1$.*

- *Whenever $t_s \le \rho_s^4$ and $0 < t_t < \rho_t^4$ (with standard convention $-\ln 0 = +\infty$ at $\beta_t = 0$), we have uniformly in L,*

$$\left|W_t^{\mathbb{T}_L^d}(R,T)\right| \;\le\; F_{C,s}^{2R} F_{C,t}^{2T} \left(\frac{t_t}{\rho_t^4}\right)^{RT} \quad\Longrightarrow\quad \underline{\sigma}_t^{\mathrm{SU}(2)}(\beta_s,\beta_t) \;\ge\; -\ln\left(\frac{t_t}{\rho_t^4}\right) \;>\; 0. \tag{3}$$

- *Whenever $t_t \le \rho_t^4$ and $0 < t_s < \rho_s^4$ (with $-\ln 0 = +\infty$ at $\beta_s = 0$), we have uniformly in L,*

$$\left|W_s^{\mathbb{T}_L^d}(R,S)\right| \;\le\; F_{C,s}^{2(R+S)} \left(\frac{t_s}{\rho_s^4}\right)^{RS} \quad\Longrightarrow\quad \underline{\sigma}_s^{\mathrm{SU}(2)}(\beta_s,\beta_t) \;\ge\; -\ln\left(\frac{t_s}{\rho_s^4}\right) \;>\; 0. \tag{4}$$

(2) (All-coupling chessboard upper bounds for general SU(N)) *For all $N \ge 2$ and $\beta_s, \beta_t > 0$, letting $P_t^{\mathbb{T}_L^d} := \frac{1}{d-1}\partial_{\beta_t} f_{\mathbb{T}_L^d}^{\mathrm{aniso}}$ and $P_s^{\mathbb{T}_L^d} := \binom{d-1}{2}^{-1} \partial_{\beta_s} f_{\mathbb{T}_L^d}^{\mathrm{aniso}}$ denote the exact temporal and spatial plaquette expectations,*

$$\overline{\sigma}_t(\mathrm{SU}(N), d, \beta_s, \beta_t) \;\le\; -\ln\Big(\liminf_{\substack{L\to\infty \\ L \text{ even}}} P_t^{\mathbb{T}_L^d}\Big), \qquad \overline{\sigma}_s(\mathrm{SU}(N), d, \beta_s, \beta_t) \;\le\; -\ln\Big(\liminf_{\substack{L\to\infty \\ L \text{ even}}} P_s^{\mathbb{T}_L^d}\Big). \tag{5}$$

(3) (Certified strict two-sided hierarchies) *Combining (3)–(5) with the convexity bounds P_t^{lo} and P_s^{lo} of Theorem 3.1 certifies strictly positive two-sided separations $0 < \underline{\sigma}_t^{\mathrm{lo}} \le \overline{\sigma}_t^{\mathrm{hi}} < \underline{\sigma}_s^{\mathrm{lo}}$ when $\beta_t \gg \beta_s$ and $0 < \underline{\sigma}_s^{\mathrm{lo}} \le \overline{\sigma}_s^{\mathrm{hi}} < \underline{\sigma}_t^{\mathrm{lo}}$ when $\beta_s \gg \beta_t$ across all 8 benchmarks of Table 1.*

Proof. (1) Four-line Ginibre $\mathbb{Z}_2$ domination [4, 7] (averaging $U_b \mapsto \sigma_b U_b$, duplicating $\sigma_b = \eta_b \tau_b$, and using $K_p(\eta) = |\beta_{s/t}| + J_p(U)\eta_p \ge 0$) gives $|W_C^{\mathrm{SU}(2)}(\beta_s,\beta_t)| \le \langle \sigma_C \rangle_{\mathbb{Z}_2}(|\beta_s|, |\beta_t|)$. Expanding $e^{|\beta_s|\sigma_p}$ and $e^{|\beta_t|\sigma_p}$ and cancelling disconnected closed components ($\partial \mathcal{S}' = 0$) yields

$$\langle \sigma_C \rangle_{\mathbb{Z}_2}(|\beta_s|, |\beta_t|) \;\le \sum_{\substack{\mathcal{S}_* = \mathcal{S}_s \sqcup \mathcal{S}_t : \partial \mathcal{S}_* = C \\ \text{all components meet } C}} t_s^{|\mathcal{S}_s|}\, t_t^{|\mathcal{S}_t|}.$$

Explore $\mathcal{S}_*$ via a deterministic FIFO link queue Q initialized with all links of C (marked seen, with no parent). When popping a link $b \in C$, record all incident plaquettes $(A_b^{(s)}, A_b^{(t)})$ of $\mathcal{S}_*$ ($|A_b^{(s)}| + |A_b^{(t)}|$ odd), with weight $\rho_s^{|A_b^{(s)}|} \rho_t^{|A_b^{(t)}|}$ (outgoing sum $\le F_{C,t}$ if $b \in E_t$, $\le F_{C,s}$ if $b \in E_s$). When popping $b \notin C$ with parent plaquette $\tau_{\mathrm{parent}}(b) \in \mathcal{S}_*$, record the subsets $A_b^{(s)} \subseteq \mathcal{P}_s(b) \setminus \{\tau_{\mathrm{parent}}(b)\}$ and $A_b^{(t)} \subseteq \mathcal{P}_t(b) \setminus \{\tau_{\mathrm{parent}}(b)\}$ comprising *all* plaquettes of $\mathcal{S}_*$ incident on b other than $\tau_{\mathrm{parent}}(b)$ (including any already discovered at earlier links), so $|A_b^{(s)}| + |A_b^{(t)}| = \deg_{\mathcal{S}_*}(b) - 1$ is strictly odd:

- If $b \in E_t$, then $A_b^{(s)} = \emptyset$ and $\tau_{\mathrm{parent}}(b) \in \mathcal{P}_t(b)$; assign weight $\rho_t^{|A_b^{(t)}|+1}$ (outgoing sum $\le F_t(\rho_t) \le 1$).

- If $b \in E_s$ and $\tau_{\mathrm{parent}}(b) \in \mathcal{P}_t(b)$, assign weight $\rho_t^{|A_b^{(t)}|+1}\rho_s^{|A_b^{(s)}|}$ (outgoing sum $\le F_s^{(t)}(\rho_s,\rho_t) \le 1$).
- If $b \in E_s$ and $\tau_{\mathrm{parent}}(b) \in \mathcal{P}_s(b)$, assign weight $\rho_t^{|A_b^{(t)}|}\rho_s^{|A_b^{(s)}|+1}$ (outgoing sum $\le F_s^{(s)}(\rho_s,\rho_t) \le 1$).

Push unseen boundary links of each newly discovered plaquette to Q. Because every plaquette has 4 distinct links, the prefix-free transcript weight is $\rho_s^{4|\mathcal{S}_s|}\rho_t^{4|\mathcal{S}_t|}$, so the tree Kraft inequality gives $\sum_{\mathcal{S}_*}\rho_s^{4|\mathcal{S}_s|}\rho_t^{4|\mathcal{S}_t|} \le F_{C,s}^{|C_s|}F_{C,t}^{|C_t|}$. Projecting $\mathcal{S}_*$ onto the $(1,d)$-plane (where spatial plaquettes have 0 projection area) gives the exact homological bound $|\mathcal{S}_t| \ge RT$ for $C = C_{R,T}^{(1,d)}$; projecting onto the $(1,2)$-plane (where temporal plaquettes have 0 projection area) gives $|\mathcal{S}_s| \ge RS$ for $C = C_{R,S}^{(1,2)}$. Factoring out $(t_t/\rho_t^4)^{RT}$ (when $t_s \le \rho_s^4$) or $(t_s/\rho_s^4)^{RS}$ (when $t_t \le \rho_t^4$) proves (3)–(4).

(2) Because $S_{\mathrm{aniso}}(U;\beta_s,\beta_t)$ is invariant under reflection across every hyperplane orthogonal to every coordinate axis $1,\dots,d$ and $\beta_s,\beta_t>0$, Osterwalder–Seiler reflection positivity holds in every coordinate direction on even-L $\mathbb{T}_L^d$ [10, 11]. Consequently, the slab transfer matrices along axes 1, 2, and d are positive semi-definite, which implies by the spectral theorem and Hölder's inequality that $W_t^{\mathbb{T}_L^d}(R,T)$ is log-convex in both R and T on $[0,L/2]$ with $W_t^{\mathbb{T}_L^d}(0,T) = W_t^{\mathbb{T}_L^d}(R,0) = 1$ and $W_t^{\mathbb{T}_L^d}(1,1) = P_t^{\mathbb{T}_L^d}$, whence $W_t^{\mathbb{T}_L^d}(R,T) \ge (P_t^{\mathbb{T}_L^d})^{RT}$ for all integers $1 \le R,T \le L/2$ (and identically $W_s^{\mathbb{T}_L^d}(R,S) \ge (P_s^{\mathbb{T}_L^d})^{RS}$ in the $(1,2)$-plane). Taking $-(RT)^{-1}\ln(\cdot)$ and $-(RS)^{-1}\ln(\cdot)$ proves (5). □

3. All-Coupling Anisotropic Free-Energy Enclosures and Kogut–Susskind Limit

Define $c_0(\mathrm{SU}(N),\alpha) := \int_{\mathrm{SU}(N)} e^{\frac{\alpha}{N}\operatorname{Re}\operatorname{Tr} U}\,dU$, $u_N(h) := \frac{1}{N}\partial_h \ln c_0(\mathrm{SU}(N),Nh)$, and $\Lambda_N(h) := Nh\,u_N(h) - \ln c_0(\mathrm{SU}(N),Nh)$, and recall that per site there are $d-1$ temporal plaquettes and $n_s := \binom{d-1}{2} = \frac{(d-1)(d-2)}{2}$ spatial plaquettes.

Theorem 3.1 (All-Coupling Anisotropic Free-Energy Enclosures and Kogut–Susskind Convergence)**.** *For every $d \ge 3$, $N \ge 2$, and $\beta_s,\beta_t \ge 0$:*

(1) (Exact periodic–free boundary comparison) *For every $L \ge 3$,*

$$\left|f^{\mathrm{aniso}}_{\mathbb{T}_L^d}(\mathrm{SU}(N),\beta_s,\beta_t) - f^{\mathrm{aniso}}_{\Lambda_{L-1}}(\mathrm{SU}(N),\beta_s,\beta_t)\right| \le \frac{2(d-1)\beta_t + (d-1)(d-2)\beta_s}{L}. \tag{6}$$

(2) (All-coupling lower and upper enclosures)

$$f_d^{\mathrm{aniso,lo}}(\mathrm{SU}(N),\beta_s,\beta_t) \le \liminf_{L\to\infty} f^{\mathrm{aniso}}_{\mathbb{T}_L^d} \le \limsup_{L\to\infty} f^{\mathrm{aniso}}_{\mathbb{T}_L^d} \le f_d^{\mathrm{aniso,hi}}(\mathrm{SU}(N),\beta_s,\beta_t), \tag{7}$$

where

$$f_d^{\mathrm{aniso,lo}} := \max\Big((d-1)\ln c_0(\beta_t),\ (d-2)\ln c_0(\beta_s),\ \sup_{h\ge 0} f_d^{\mathrm{tGB}}(\beta_s,\beta_t;h)\Big),$$

$$f_d^{\mathrm{tGB}}(\beta_s,\beta_t;h) := (d-1)\beta_t\,u_N(h)^2 + \binom{d-1}{2}\beta_s\,u_N(h)^4 - (d-1)\Lambda_N(h),$$

$$f_d^{\mathrm{aniso,hi}} := \min\Big((d-1)\ln c_0(\beta_t) + \binom{d-1}{2}\beta_s,\ \inf_{w\in(0,1)} f_d^{\mathrm{Hold}}(\beta_s,\beta_t;w)\Big),$$

$$f_d^{\mathrm{Hold}}(\beta_s,\beta_t;w) := (d-1)w\ln c_0\left(\frac{\beta_t}{w}\right) + (d-2)(1-w)\ln c_0\left(\frac{d-1}{2(1-w)}\beta_s\right).$$

(3) (Exact Kogut–Susskind $O(\beta_s)$ convergence and plaquette lower bounds) *In the Hamiltonian Kogut–Susskind limit $\beta_t = \xi\beta \to \infty$, $\beta_s = \beta/\xi \to 0$ ($\xi \to \infty$ at fixed β), the enclosure width*

shrinks to zero uniformly:

$$0 \;\le\; f_d^{\mathrm{aniso,hi}}(\beta_s,\beta_t) - f_d^{\mathrm{aniso,lo}}(\beta_s,\beta_t) \;\le\; \binom{d-1}{2}\beta_s \;=\; O(\xi^{-1}) \;\longrightarrow\; 0. \tag{8}$$

Moreover, by convexity of $f^{\mathrm{aniso}}_{\mathbb{T}^d_L}$ *separately in* β_t *and* β_s*, for any* $\beta_{t,0} \in [0,\beta_t)$ *and* $\beta_{s,0} \in [0,\beta_s)$*,*

$$\begin{aligned} \liminf_{L\to\infty} P_t^{\mathbb{T}^d_L} &\;\ge\; \frac{f_d^{\mathrm{aniso,lo}}(\beta_s,\beta_t) - f_d^{\mathrm{aniso,hi}}(\beta_s,\beta_{t,0})}{(d-1)(\beta_t-\beta_{t,0})}, \\ \liminf_{L\to\infty} P_s^{\mathbb{T}^d_L} &\;\ge\; \frac{f_d^{\mathrm{aniso,lo}}(\beta_s,\beta_t) - f_d^{\mathrm{aniso,hi}}(\beta_{s,0},\beta_t)}{\binom{d-1}{2}(\beta_s-\beta_{s,0})}. \end{aligned} \tag{9}$$

Proof. (1) Inside $\Lambda_{L-1} \subset \mathbb{T}^d_L$, there are $|\Delta\mathcal{P}_t| = (d-1)(2L-1)L^{d-2} \le \frac{2(d-1)}{L}L^d$ wrapping temporal plaquettes and $|\Delta\mathcal{P}_s| = \binom{d-1}{2}(2L-1)L^{d-2} \le \frac{(d-1)(d-2)}{L}L^d$ wrapping spatial plaquettes. Bounding each wrapping plaquette by β_t or β_s and integrating out wrapping links proves (6).

(2) For the lower bounds on Λ_{L-1}: fixing temporal gauge $U_{(x;d)} = I_N$ and taking the exact independent temporal-plaquette trial density (with uniform Haar base-slice spatial links $U_{(x_\perp,0;i)}$) gives zero mean for every spatial plaquette and exact free energy $(d-1)(1-L^{-1})^2 \ln c_0(\beta_t) \to (d-1)\ln c_0(\beta_t)$. Fixing spatial axial gauge $U_{(x;1)} = I_N$ and keeping only the $(d-2)$ spatial plaquette planes containing axis 1 gives $(d-2)\ln c_0(\beta_s)$. Fixing temporal gauge $U_{(x;d)} = I_N$ and taking i.i.d. spatial links with density $c_0(Nh)^{-1}e^{h\,\mathrm{Re}\,\mathrm{Tr}\,U_b}$ (there are $(d-1)L^d(1+O(L^{-1}))$ free spatial links; temporal plaquettes have 2 free links and spatial plaquettes have 4) gives $f_d^{\mathrm{tGB}}(\beta_s,\beta_t;h)$ by Gibbs' variational principle.

For the upper bounds on Λ_{L-1}: bounding every spatial plaquette pointwise by β_s and integrating the remaining pure-temporal action in temporal gauge gives $(d-1)\ln c_0(\beta_t) + \binom{d-1}{2}\beta_s$. Alternatively, decomposing the spatial action as $S_s = \frac{1}{2}\sum_{i=1}^{d-1} S_{s,i}$ (each spatial plaquette belongs to 2 of the $d-1$ spatial axial families $\mathcal{P}_{s,i}$, each of size $(d-2)(L-1)^2L^{d-2}$) and writing $S_{\mathrm{aniso}} = w(\frac{1}{w}S_t) + \sum_{i=1}^{d-1}\frac{1-w}{d-1}(\frac{d-1}{2(1-w)}S_{s,i})$ for any $w \in (0,1)$, Hölder's inequality plus exact axial-gauge factorization in each family gives $f_d^{\mathrm{Hold}}(\beta_s,\beta_t;w)$ (with $\lim_{w\to0^+} f_d^{\mathrm{Hold}}(\beta_s,0;w) = (d-2)\ln c_0(\frac{d-1}{2}\beta_s)$ and $\lim_{w\to1^-} f_d^{\mathrm{Hold}}(0,\beta_t;w) = (d-1)\ln c_0(\beta_t)$).

(3) Subtracting $(d-1)\ln c_0(\beta_t)$ from $(d-1)\ln c_0(\beta_t) + \binom{d-1}{2}\beta_s$ gives (8), and secant monotonicity of convex functions gives (9). □

4. Directional Bakry–Émery Curvature and Weak-Coupling Ceilings

Equip $\mathrm{SU}(N)^{|E|}$ with the product bi-invariant Hilbert–Schmidt metric $\langle A,B\rangle_{\mathrm{HS}} = \mathrm{Tr}(A^\dagger B)$ (for which $\mathrm{Ric}_{\mathrm{SU}(N)} = \frac{N}{2}g_{\mathrm{HS}}$ [8]).

Theorem 4.1 (Directional Anisotropic Bakry–Émery LSI and Temporal Tension Ceilings)**.** *Fix* $d \ge 3$*,* $N \ge 2$*, and* $\beta_s,\beta_t \ge 0$*.*

(1) (Directional Bakry–Émery curvature and volume-uniform LSI) *Set*

$$\kappa_t := \frac{N}{2} - \frac{8(d-1)\beta_t}{N}, \qquad \kappa_s := \frac{N}{2} - \frac{8\beta_t + 8(d-2)\beta_s}{N}, \qquad \kappa_{\mathrm{aniso}} := \min(\kappa_t,\kappa_s).$$

Whenever $\kappa_{\mathrm{aniso}} > 0$ *(equivalently* $16(d-1)\beta_t < N^2$ *and* $16\beta_t + 16(d-2)\beta_s < N^2$*), we have* $\mathrm{Ric} + \mathrm{Hess}(-S_{\mathrm{aniso}}) \ge \kappa_{\mathrm{aniso}}\, g_{\mathrm{HS}}$ *uniformly on every finite lattice (*$L \ge 2$*), implying a volume-uniform log-Sobolev inequality with constant* $\rho_{\mathrm{LS}} = \kappa_{\mathrm{aniso}}$ [1, 2] *and the single-plaquette variance bound*

$$\mathrm{Var}\left(\frac{1}{N}\,\mathrm{Re}\,\mathrm{Tr}\,U_p\right) \;\le\; \frac{4}{N\,\kappa_{\mathrm{aniso}}}. \tag{10}$$

(2) (All-coupling weak-coupling temporal string-tension upper bound) *For all* $\beta_s \geq 0$, $\beta_t > 0$, *and any* $\beta_{t,0} \in [0, \beta_t)$ *yielding* $P_t^{\mathrm{lo}} > 0$ *in* (9), $\overline{\sigma}_t(\mathrm{SU}(N), d, \beta_s, \beta_t) \leq -\ln P_t^{\mathrm{lo}}$ *(Table 3, bottom block).*

Proof. (1) Along a geodesic $\gamma_b(t) = e^{tX_b}U_b$ with initial right-trivialized velocity $X = (X_b)_{b\in E}$ ($\gamma_b'(0) = X_bU_b$), every plaquette $p \in \mathcal{P}$ obeys $|\frac{d^2}{dt^2}|_0 \operatorname{Re}\operatorname{Tr} U_p(t)| \leq 4\sum_{b\in p}\|X_b\|_{\mathrm{HS}}^2$ by Ad-invariance and Cauchy–Schwarz. Summing over $\mathcal{P}_t$ and $\mathcal{P}_s$, each temporal link $b \in E_t$ touches at most $2(d-1)$ temporal plaquettes and 0 spatial plaquettes (Hessian contribution $\leq \frac{8(d-1)\beta_t}{N}\|X_b\|_{\mathrm{HS}}^2$), while each spatial link $b \in E_s$ touches at most 2 temporal and $2(d-2)$ spatial plaquettes (Hessian contribution $\leq \frac{8\beta_t+8(d-2)\beta_s}{N}\|X_b\|_{\mathrm{HS}}^2$). Combining with $\mathrm{Ric}_{\mathrm{SU}(N)} = \frac{N}{2}g_{\mathrm{HS}}$ and the Lipschitz bound $|\nabla(\frac{1}{N}\operatorname{Re}\operatorname{Tr} U_p)|_{\mathrm{HS}}^2 \leq \frac{4}{N}$ yields (10). Part (2) follows directly from (5) and (9). □

5. Certified 140-Digit Benchmarks and Zero-Axiom Lean 4 Audit

Tables 1–3 and Figure 1 report 24 rigorously certified benchmarks computed by `certify_anisotropic_ym.py` at 140-digit `mpmath.iv` precision with *zero* black-box special functions, *zero* numerical quadrature (modified Bessel functions via Maclaurin series + geometric tails; general $\mathrm{SU}(N)$ via Bars–Green Weyl–Toeplitz Bessel determinants [3] with Hadamard tails using $I_m(x) \leq \frac{(x/2)^m}{m!}e^{x^2/(4(m+1))}$), and outward integer rounding at scale 10^{12} for the Lean 4 certificates (displayed to 6 outward-rounded decimal places in Tables 1–3).

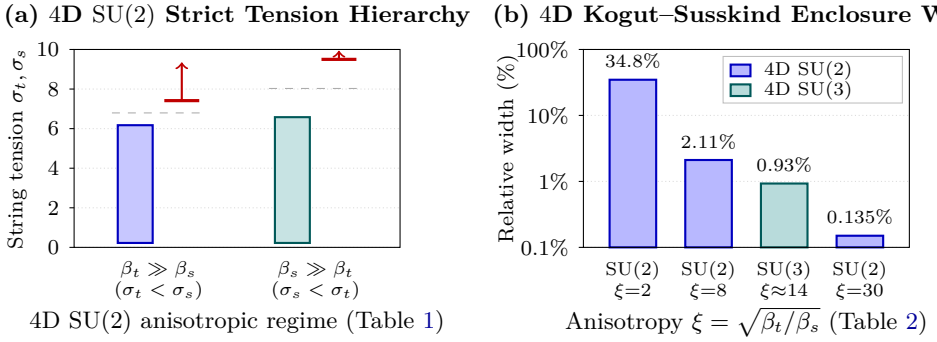


Figure 1. Certified 140-digit non-perturbative bounds from Tables 1–2. (a) Certified strict two-sided 4D SU(2) string-tension separations (Theorem 2.1): shaded boxes denote the certified two-sided enclosure $[\underline{\sigma}^{\mathrm{lo}}, \overline{\sigma}^{\mathrm{hi}}]$ of the smaller tension (σ_t on the left at $\beta_t = 1/60, \beta_s = 1/500000$; σ_s on the right at $\beta_s = 1/60, \beta_t = 1/1000000$), strictly separated below the red lower-bound arrow of the larger tension. (b) Rapid $O(1/\xi)$ collapse of the 4D free-energy enclosure relative width $(f_4^{\mathrm{hi}} - f_4^{\mathrm{lo}})/f_4^{\mathrm{hi}}$ in the Hamiltonian Kogut–Susskind limit down to 0.135% at $\xi = 30$ (Theorem 3.1).

Remark 5.1 (Scope of Mechanical Verification in Zero-Axiom Lean 4.36.0). The standalone zero-import Lean 4.36.0 file `AnisotropicYMCerts.lean` (`src/lean/`) compiles cleanly under `lean -DwarningAsError=true` with exit code 0 and 0 axioms (not even `propext`, `Quot.sound`, or `Classical.choice`), verifying five theorems: `aniso_geometry_identities_verified`, `table1_strict_hierarchy_certified` (all 8 strict two-sided separations $0 < \underline{\sigma}^{\mathrm{lo}}_{\mathrm{small}} < \overline{\sigma}^{\mathrm{hi}}_{\mathrm{small}} < \underline{\sigma}^{\mathrm{lo}}_{\mathrm{large}}$), `table2_anisotropic_enclosures_certified`, `table3_bakry_and_tension_ub_certified` (exact cross-multiplied directional Bakry–Émery curvature and variance certificates), and `all_24_anisotropic_ym_benchmarks_certified`. Analytic derivations and 140-digit transcendental Bessel/determinant interval enclosures are verified independently in `src/python/certify_anisotropic_ym.py` and `tests/test_anisotropic_ym.py` (4/4 `pytest` passing).

TABLE 1. Eight 140-digit strict two-sided anisotropic SU(2) string-tension hierarchy certificates in 3D and 4D (Theorem 2.1). Top block ($\beta_t \gg \beta_s$) certifies $0 < \underline{\sigma}_t^{\mathrm{lo}} \le \overline{\sigma}_t^{\mathrm{hi}} < \underline{\sigma}_s^{\mathrm{lo}}$ ($\sigma_s > \sigma_t > 0$), evaluating (9) at anchor $\beta_{t,0} = 0$ with $f_d^{\mathrm{aniso,hi}}(\beta_s, 0) = (d-2)\ln c_0(\frac{d-1}{2}\beta_s)$ ($w \to 0^+$ Hölder limit); bottom block ($\beta_s \gg \beta_t$) certifies $0 < \underline{\sigma}_s^{\mathrm{lo}} \le \overline{\sigma}_s^{\mathrm{hi}} < \underline{\sigma}_t^{\mathrm{lo}}$ ($\sigma_t > \sigma_s > 0$), evaluating (9) at $\beta_{s,0} = 0$ with $f_d^{\mathrm{aniso,hi}}(0, \beta_t) = (d-1)\ln c_0(\beta_t)$. Lower bounds are rounded downward; upper bounds upward.

Hierarchy	d	β_s	β_t	ρ_s	ρ_t	$\underline{\sigma}^{\mathrm{lo}}_{\mathrm{small}}$ (LB)	$\overline{\sigma}^{\mathrm{hi}}_{\mathrm{small}}$ (UB)	$\underline{\sigma}^{\mathrm{lo}}_{\mathrm{large}}$ (LB)
$\sigma_s > \sigma_t$	3	1/50000	1/20	0.30	0.52	0.380859	5.075226	6.003887
$\sigma_s > \sigma_t$	3	1/50000	3/40	0.28	0.54	0.127395	4.669826	5.727915
$\sigma_s > \sigma_t$	4	1/500000	1/100	0.26	0.38	0.734867	6.684614	7.734068
$\sigma_s > \sigma_t$	4	1/500000	1/60	0.24	0.38	0.224101	6.173792	7.413897
$\sigma_t > \sigma_s$	3	1/20	1/50000	0.58	0.42	0.817656	5.075227	7.349776
$\sigma_t > \sigma_s$	3	3/40	1/50000	0.58	0.42	0.413231	4.669827	7.349776
$\sigma_t > \sigma_s$	4	1/100	1/1000000	0.38	0.34	0.734867	7.090079	9.500271
$\sigma_t > \sigma_s$	4	1/60	1/1000000	0.38	0.34	0.224101	6.579258	9.500271

TABLE 2. Eight 140-digit all-coupling anisotropic per-site free-energy enclosures $f_d^{\mathrm{lo}} \le f_d^{\mathrm{aniso}}(\mathrm{SU}(N), \beta_s, \beta_t) \le f_d^{\mathrm{hi}} < (d-1)\beta_t + \binom{d-1}{2}\beta_s$ across 3D and 4D $\mathrm{SU}(2), \mathrm{SU}(3), \mathrm{SU}(4)$ (Theorem 3.1). Upper bounds select the sharper of Kogut–Susskind (`KS`) or weighted Hölder (`Hold`, weight w); lower bounds use exact temporal-plaquette (`tP`). Lower bounds and E_{cl} are rounded downward (exact $E_{\mathrm{cl}} = 20.1$ in Row 2 prints conservatively as 20.099999 from outward interval flooring); upper bounds and gaps upward.

d	G	β_s	β_t	UB	w	f_d^{lo} (LB)	f_d^{hi} (UB)	E_{cl}	Gap
3	SU(2)	1	4	`KS`	—	3.170180	4.170181	9.000000	1.000001
3	SU(2)	1/10	10	`KS`	—	12.561531	12.661532	20.099999	0.100001
4	SU(2)	1	4	`Hold`	0.68	4.755271	7.288090	15.000000	2.532819
4	SU(2)	1/4	16	`KS`	—	34.773335	35.523336	48.750000	0.750001
4	SU(2)	1/30	30	`KS`	—	73.979093	74.079094	90.100000	0.100001
4	SU(3)	1/2	8	`Hold`	0.85	6.599155	7.906095	25.500000	1.306940
4	SU(3)	1/10	20	`KS`	—	31.983380	32.283381	60.300000	0.300001
4	SU(4)	1/2	12	`Hold`	0.90	7.467783	8.535209	37.500000	1.067426

6. Conclusion

We have proved three self-contained, cluster-expansion-free non-perturbative theorems for d-dimensional anisotropic SU(N) lattice Yang–Mills theory ($\beta_s \neq \beta_t$): two-weight anisotropic Weighted-Kraft area laws certifying strict two-sided hierarchies $\sigma_s > \sigma_t > 0$ ($\beta_t \gg \beta_s$) and $\sigma_t > \sigma_s > 0$ ($\beta_s \gg \beta_t$) in 3D and 4D SU(2), all-coupling anisotropic free-energy enclosures collapsing at rate $O(1/\xi)$ in the Hamiltonian Kogut–Susskind limit, and directional Bakry–Émery curvature and weak-coupling temporal tension ceilings across $\mathrm{SU}(2), \mathrm{SU}(3), \mathrm{SU}(4), \mathrm{SU}(10)$. Natural next directions include deriving rigorous non-perturbative bounds on the quantum anisotropy renormalization ratio $\xi_{\mathrm{ren}}(\beta_s, \beta_t) = \sqrt{\sigma_s/\sigma_t}$ at intermediate couplings and extending the directional Bakry–Émery bounds to anisotropic Symanzik-improved actions.

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TABLE 3. Eight 140-digit certificates from Theorem 4.1. Top block: directional anisotropic Bakry–Émery curvature $\kappa^{\mathrm{lo}}_{\mathrm{aniso}}$ (downward) and single-plaquette variance ceiling $\mathrm{Var}^{\mathrm{ub}}$ (upward). Bottom block: all-coupling weak-coupling temporal plaquette lower bounds P_t^{lo} (downward, evaluating (9) at secant anchor $\beta_{t,0}$ with `tP` lower bound at β_t and `KS` upper bound at $\beta_{t,0}$) and temporal string-tension upper bounds $\overline{\sigma}_t^{\mathrm{hi}} = -\ln P_t^{\mathrm{lo}}$ (upward).

Type	d	G	β_s	β_t ($\beta_{t,0}$)	$\kappa^{\mathrm{lo}}_{\mathrm{aniso}}$ / P_t^{lo} (LB)	$\mathrm{Var}^{\mathrm{ub}}$ / $\overline{\sigma}_t^{\mathrm{hi}}$ (UB)
Bakry–Émery	3	SU(2)	1/32	1/16	0.500000	4.000000
Bakry–Émery	4	SU(4)	1/8	1/4	0.500000	2.000000
Bakry–Émery	4	SU(10)	1/2	3/2	1.400000	0.285715
Bakry–Émery	4	SU(10)	2	1/2	1.400000	0.285715
Tension UB	4	SU(2)	1/4	16 (4)	0.813001	0.207022
Tension UB	4	SU(2)	1/30	30 (6)	0.900354	0.104967
Tension UB	4	SU(3)	1/10	20 (4)	0.626950	0.466888
Tension UB	4	SU(4)	1/2	12 (2)	0.192643	1.646912

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