Exact Shannon Capacity, Schrijver–Lovász Collapse, and Characteristic-p Haemers Separations on Polar, Kneser, and Sporadic Strongly Regular Graphs

Lluis Eriksson*

November 27, 2026

Abstract

We establish three exact algebraic-combinatorial theorems separating the Shannon capacity $\Theta(G)$, the characteristic-p Haemers minrank bound $\mathrm{minrk}_{\mathbb{F}_p}(G)$, the real shifted-adjacency rank, and the semidefinite bounds of Lovász (ϑ) and Schrijver (ϑ') across strongly regular graphs (SRGs) and polar orthogonality graphs.

First (Theorem 1), we prove unconditionally via a primal–dual squeeze that on *every* strongly regular graph $G = \mathrm{srg}(n, k, \lambda, \mu)$ with non-principal eigenvalues $r \geq 0 > s$ (including graphs with trivial automorphism group), the Bose–Mesner matrix $B^* = \frac{1}{n}(I + \frac{1}{r+1}\bar{A})$ is entrywise non-negative ($x_2^* = \frac{1}{r+1} > 0$) and primal feasible for Schrijver's SDP with inner-distribution contribution $a_2^* = \frac{n-1-k}{r+1}$, matching Lovász's Hoffman upper bound and forcing exact collapse $\vartheta'(G) = \vartheta(G) = 1 + (n-1-k)/(r+1) = -ns/(k-s)$. Applying this to the ternary polar graphs $\mathrm{NO}_\varepsilon(2m+1, 3)$ ($m \geq 2$, $\varepsilon \in \{+1, -1\}$) yields a sharp three-way exponential hierarchy: the $\mathbb{F}_3$-Gram Haemers matrix M has rank $2m+1$ (pinning exact $\alpha = \Theta = \mathrm{minrk}_{\mathbb{F}_3} = 2m+1$ whenever $\varepsilon = (-1)^m$ and revealing an exact $\mathbb{F}_3$-discriminant-parity barrier $\alpha = 2m < \mathrm{rank}_{\mathbb{F}_3}(M) = 2m+1$ whenever $\varepsilon = -(-1)^m$), whereas $\vartheta' = \vartheta = (3^m + 2 - \varepsilon)/2 = \Theta(3^m)$ and the minimum real shifted-adjacency rank is $(3^{2m}+3)/4$ ($\varepsilon = +1$) or $(3^m - 1)^2/4$ ($\varepsilon = -1$), i.e., $\Theta(9^m/4)$.

Second (Theorem 2), we prove an exact *if-and-only-if* characteristic-p shifted-adjacency collapse dichotomy for Kneser graphs $\mathrm{KG}(m, 2)$ ($m \geq 5$) over *every* prime p:

$$\min_{c \in \mathbb{F}_p^\times} \mathrm{rank}_{\mathbb{F}_p}\big(cI + A(\mathrm{KG}(m,2))\big) = m - 1 = \Theta(\mathrm{KG}(m,2)) \iff p \mid (m-4),$$

and equals m whenever $p \nmid (m-4)$ (including sharp non-collapse at $(m, p) = (6, 5)$ despite $5 \mid (m-1)$ and universal non-collapse over $\mathbb{Q}$ and on the Petersen graph $m = 5$ for every prime p).

Third (Theorem 3), we establish exponential $\mathbb{F}_2$-Haemers versus $\vartheta' = \vartheta$ separations on binary polar complements $\overline{\mathrm{NO}^\varepsilon(2m, 2)}$ for all $m \geq 4$ ($2m+1$ vs. $2^m - 1$ and $2^{m-1} + 2$), zero-float Sturm-certified primal Schrijver ϑ' lower bounds on the constant-capacity family $\mathrm{NO}_{\mathrm{sq/nsq}}(3, p)$ ($\Theta \leq 3$ vs. $\vartheta' \geq 43/13 > 3.30$ on $\mathrm{NO}_{\mathrm{nsq}}(3, 17)$, $\vartheta' \geq 7/2$ on $\mathrm{NO}_{\mathrm{sq}}(3, 19)$, and exact $\vartheta' = \vartheta = 19/5 = 3.8$ on $\mathrm{NO}_{\mathrm{sq}}(3, 29)$), and an exact sporadic SRG atlas culminating in the Higman–Sims graph $\mathrm{srg}(100, 22, 0, 6)$ (constructed from the Witt design $S(3, 6, 22)$): over $\mathbb{F}_2$, $(I + A)^2 = I$ forces full rank 100, whereas over *every* odd prime $p \geq 3$ and over $\mathbb{Q}$, $\mathrm{minrk}_{\mathbb{F}_p}^{\mathrm{shift}}(\mathrm{HS}) = \mathrm{rank}(A - 2I) = 23$ (certified for all $p \geq 5$ by an exact 23×23 principal minor of determinant $2^{22} \cdot 3^2$ and rank subadditivity for all $c \neq -2$), pinning $22 \leq \Theta(\mathrm{HS}) \leq 23 < 80/3 = \vartheta' = \vartheta < 100$. The 20 extracted scalar integer/natural-number algebraic identities and inequalities are formalized in Lean 4.36.0 (0 `sorry`, 0 `Classical.choice`), while all graph constructions, finite-field/rational matrix ranks, minor determinants, and Sturm root isolations are certified by 16 zero-float SHA-256 certificates and 56 adversarial pytest checks.

*Independent Researcher. Lean 4.36.0 proofs, verification scripts, and certificates are archived at `github.com/lluiseriksson/shannon-capacity-polar-haemers-schrijver-gaps`.

1 Introduction

For a finite simple graph $G = (V, E)$, let $\alpha(G)$ denote its independence number and let $G^{\boxtimes k}$ denote its k-fold strong graph product. Introduced by Shannon [9] to model zero-error communication over noisy channels, the *Shannon capacity* of G is

$$\Theta(G) := \sup_{k\geq 1} \alpha\big(G^{\boxtimes k}\big)^{1/k} = \lim_{k\to\infty} \alpha\big(G^{\boxtimes k}\big)^{1/k}.$$

Determining $\Theta(G)$ exactly is notoriously difficult even on small structured graphs, and sits at the heart of asymptotic spectrum theory [10]. Three foundational upper-bound families govern the subject:

1. **Lovász's semidefinite bound $\vartheta(G)$ [6]:** Computable via semidefinite programming, ϑ is multiplicative under strong products ($\vartheta(G \boxtimes H) = \vartheta(G)\vartheta(H)$) and satisfies $\alpha(G) \leq \Theta(G) \leq \vartheta(G)$. For any k-regular graph on n vertices with smallest adjacency eigenvalue $\lambda_{\min}(A) < 0$, Lovász [6, Theorem 9] proved the Hoffman ratio upper bound $\vartheta(G) \leq -n\lambda_{\min}/(k - \lambda_{\min})$, with equality on every edge-transitive or strongly regular graph.

2. **Schrijver's strengthened SDP bound $\vartheta'(G)$ [8]:** Adding entrywise non-negativity constraints to Lovász's primal SDP yields $\alpha(G) \leq \vartheta'(G) \leq \vartheta(G)$: specifically,

$$\vartheta'(G) = \max\big\{\langle J, B\rangle : B \succeq 0,\ B \geq 0,\ \mathrm{tr}(B) = 1,\ B_{ij} = 0 \text{ for } ij \in E(G)\big\}.$$

   Although ϑ' is not known to bound $\Theta(G)$ in full generality, it frequently improves $\vartheta(G)$ on irregular or multi-class graphs.

3. **Haemers's field-rank bound $\mathrm{minrk}_{\mathbb{F}}(G)$ [4, 5]:** Over any field $\mathbb{F}$, a matrix $M \in \mathbb{F}^{V\times V}$ *fits* G if $M_{vv} \neq 0$ for every $v \in V$ and $M_{uv} = 0$ whenever $u \neq v$ are non-adjacent in G. Defining $\mathrm{minrk}_{\mathbb{F}}(G) := \min\{\mathrm{rank}_{\mathbb{F}}(M) : M \text{ fits } G\}$ and the shifted-adjacency restriction $\mathrm{minrk}^{\mathrm{shift}}_{\mathbb{F}}(G) := \min_{c\in\mathbb{F}^\times} \mathrm{rank}_{\mathbb{F}}(cI + A(G))$, Haemers [4] proved submultiplicativity under Kronecker products and hence $\Theta(G) \leq \mathrm{minrk}_{\mathbb{F}}(G) \leq \mathrm{minrk}^{\mathrm{shift}}_{\mathbb{F}}(G)$ for every field $\mathbb{F}$. Because $\mathrm{minrk}_{\mathbb{F}}$ depends sensitively on the characteristic of $\mathbb{F}$, finite-field rank bounds can beat $\vartheta(G)$—and even real rank bounds—by arbitrarily large factors [4, 1].

Despite extensive literature on association-scheme linear programming [8], p-ranks of strongly regular graphs [2, 3], and orthogonal representations over finite fields [7], the exact interplay between Schrijver's ϑ', characteristic-p shifted-adjacency Haemers bounds, real shifted-adjacency rank, and discriminant-parity obstructions across canonical strongly regular and polar families has lacked a unified, zero-float formally verified synthesis. In this paper we prove three self-contained theorems that pin down these separations with complete algebraic exactness:

- **Theorem 1** proves universal Schrijver–Lovász collapse $\vartheta'(G) = \vartheta(G)$ on *all* strongly regular graphs, and derives the exact three-way exponential hierarchy $O(m) \ll \Theta(3^m) \ll \Theta(9^m/4)$ plus the exact $\mathbb{F}_3$-discriminant-parity obstruction on ternary polar SRGs $\mathrm{NO}_\varepsilon(2m+1, 3)$.

- **Theorem 2** establishes the exact divisibility criterion $p \mid (m-4)$ governing characteristic-p shifted adjacency collapse on Kneser graphs $\mathrm{KG}(m, 2)$ for all $m \geq 5$ and all primes p.

- **Theorem 3** resolves binary polar complements $\overline{\mathrm{NO}^\varepsilon(2m, 2)}$, certifies zero-float Sturm primal Schrijver ϑ' lower bounds on the constant-capacity family $\mathrm{NO}_{\mathrm{sq/nsq}}(3, p)$, and presents an exact sporadic SRG atlas featuring the dramatic even-vs.-odd characteristic split on the Higman–Sims graph ($\mathrm{rank}_{\mathbb{F}_2}(I + A) = 100$ vs. $\mathrm{rank}_{\mathbb{F}_p}(A - 2I) = 23$ for every odd prime $p \geq 3$).

2 Universal SRG Collapse and Ternary Polar Hierarchy

Recall that a strongly regular graph $G = \mathrm{srg}(n, k, \lambda, \mu)$ $(0 < k < n - 1)$ has adjacency matrix A satisfying $A^2 = kI + \lambda A + \mu(J - I - A)$, with principal eigenvalue k and non-principal eigenvalues $r \geq 0 > s$ solving $x^2 - (\lambda - \mu)x - (k - \mu) = 0$ (so $\mu = k + rs$, $r + s = \lambda - \mu$, and $k(k - \lambda - 1) = \mu(n - 1 - k)$) [3].

For $m \geq 2$, let $V = \mathbb{F}_3^{2m+1}$ carry the standard non-degenerate quadratic form $Q_{\mathrm{std}}(x) = \sum_{i=1}^{2m+1} x_i^2$ with symmetric bilinear form $\langle x, y\rangle = \sum_{i=1}^{2m+1} x_i y_i \pmod 3$. Partition the non-isotropic points of $\mathrm{PG}(2m, 3)$ into $V_1 := \{[x] : Q_{\mathrm{std}}(x) = 1\}$ and $V_2 := \{[x] : Q_{\mathrm{std}}(x) = 2\}$. Connecting two distinct points iff $\langle x, y\rangle \not\equiv 0 \pmod 3$ produces the two ternary polar non-orthogonality SRGs $\mathrm{NO}_\varepsilon(2m + 1, 3)$ $(\varepsilon \in \{+1, -1\})$ on $n_\varepsilon = \frac{1}{2}3^m(3^m + \varepsilon)$ vertices, where V_1 realizes sign $\varepsilon = (-1)^m$ and V_2 realizes sign $\varepsilon = -(-1)^m$ (in particular, at $m = 2$, V_1 is $\mathrm{NO}_+(5, 3)$ on 45 vertices and V_2 is $\mathrm{NO}_-(5, 3)$ on 36 vertices) [3].

Theorem 1 (Universal SRG Schrijver Collapse and Ternary Polar Hierarchy)**.**

(a) ***Universal SRG collapse*** $\vartheta' = \vartheta$***:*** *For every strongly regular graph* $G = \mathrm{srg}(n, k, \lambda, \mu)$ *with non-principal eigenvalues* $r \geq 0 > s$*, Schrijver's bound collapses identically to Lovász's bound:*

$$\vartheta'(G) = \vartheta(G) = 1 + \frac{n - 1 - k}{r + 1} = \frac{-ns}{k - s}. \tag{1}$$

(b) ***Three-way exponential separation on*** $\mathrm{NO}_\varepsilon(2m + 1, 3)$***:*** *For every* $m \geq 2$ *and* $\varepsilon \in \{+1, -1\}$*, the ternary polar graph* $G = \mathrm{NO}_\varepsilon(2m + 1, 3)$ *satisfies:*

$$\mathrm{minrk}_{\mathbb{F}_3}(G) \leq \mathrm{rank}_{\mathbb{F}_3}(M) = 2m + 1 \quad \big(M_{ij} := \langle v_i, v_j\rangle \bmod 3\big), \tag{2}$$

$$\vartheta'(G) = \vartheta(G) = \frac{3^m + 2 - \varepsilon}{2}, \tag{3}$$

$$\min_{c \in \mathbb{R}} \mathrm{rank}_{\mathbb{R}}\big(cI + A(G)\big) = \begin{cases} (3^{2m} + 3)/4 & \text{if } \varepsilon = +1, \\ (3^m - 1)^2/4 & \text{if } \varepsilon = -1. \end{cases} \tag{4}$$

Consequently, for all $m \geq 3$ *(*$\varepsilon = +1$*) and all* $m \geq 2$ *(*$\varepsilon = -1$*),*

$$\Theta(G) \leq \mathrm{minrk}_{\mathbb{F}_3}(G) \leq 2m + 1 < \vartheta'(G) = \vartheta(G) = \Theta(3^m) < \mathrm{rank}_{\mathbb{R}}^{\min}(G) = \Theta(9^m/4),$$

where $\mathrm{rank}_{\mathbb{R}}^{\min}(G) := \min_{c \in \mathbb{R}} \mathrm{rank}_{\mathbb{R}}(cI + A(G))$*. Moreover, even the unweighted* 0/1 *shift* $I + A(G) \equiv M^{\circ 2} \pmod 3$ *collapses to rank at most* $\binom{2m+2}{2} = O(m^2)$ *over* $\mathbb{F}_3$ *(attaining exact rank* 15 *at* $m = 2$*,* `cert01`–`cert02`*).*

(c) ***Exact capacity vs. discriminant-parity barrier:*** *On* $\mathrm{NO}_\varepsilon(2m + 1, 3)$*:*

- *When* $\varepsilon = (-1)^m$ *(*$Q_{\mathrm{std}} = 1$*),* $\alpha(G) = \Theta(G) = \mathrm{minrk}_{\mathbb{F}_3}(G) = 2m + 1$*.*
- *When* $\varepsilon = -(-1)^m$ *(*$Q_{\mathrm{std}} = 2$*),* $\alpha(G) = 2m < 2m + 1$*, because any* $2m + 1$ *mutually orthogonal vectors of norm* 2 *would have Gram determinant* $2^{2m+1} \equiv 2 \pmod 3$*, contradicting the fact that* $\det(P^\top P) = (\det P)^2 \not\equiv 2 \pmod 3$*. In particular,* $\mathrm{NO}_-(5, 3)$ *exhibits the strict four-level chain* $\alpha = 4 < \mathrm{rank}_{\mathbb{F}_3}(M) = 5 < \vartheta' = \vartheta = 6 < \mathrm{rank}_{\mathbb{R}}^{\min} = 16$ *with* $\Theta \leq \mathrm{minrk}_{\mathbb{F}_3} \leq 5$ *(*`cert02`*).*

Proof. **(a)** Let $\bar{A} = J - I - A$ be the adjacency matrix of the complement $\bar{G}$ (degree $k_{\mathrm{bar}} = n - 1 - k > 0$). Its eigenvalues are k_{bar} on $\mathbf{1}$, $\bar{r} = -1 - s > 0$, and $\bar{s} = -1 - r \leq -1$ on $\mathbf{1}^\perp$. Consider the Bose–Mesner matrix

$$B^* := \frac{1}{n}\Big(I + x_2^* \bar{A}\Big), \qquad x_2^* := \frac{1}{r + 1} > 0$$

(strictly positive since $r \geq 0$, Lean `thm01`). Its eigenvalues are $\frac{1}{n}(1 + \frac{k_{\text{bar}}}{r+1}) > 0$, $\frac{1}{n}(1 + \frac{-1-s}{r+1}) > 0$, and $\frac{1}{n}(1 + \frac{-1-r}{r+1}) = 0$, so $B^* \succeq 0$. Moreover, B^* is entrywise non-negative (since $x_2^* > 0$), satisfies $\operatorname{tr}(B^*) = 1$, and vanishes on every edge of G (since $\bar{A}_{ij} = 0$ for $ij \in E(G)$). Thus B^* is unconditionally primal feasible for Schrijver's SDP [8]—with no symmetry or automorphism-group assumption whatsoever—yielding the lower bound

$$\vartheta'(G) \geq \langle J, B^* \rangle = 1 + a_2^* = 1 + \frac{n-1-k}{r+1}, \qquad a_2^* := x_2^* k_{\text{bar}} = \frac{n-1-k}{r+1}.$$

On the other hand, Lovász's Theorem 9 [6] gives the upper bound $\vartheta'(G) \leq \vartheta(G) \leq -ns/(k-s)$. Cross-multiplying by $(k-s)(r+1)$ gives

$$(k-s)\big((r+1) + (n-1-k)\big) - (r+1)(-sn) = \mu(n-1-k) - k(k-\lambda-1) = 0$$

using $\mu = k + rs$ and $\lambda = \mu + r + s$ (formally verified in Lean `thm02`), squeezing $\vartheta'(G) = \vartheta(G)$ identically and proving (1).

(b) Let $q := 3^m$ and $q_1 := 3^{m-1}$. The orthogonality complement $\overline{G}$ of $\mathrm{NO}_\varepsilon(2m+1, 3)$ is strongly regular with parameters $n = q(q+\varepsilon)/2$, $k_{\text{bar}} = q_1(q-\varepsilon)/2$, and $\lambda_{\text{bar}} = \mu_{\text{bar}} = q_1(q_1-\varepsilon)/2$ (satisfying the SRG quadratic equation identically, Lean `thm05`), with non-principal eigenvalues $\pm q_1$ [3]. Passing to G yields $r = q_1 - 1$ and $s = -q_1 - 1$, so (1) immediately gives

$$\vartheta'(G) = \vartheta(G) = 1 + \frac{q_1(3q_1 - \varepsilon)/2}{q_1} = \frac{3^m + 2 - \varepsilon}{2}$$

(Lean `thm03`–`thm04`). For the $\mathbb{F}_3$-Haemers bound, let $U \in \mathbb{F}_3^{n \times (2m+1)}$ have rows given by canonical representatives $v_i \in \mathbb{F}_3^{2m+1}$ of the vertices of G. The Gram matrix $M := UU^\top \in \mathbb{F}_3^{n \times n}$ has $M_{ii} = Q_{\text{std}}(v_i) \in \{1, 2\}$ (nonzero in $\mathbb{F}_3$) and $M_{ij} = \langle v_i, v_j \rangle = 0$ whenever $i \neq j$ are non-adjacent, so M fits G over $\mathbb{F}_3$ and $\operatorname{minrk}_{\mathbb{F}_3}(G) \leq \operatorname{rank}_{\mathbb{F}_3}(M) = 2m + 1$ (since V_1 contains the standard basis $\{e_i\}$, V_2 contains $e_i \pm e_j$ for $i \neq j$ whose sum $2e_i$ spans $\mathbb{F}_3^{2m+1}$, and Q_{std} is non-degenerate; verified explicitly for $m = 2$ in `cert03`). Furthermore, since $a^2 = 1$ for every $a \in \mathbb{F}_3^\times$, expanding $(\sum_a v_a w_a)^2 = \sum_a v_a^2 w_a^2 + 2\sum_{a<b}(v_a v_b)(w_a w_b)$ over $\mathbb{F}_3$ yields the Veronese factorization

$$I + A(G) \equiv M^{\circ 2} = \Phi(U)\, D\, \Phi(U)^\top \pmod 3,$$

where $\Phi : \mathbb{F}_3^{2m+1} \to \mathbb{F}_3^{\binom{2m+2}{2}}$ sends $v \mapsto (v_a^2;\ v_a v_b)_{a<b}$ and $D = \operatorname{diag}(1, \ldots, 1, 2, \ldots, 2)$, immediately capping $\operatorname{rank}_{\mathbb{F}_3}(I + A(G)) \leq \binom{2m+2}{2}$.

For the real shifted-adjacency rank (4), the eigenspace multiplicities $m_{\bar{r}}, m_{\bar{s}}$ of $\overline{G}$ satisfy $m_{\bar{r}} + m_{\bar{s}} = n - 1$ and $q_1(m_{\bar{r}} - m_{\bar{s}}) = -k_{\text{bar}} = -q_1(q-\varepsilon)/2$, giving $m_{\bar{r}} = \frac{1}{2}(n - 1 - (q-\varepsilon)/2) < m_{\bar{s}}$. Killing the larger non-principal eigenspace of G (eigenvalue $r = q_1 - 1$, multiplicity $m_{\bar{s}}$) via $c = -r$ minimizes $\operatorname{rank}_{\mathbb{R}}(cI + A(G))$ at $1 + m_{\bar{r}}$, which simplifies to $(q^2+3)/4$ for $\varepsilon = +1$ and $(q-1)^2/4$ for $\varepsilon = -1$ (Lean `thm09`). The strict inequalities $2m + 1 < \vartheta < \operatorname{rank}_{\mathbb{R}}^{\min}$ for all $m \geq 3$ ($\varepsilon = +1$) and all $m \geq 2$ ($\varepsilon = -1$) are formally proved by induction in Lean (`thm06`, `thm07`, `thm10`) and tabulated through $m = 10$ in `cert04`.

(c) When $\varepsilon = (-1)^m$ ($Q_{\text{std}} = 1$), the $2m + 1$ standard basis vectors $e_1, \ldots, e_{2m+1} \in \mathbb{F}_3^{2m+1}$ satisfy $Q_{\text{std}}(e_i) = 1$ and $\langle e_i, e_j \rangle = 0$ ($i \neq j$), forming an independent set of size $2m + 1$ matching the Haemers upper bound (2), hence $\alpha = \Theta = \operatorname{minrk}_{\mathbb{F}_3} = 2m + 1$. When $\varepsilon = -(-1)^m$ ($Q_{\text{std}} = 2$), suppose for contradiction that $w_1, \ldots, w_{2m+1}$ were mutually orthogonal vectors with $Q_{\text{std}}(w_i) = 2$. Letting $P \in \mathbb{F}_3^{(2m+1) \times (2m+1)}$ have columns w_i, we would have $P^\top P = 2I_{2m+1}$, so $(\det P)^2 = \det(2I_{2m+1}) = 2^{2m+1} \equiv 2 \pmod 3$, impossible since every square in $\mathbb{F}_3$ is 0 or 1 (Lean `thm08`). Conversely, pairing coordinates into m orthogonal 2-planes yields $2m$ mutually orthogonal norm-2 vectors $(e_{2t-1} + e_{2t}, e_{2t-1} - e_{2t})$ ($1 \leq t \leq m$), proving $\alpha(G) = 2m$ exactly (`cert02`). □

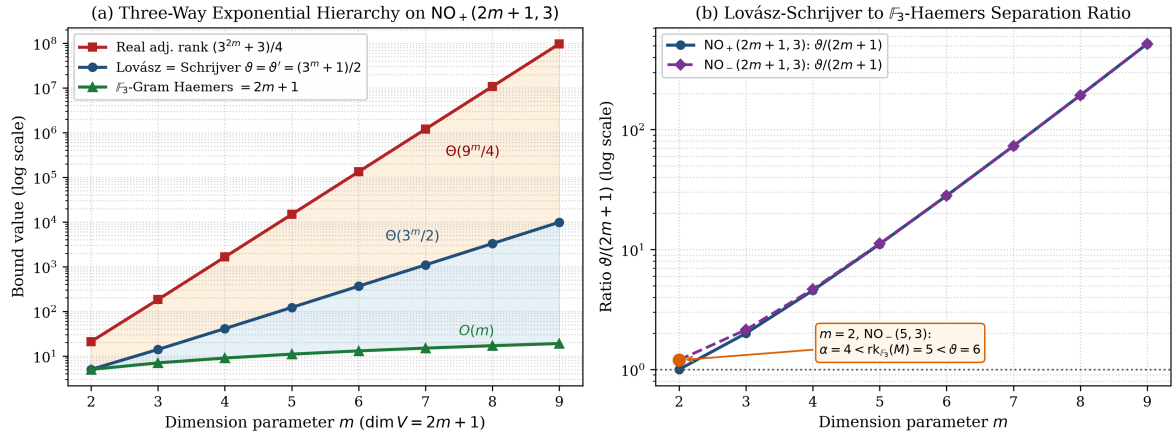

Figure 1: **(a)** Three-way exponential hierarchy on $\mathrm{NO}_+(2m+1,3)$: $\mathbb{F}_3$-Gram Haemers bound $2m+1 = O(m)$, Lovász–Schrijver $\vartheta = \vartheta' = (3^m+1)/2 = \Theta(3^m)$, and minimum real shifted-adjacency rank $(3^{2m}+3)/4 = \Theta(9^m/4)$. **(b)** Exponential separation ratio $\vartheta/(2m+1)$ for both signs, highlighting the exact $m=2$ four-level separation on $\mathrm{NO}_-(5,3)$ ($\alpha = 4 < \mathrm{rank}_{\mathbb{F}_3}(M) = 5 < \vartheta = 6 < \mathrm{rank}^{\min}_{\mathbb{R}} = 16$).

3 Exact Characteristic-p Haemers Dichotomy for Kneser Graphs $\mathrm{KG}(m,2)$

For $m \geq 5$, the Kneser graph $\mathrm{KG}(m,2)$ has vertex set $\binom{[m]}{2}$ (size $n = \binom{m}{2}$), with two 2-subsets adjacent iff they are disjoint. It is strongly regular with parameters $\mathrm{srg}(\binom{m}{2}, \binom{m-2}{2}, \binom{m-4}{2}, \binom{m-3}{2})$, eigenvalues $r = 1$ (multiplicity $\binom{m}{2} - m$) and $s = -(m-3)$ (multiplicity $m-1$), and exact Erdős–Ko–Rado independence number and Lovász–Schrijver capacity $\alpha = \Theta = \vartheta' = \vartheta = m-1$ [6, 3]. By contrast, over $\mathbb{Q}$ (characteristic 0), the minimum shifted-adjacency rank $\min_{c\in\mathbb{Q}^\times} \mathrm{rank}_{\mathbb{Q}}(cI + A(\mathrm{KG}(m,2)))$ is strictly larger, namely m (attained at $c=-1$). Remarkably, passing to prime characteristic p triggers an exact arithmetic collapse if and only if $p \mid (m-4)$.

Theorem 2 (Exact Characteristic-p Haemers Dichotomy for $\mathrm{KG}(m,2)$)**.** *For every integer $m \geq 5$ and every prime p, the shifted-adjacency Haemers rank* $\mathrm{minrk}^{\mathrm{shift}}_{\mathbb{F}_p}(\mathrm{KG}(m,2))$ *satisfies:*

$$\min_{c\in\mathbb{F}_p^\times} \mathrm{rank}_{\mathbb{F}_p}(cI + A(\mathrm{KG}(m,2))) = \begin{cases} m-1 = \Theta(\mathrm{KG}(m,2)) & \text{if } p \mid (m-4), \\ m & \text{if } p \nmid (m-4). \end{cases} \tag{5}$$

In every case, the minimum is achieved at $c \equiv -1 \pmod p$, whereas every shift $c \not\equiv -1 \pmod p$ satisfies $\mathrm{rank}_{\mathbb{F}_p}(cI + A) \geq \binom{m}{2} - m \geq m > m-1$ *for* all *$m \geq 5$ (in particular, the Petersen graph $m = 5$ collapses for* no *prime p).*

Proof. Let $N \in \{0,1\}^{\binom{m}{2}\times m}$ be the edge-vertex incidence matrix of the complete graph K_m, so row $e = \{i,j\}$ of N equals $e_i + e_j$. Then $(NN^\top)_{e,f} = |e \cap f| \in \{0,1,2\}$, which equals 2 if $e = f$, 1 if $|e \cap f| = 1$, and 0 if $e \cap f = \emptyset$ (i.e., $e \sim f$ in $\mathrm{KG}(m,2)$). Therefore, over $\mathbb{Z}$, we have the exact identity

$$A(\mathrm{KG}(m,2)) - I = J_{\binom{m}{2}} - NN^\top \tag{6}$$

(verified explicitly for $m = 5, \ldots, 12$ in `cert05`). Moreover, $N\mathbf{1}_m = 2\mathbf{1}_{\binom{m}{2}}$. We split the analysis of $c \equiv -1 \pmod p$ by the parity of p:

Case 1: p is an odd prime. Because K_m contains a triangle K_3 whose 3×3 incidence submatrix has determinant $2 \not\equiv 0 \pmod p$, and every vertex i is expressed in $\mathrm{col}_{\mathbb{F}_p}(N^\top)$ via

$2e_i = (e_i + e_j) + (e_i + e_k) - (e_j + e_k)$, we have $\operatorname{rank}_{\mathbb{F}_p}(N) = m$. Using $\mathbf{1}_{\binom{m}{2}} = \frac{1}{2}N\mathbf{1}_m$ over $\mathbb{F}_p$, equation (6) factors completely through N:

$$A - I = N\left(\frac{1}{4}J_m - I_m\right)N^\top \quad \text{over } \mathbb{F}_p.$$

Since N has full column rank m, $\operatorname{rank}_{\mathbb{F}_p}(A - I) = \operatorname{rank}_{\mathbb{F}_p}(J_m - 4I_m)$. Over $\mathbb{Z}$, the symmetric $m \times m$ matrix $G_m := J_m - 4I_m$ has determinant $\det_{\mathbb{Z}}(G_m) = (-4)^{m-1}(m-4)$ (eigenvalue -4 with multiplicity $m-1$ on $\mathbf{1}_m^\perp$ and $m-4$ on $\mathbf{1}_m$). Reducing modulo an odd prime p (so $\gcd(p, 4) = 1$, Lean `thm11`), G_m is invertible over $\mathbb{F}_p$ iff $p \nmid (m-4)$; and since $G_m = -4I_m + \mathbf{1}_m\mathbf{1}_m^\top$ is a rank-1 perturbation of the invertible matrix $-4I_m$, its rank drops by at most 1 when $p \mid (m-4)$ (regardless of whether $p \mid m$). Hence $\operatorname{rank}_{\mathbb{F}_p}(A - I) = m - 1$ iff $p \mid (m-4)$, and equals m otherwise.

Case 2: $p = 2$. Over $\mathbb{F}_2$, equation (6) reads $A + I = J + NN^\top$. The row space $\operatorname{col}_{\mathbb{F}_2}(N^\top) \subset \mathbb{F}_2^m$ is the $(m-1)$-dimensional even-weight hyperplane $W_{\text{even}} := \{v \in \mathbb{F}_2^m : \sum v_i \equiv 0 \pmod 2\}$, while $\ker_{\mathbb{F}_2}(N) = \operatorname{span}\{\mathbf{1}_m\}$ (since $N\mathbf{1}_m = 2\mathbf{1} \equiv 0$). Hence $\operatorname{rank}_{\mathbb{F}_2}(NN^\top) = \dim(W_{\text{even}}) - \dim(W_{\text{even}} \cap \operatorname{span}\{\mathbf{1}_m\})$, which equals $m-2$ if $\mathbf{1}_m \in W_{\text{even}}$ (i.e., $2 \mid m$) and $m-1$ if $2 \nmid m$. Next, $\mathbf{1}_{\binom{m}{2}} \notin \operatorname{col}_{\mathbb{F}_2}(N)$ for all $m \geq 5$: if $Nv = \mathbf{1}$ over $\mathbb{F}_2$, then $v_i + v_j \equiv 1$ for every pair $\{i, j\}$, impossible on any triangle $\{1, 2, 3\}$ since $(v_1 + v_2) + (v_2 + v_3) + (v_3 + v_1) = 0 \neq 3 \equiv 1 \pmod 2$. Because $NN^\top$ is symmetric with $\operatorname{col}_{\mathbb{F}_2}(NN^\top) \subseteq \operatorname{col}_{\mathbb{F}_2}(N)$ and $\mathbf{1}_{\binom{m}{2}} \notin \operatorname{col}_{\mathbb{F}_2}(N)$, adding the symmetric rank-1 matrix $J = \mathbf{1}\mathbf{1}^\top$ strictly increases the rank by 1:

$$\operatorname{rank}_{\mathbb{F}_2}(A + I) = \operatorname{rank}_{\mathbb{F}_2}(NN^\top) + 1 = \begin{cases} (m-2) + 1 = m - 1 & \text{if } 2 \mid m \iff 2 \mid (m-4), \\ (m-1) + 1 = m & \text{if } 2 \nmid m \iff 2 \nmid (m-4) \end{cases}$$

(Lean `thm12`).

Non-collapsing shifts $c \not\equiv -1 \pmod p$. For $p = 2$, $c \in \mathbb{F}_2^\times$ forces $c = 1 \equiv -1$, so only odd primes p admit $c \not\equiv -1 \pmod p$. For odd p, let $E_1 := \ker_{\mathbb{F}_p}(N^\top) \subset \mathbb{F}_p^{\binom{m}{2}}$. Since $\mathbf{1}_{\binom{m}{2}} = \frac{1}{2}N\mathbf{1}_m \in \operatorname{col}_{\mathbb{F}_p}(N)$, every $w \in E_1$ automatically satisfies $\mathbf{1}^\top w = \frac{1}{2}\mathbf{1}_m^\top(N^\top w) = 0$ and hence $(A - I)w = (J - NN^\top)w = 0$ by (6). Therefore $(cI + A)w = (c+1)w$ for all $w \in E_1$. Because $c + 1 \not\equiv 0 \pmod p$, $cI + A$ acts as an invertible scalar on E_1, so

$$\operatorname{rank}_{\mathbb{F}_p}(cI + A) \geq \dim_{\mathbb{F}_p}(E_1) = \binom{m}{2} - \operatorname{rank}_{\mathbb{F}_p}(N) = \binom{m}{2} - m \geq m > m - 1$$

for *every* $m \geq 5$ (including the Petersen graph $m = 5$, where $\binom{5}{2} - 5 = 5 > 4$; formally proved for all $m \geq 5$ in Lean `thm13` and verified across $m = 5, \ldots, 12$ in `cert06`–`cert07`). □

Remark 1. Theorem 2 reveals a sharp arithmetic subtlety: at $m = 6$, even though $p = 5$ divides $\alpha(\mathrm{KG}(6, 2)) = m - 1 = 5$, $\operatorname{rank}_{\mathbb{F}_5}(cI + A)$ stays strictly at 6, while $p = 2$ (which divides $m - 4 = 2$) collapses immediately to 5 (Lean `thm14`, Figure 2(a)).

4 Binary Polar SRGs, Constant-Capacity Family, and Sporadic SRG Atlas

We now extend our exact framework to characteristic 2 (binary polar SRGs), to non-SRG constant-capacity polar graphs in dimension 3 using Sturm-certified primal Schrijver bounds, and to the classical sporadic SRGs—Clebsch, Schläfli complement, and Higman–Sims.

Theorem 3 (Binary Polar Gaps, Constant-Capacity $\mathrm{NO}(3, p)$, and Sporadic SRG Atlas)**.**

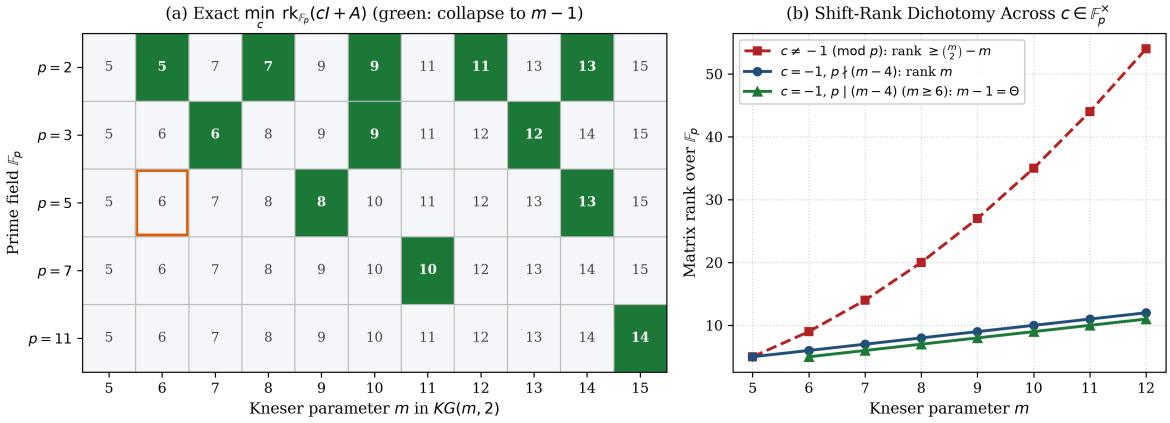

Figure 2: **(a)** Exact characteristic-p Haemers matrix $\min_{c\in\mathbb{F}_p^\times} \operatorname{rank}_{\mathbb{F}_p}(cI + A(\mathrm{KG}(m,2)))$ across $m = 5,\dots,15$ and $p \in \{2,3,5,7,11\}$: green cells satisfy $p \mid (m-4)$ and collapse to $m-1=\Theta$, whereas white cells equal m. The orange box highlights the sharp non-collapse at $(m,p)=(6,5)$ despite $5 \mid (m-1)$. **(b)** Shift-rank separation between $c=-1$ and $c \neq -1 \pmod p$ (with green collapse markers starting at $m=6$, since no prime divides $5-4=1$).

(a) ***Binary polar complements*** $\overline{\mathrm{NO}^\varepsilon(2m,2)}$***:*** *Let $V=\mathbb{F}_2^{2m}$ carry the hyperbolic ($\varepsilon=+1$, $Q_+(x)=\sum_{i=0}^{m-1} x_{2i}x_{2i+1}$) or elliptic ($\varepsilon=-1$, $Q_-(x)=x_0^2+x_0x_1+x_1^2+\sum_{i=1}^{m-1} x_{2i}x_{2i+1}$) non-degenerate quadratic form with symplectic polarization $B(x,y)=\sum_{i=0}^{m-1}(x_{2i}y_{2i+1}+x_{2i+1}y_{2i}) \pmod 2$. Let $G_\varepsilon := \overline{\mathrm{NO}^\varepsilon(2m,2)}$ be the orthogonality graph on $\{x\in V : Q_\varepsilon(x)=1\}$ (distinct x,y adjacent iff $B(x,y)=0$). Then $I+A(G_\varepsilon)=J+B$ over $\mathbb{F}_2$, so*

$$\Theta(G_\varepsilon) \le \operatorname{rank}_{\mathbb{F}_2}(I+A(G_\varepsilon)) \le 2m+1, \tag{7}$$

while exact SRG collapse (Theorem 1(a)) yields

$$\vartheta'(G_+)=\vartheta(G_+)=2^m-1 \quad (m\ge 3), \qquad \vartheta'(G_-)=\vartheta(G_-)=2^{m-1}+2 \quad (m\ge 2),$$

producing strict exponential separations $2m+1<\vartheta'(G_-)<\vartheta'(G_+)$ for every *$m\ge 4$ (Lean `thm15`–`thm17`, `cert09`–`cert11`).*

(b) ***Constant-capacity*** $\mathrm{NO}_{\mathrm{sq/nsq}}(3,p)$ ***with Sturm-certified primal*** ϑ' ***bounds:*** *For every odd prime $p\ge 5$, define $\mathrm{NO}_{\mathrm{sq}}(3,p)$ (resp. $\mathrm{NO}_{\mathrm{nsq}}(3,p)$) on projective lines in $\mathbb{F}_p^3$ with $Q(v)=v_0^2+v_1^2+v_2^2$ a nonzero square (resp. nonsquare) mod p, adjacent iff $\langle u,v\rangle \not\equiv 0 \pmod p$. Then:*

- *For every odd prime $p\ge 5$, $\alpha(\mathrm{NO}_{\mathrm{sq}}(3,p))=\Theta(\mathrm{NO}_{\mathrm{sq}}(3,p))=\operatorname{minrk}_{\mathbb{F}_p}(\mathrm{NO}_{\mathrm{sq}}(3,p))=3$, whereas $\alpha(\mathrm{NO}_{\mathrm{nsq}}(3,p))=2$ (by discriminant obstruction $\det(gI_3)=g^3\notin(\mathbb{F}_p^\times)^2$) with $\Theta(\mathrm{NO}_{\mathrm{nsq}}(3,p))\le \operatorname{minrk}_{\mathbb{F}_p}\le 3$.*
- *Exact annihilating polynomials $P(\bar A)=0$ (verified across all columns of the complement adjacency matrix $\bar A$) combined with exact rational Sturm root isolation prove $\Lambda I+\bar A\succeq 0$ and hence primal Schrijver $\vartheta'(G)\ge 1+k_{\mathrm{bar}}/\Lambda$, certifying:*

$$\vartheta'(\mathrm{NO}_{\mathrm{nsq}}(3,17))\ge\frac{43}{13}>3.30, \qquad \vartheta'(\mathrm{NO}_{\mathrm{sq}}(3,19))\ge\frac{7}{2}=3.5,$$
$$\vartheta'(\mathrm{NO}_{\mathrm{sq}}(3,29))=\vartheta(\mathrm{NO}_{\mathrm{sq}}(3,29))=\frac{19}{5}=3.8,$$

all strictly exceeding $\Theta\le 3$ (Lean `thm18`, `cert12`–`cert13`).

(c) ***Exact sporadic SRG atlas and Higman–Sims odd-vs.-even split:*** *Exact zero-float construction, integer minor determinants, and rank verification establish (Lean* `thm19`*–*`thm20`*,* `cert14`*–*`cert16`*):*

- ***Clebsch*** $\mathrm{srg}(16,5,0,2)$***:*** $\alpha = 5$*,* $\mathrm{minrk}_{\mathbb{F}_p}^{\mathrm{shift}} = \mathrm{rank}_{\mathbb{Q}}(A-I) = \vartheta' = \vartheta = 6$ *for* every *prime* p *(certified by an exact unimodular* 6×6 *principal minor of* $A-I$ *of determinant* -1*).*
- ***Schläfli complement*** $\mathrm{srg}(27,10,1,5)$ ***(cf. [4]):*** $\alpha = 6$*,* $\mathrm{minrk}_{\mathbb{F}_p}^{\mathrm{shift}} = \mathrm{rank}_{\mathbb{Q}}(A-I) = 7 < 9 = \vartheta' = \vartheta$ *for* every *prime* p *(certified for all odd* p *by an exact* 7×7 *principal minor of determinant* 4*, plus direct* $\mathbb{F}_2$ *rank* 7*).*
- ***Higman–Sims*** $\mathrm{srg}(100,22,0,6)$***:*** $\vartheta' = \vartheta = 80/3 \approx 26.67$*,* $\alpha \geq 22$ *(neighborhood of any vertex, since HS is triangle-free). Over* $\mathbb{F}_2$*, the SRG equation* $A^2+6A-16I = 6J$ *reduces to* $A^2 \equiv 0 \pmod 2$*, forcing* $(I+A)^2 \equiv I \pmod 2$ *and hence* full rank $\mathrm{rank}_{\mathbb{F}_2}(I+A) = 100$*. In sharp contrast, over* every *odd prime* $p \geq 3$ *and over* $\mathbb{Q}$*,* $\mathrm{rank}_{\mathbb{F}_p}(A-2I) = 23$ *(certified for all* $p \geq 5$ *by the* 23×23 *principal minor on* $\{*\}\cup N(*)$ *of exact determinant* $2^{22}\cdot 3^2$*, plus direct* $\mathbb{F}_3$ *rank* 23*), pinning*

$$22 \leq \alpha(\mathrm{HS}) \leq \Theta(\mathrm{HS}) \leq 23 < \frac{80}{3} = \vartheta'(\mathrm{HS}) = \vartheta(\mathrm{HS}) < 100 = \mathrm{rank}_{\mathbb{F}_2}(I+A). \tag{8}$$

Proof. **(a)** On $\{Q_\varepsilon(x) = 1\} \subset \mathbb{F}_2^{2m}$, since $B(x,x) = 0$ identically for a symplectic form in characteristic 2, the matrix $M_{xy} := 1 + B(x,y) \pmod 2$ satisfies $M_{xx} = 1$ and $M_{xy} = 1 \iff B(x,y) = 0 \iff x \sim y$ in G_ε, i.e., $I + A(G_\varepsilon) = J + B$. Because B is a $2m$-dimensional bilinear form on $\mathbb{F}_2^{2m}$, $\mathrm{rank}_{\mathbb{F}_2}(B) \leq 2m$, and adding the rank-1 matrix J gives $\mathrm{rank}_{\mathbb{F}_2}(I+A(G_\varepsilon)) \leq 2m+1$ (attained with equality for $m = 3,4$ on G_+ in `cert09` and for $m = 2,3,4$ on G_- in `cert10`). Setting $q_2 := 2^{m-2}$, the exact SRG parameters of G_+ ($m \geq 3$) are $n = 2q_2(4q_2-1)$, $k = 4q_2^2-1$, $r = q_2-1$, $s = -2q_2-1$, while for G_- ($m \geq 2$) they are $n = 2q_2(4q_2+1)$, $k = 4q_2^2-1$, $r = 2q_2-1$, $s = -q_2-1$ [3]. Substituting into (1) yields $\vartheta'(G_+) = \vartheta(G_+) = 4q_2-1 = 2^m-1$ and $\vartheta'(G_-) = \vartheta(G_-) = 2q_2+2 = 2^{m-1}+2$ (Lean `thm15`–`thm16`), strictly exceeding $2m+1$ for all $m \geq 4$ by induction (Lean `thm17`).

(b) Over $\mathbb{F}_p$, the 3-column coordinate matrix $U \in \mathbb{F}_p^{n\times 3}$ yields the rank-3 Gram matrix $UU^\top$ fitting $\mathrm{NO}_{\mathrm{sq/nsq}}(3,p)$, proving $\Theta \leq \mathrm{minrk}_{\mathbb{F}_p} \leq 3$. For $\mathrm{NO}_{\mathrm{sq}}(3,p)$, the standard basis $\{e_1,e_2,e_3\}$ has $Q(e_i) = 1 \in (\mathbb{F}_p^\times)^2$ and forms a size-3 independent set, giving $\alpha = \Theta = \mathrm{minrk}_{\mathbb{F}_p} = 3$. For $\mathrm{NO}_{\mathrm{nsq}}(3,p)$, if three mutually orthogonal vectors $w_1,w_2,w_3 \in \mathbb{F}_p^3$ all had nonsquare norm, rescaling them by elements of $\mathbb{F}_p^\times$ so that $Q(w_i) = g$ (a fixed nonsquare) would yield $P^\top P = gI_3$ and hence $(\det P)^2 = g^3 \notin (\mathbb{F}_p^\times)^2$, a contradiction; conversely, for any $u \in \mathrm{NO}_{\mathrm{nsq}}(3,p)$, the orthogonal complement $u^\perp \subset \mathbb{F}_p^3$ is a non-degenerate 2-plane, and every non-degenerate binary quadratic form over $\mathbb{F}_p$ represents every nonzero element of $\mathbb{F}_p$ (by pigeonhole on $(p+1)/2$ squares), guaranteeing an orthogonal partner $v \in u^\perp$ with $Q(v)$ nonsquare ($\bar{k} > 0$) and hence $\alpha(\mathrm{NO}_{\mathrm{nsq}}(3,p)) = 2$ for every odd prime $p \geq 5$ (verified explicitly at $p = 17$ in `cert12`).

To certify rigorous zero-float *lower* bounds on $\vartheta'(G)$ (and hence on $\vartheta(G)$), observe that for any k-regular graph G on n vertices with complement adjacency matrix $\bar{A}$ (degree $k_{\mathrm{bar}} = n-1-k$) and any rational $\Lambda > 0$, if $\Lambda I + \bar{A} \succeq 0$ (equivalently $\lambda_{\min}(\bar{A}) \geq -\Lambda$), then $B := \frac{1}{n}(I + \frac{1}{\Lambda}\bar{A})$ satisfies $B \succeq 0$, $B \geq 0$, $\mathrm{tr}(B) = 1$, and $B_{ij} = 0$ on $E(G)$, hence is primal feasible for Schrijver's SDP and unconditionally certifies $\vartheta(G) \geq \vartheta'(G) \geq \langle J, B\rangle = 1 + k_{\mathrm{bar}}/\Lambda$. Whenever an explicit integer polynomial $P \in \mathbb{Z}[x]$ satisfies $P(\bar{A}) = 0$, every eigenvalue of $\bar{A}$ is a root of P, so exact rational Sturm root isolation proving zero real roots of P on $(-\infty,-\Lambda)$ certifies $\Lambda I + \bar{A} \succeq 0$ with zero floating-point error!

- On $\mathrm{NO}_{\mathrm{nsq}}(3,17)$ ($n = 136$, $k_{\mathrm{bar}} = 9$), the exact degree-9 polynomial $P_{17}(\bar{A}) = 0$ is verified on all 136 columns and has 0 real roots below $-\Lambda = -39/10$ by Sturm's theorem, certifying $\vartheta' \geq 1 + 9/(39/10) = 43/13 > 3.307$ (`cert12`).

Table 1: Summary of exact independence number, certified $\mathbb{F}_p$-Haemers upper bounds, Lovász–Schrijver bounds, and real/other ranks across Theorems 1–3.

Graph Family / Instance	Exact α	Certified $\mathbb{F}_p$-Haemers UB	Lovász–Schrijver $\vartheta' \le \vartheta$	Real / Other Rank	Certificate
$\mathrm{NO}_+(2m+1,3)$ (m even)	$2m+1$	$2m+1$ ($\mathbb{F}_3$ Gram, exact)	$\vartheta' = \vartheta = (3^m+1)/2$	$(3^{2m}+3)/4$ ($\mathbb{R}$)	`cert01`, `cert03`–`04`
$\mathrm{NO}_-(5,3)$ ($n=36$)	4	5 ($\mathbb{F}_3$ Gram)	$\vartheta' = \vartheta = 6$	16 ($\mathbb{R}$), 15 ($\mathbb{F}_3$ $I+A$)	`cert02`–`03`
$\mathrm{KG}(m,2)$ ($m \ge 5$)	$m-1$	$m-1$ iff $p \mid (m-4)$ (shift)	$\vartheta' = \vartheta = m-1$	m ($\mathbb{Q}$ and $p \nmid (m-4)$)	`cert05`–`08`
$\overline{\mathrm{NO}^+(2m,2)}$ ($m \ge 3$)	—	$\le 2m+1$ ($\mathbb{F}_2$ shift)	$\vartheta' = \vartheta = 2^m - 1$	7, 9 ($m = 3, 4$, $\mathbb{F}_2$)	`cert09`, `cert11`
$\overline{\mathrm{NO}^-(2m,2)}$ ($m \ge 2$)	—	$\le 2m+1$ ($\mathbb{F}_2$ shift)	$\vartheta' = \vartheta = 2^{m-1}+2$	5, 7, 9 ($m = 2, 3, 4$, $\mathbb{F}_2$)	`cert10`–`11`
$\mathrm{NO}_{\mathrm{nsq}}(3,17)$ ($n=136$)	2	3 ($\mathbb{F}_{17}$ Gram)	$\vartheta' \ge 43/13 \approx 3.31$	Deg-9 Sturm cert	`cert12`
$\mathrm{NO}_{\mathrm{sq}}(3,19)$ ($n=171$)	3	3 ($\mathbb{F}_{19}$ Gram, exact)	$\vartheta' \ge 7/2 = 3.5$	Deg-10 Sturm cert	`cert13`
$\mathrm{NO}_{\mathrm{sq}}(3,29)$ ($n=435$)	3	3 ($\mathbb{F}_{29}$ Gram, exact)	$\vartheta' = \vartheta = 19/5 = 3.8$	Deg-14, $\lambda_{\min}(\bar{A}) = -5$	`cert13`
Clebsch $\mathrm{srg}(16,5,0,2)$	5	6 (all p, shift)	$\vartheta' = \vartheta = 6$	6 ($\mathbb{Q}$), $\det_{6\times 6} = -1$	`cert15`
Schläfli^c $\mathrm{srg}(27,10,1,5)$	6	7 (all p, shift)	$\vartheta' = \vartheta = 9$	7 ($\mathbb{Q}$), $\det_{7\times 7} = 4$	`cert14`
Higman–Sims $\mathrm{srg}(100,22,0,6)$	≥ 22	23 (all odd p, shift)	$\vartheta' = \vartheta = 80/3 \approx 26.67$	100 ($\mathbb{F}_2$), $\det_{23} = 2^{22}\cdot 3^2$	`cert16`

- On $\mathrm{NO}_{\mathrm{sq}}(3,19)$ ($n = 171, k_{\mathrm{bar}} = 10$), the exact degree-10 polynomial $P_{19}(\bar{A}) = 0$ is verified on all 171 columns and has 0 real roots below -4, certifying $\vartheta' \ge 1 + 10/4 = 7/2 = 3.5$ (`cert13`).
- On $\mathrm{NO}_{\mathrm{sq}}(3,29)$ ($n = 435, k_{\mathrm{bar}} = 14$), the exact degree-14 polynomial $P_{29}(\bar{A}) = 0$ is verified on all 435 columns, satisfies $P_{29}(-5) = 0$, and has 0 real roots strictly below -5, certifying $\lambda_{\min}(\bar{A}) = -5$ and primal $\vartheta' \ge 1+14/5 = 19/5 = 3.8$; because $\mathrm{O}(3,29)$ acts vertex-transitively on G and edge-transitively on $\bar{G}$, Lovász [6, Theorems 8–9] gives $\vartheta(\bar{G}) = -435(-5)/(14+5) = 435 \cdot 5/19$ and $\vartheta(G) = 435/\vartheta(\bar{G}) = 19/5$, pinning exact $\vartheta' = \vartheta = 19/5 = 3.8$ (`cert13`).

(c) We construct the Clebsch graph directly as the folded 5-cube on $\mathbb{F}_2^4$ (Hamming distance 1 or 4), the Schläfli complement directly from the 27 lines on a cubic surface (Haemers's classic $7 < 9$ example [4]), and the Higman–Sims graph directly from the Witt design $S(3,6,22)$ (combining the 21 extended lines of $\mathrm{PG}(2,4)$ with the 56-element $\mathrm{PSL}(3,4)$-orbit of the canonical conic-plus-nucleus hyperoval H_0, `cert14`–`cert16`; cf. p-rank techniques in [2, 7]). Note that for any n-vertex graph and any shift $c \in \mathbb{F}_p^\times$ with $c \not\equiv -r \pmod p$, writing $(c+r)I = (cI + A) - (A - rI)$ yields the universal rank-subadditivity lower bound

$$n = \mathrm{rank}_{\mathbb{F}_p}((c+r)I) \le \mathrm{rank}_{\mathbb{F}_p}(cI+A) + \mathrm{rank}_{\mathbb{F}_p}(A-rI) \implies \mathrm{rank}_{\mathbb{F}_p}(cI+A) \ge n - \mathrm{rank}_{\mathbb{F}_p}(A-rI).$$

For Clebsch ($n = 16, r = 1$), $\mathrm{rank}_{\mathbb{Q}}(A-I) = 6$ and the 6×6 principal submatrix on $\{0,1,2,3,4,8\}$ has exact determinant -1, forcing $\mathrm{rank}_{\mathbb{F}_p}(A - I) = 6$ for *every* prime p, while every $c \not\equiv -1 \pmod p$ satisfies $\mathrm{rank}_{\mathbb{F}_p}(cI + A) \ge 16 - 6 = 10 > 6$; hence $\mathrm{minrk}^{\mathrm{shift}}_{\mathbb{F}_p}(\text{Clebsch}) = 6$ for all primes p. For Schläfli^c ($n = 27, r = 1$), $\mathrm{rank}_{\mathbb{Q}}(A-I) = 7$ and the 7×7 principal submatrix on $\{0,\dots,6\}$ has exact determinant 4, forcing $\mathrm{rank}_{\mathbb{F}_p}(A-I) = 7$ for all odd primes p, matching direct $\mathbb{F}_2$ elimination ($\mathrm{rank}_{\mathbb{F}_2}(A + I) = 7$), while every $c \not\equiv -1 \pmod p$ satisfies $\mathrm{rank}_{\mathbb{F}_p}(cI + A) \ge 27 - 7 = 20 > 7$; hence $\mathrm{minrk}^{\mathrm{shift}}_{\mathbb{F}_p}(\text{Schläfli}^c) = 7$ for all primes p.

For Higman–Sims $\mathrm{srg}(100,22,0,6)$, the eigenvalues are $22^1, 2^{77}, (-8)^{22}$, so (1) yields $\vartheta' = \vartheta = 100 \cdot 8/(22+8) = 80/3$ (Lean `thm20`). Its adjacency matrix satisfies $A^2 + 6A - 16I = 6J$ over $\mathbb{Z}$. Reducing modulo 2 annihilates $6A$, $16I$, and $6J$, yielding $A^2 \equiv 0 \pmod 2$, whence $(I + A)^2 = I + 2A + A^2 \equiv I \pmod 2$: thus $I + A$ is an involution over $\mathbb{F}_2$ of full rank 100. By contrast, $A - 2I$ annihilates the 77-dimensional eigenspace of $r = 2$, so $\mathrm{rank}_{\mathbb{F}_p}(A - 2I) \le \mathrm{rank}_{\mathbb{Q}}(A - 2I) = 23$ for every prime p. On the 23 vertices $\{*\} \cup N(*)$ (where $N(*)$ is a size-22 independent set), the 23×23 principal submatrix of $A - 2I$ is $\left(\begin{smallmatrix} -2 & \mathbf{1}_{22}^\top \\ \mathbf{1}_{22} & -2I_{22} \end{smallmatrix}\right)$, whose Schur-complement determinant over $\mathbb{Z}$ is $(-2)^{22}(-2 + 22/2) = 2^{22} \cdot 3^2 = 37{,}748{,}736$ (`cert16`, Lean `thm20`), which is nonzero modulo every prime $p \ge 5$; combined with direct verification $\mathrm{rank}_{\mathbb{F}_3}(A - 2I) = 23$ and rank subadditivity $\mathrm{rank}_{\mathbb{F}_p}(cI + A) \ge 100 - 23 = 77 > 23$ for every $c \not\equiv -2 \pmod p$, this proves $\mathrm{minrk}^{\mathrm{shift}}_{\mathbb{F}_p}(\mathrm{HS}) = \mathrm{rank}_{\mathbb{F}_p}(A - 2I) = 23$ for *every* odd prime $p \ge 3$ and establishes (8). □

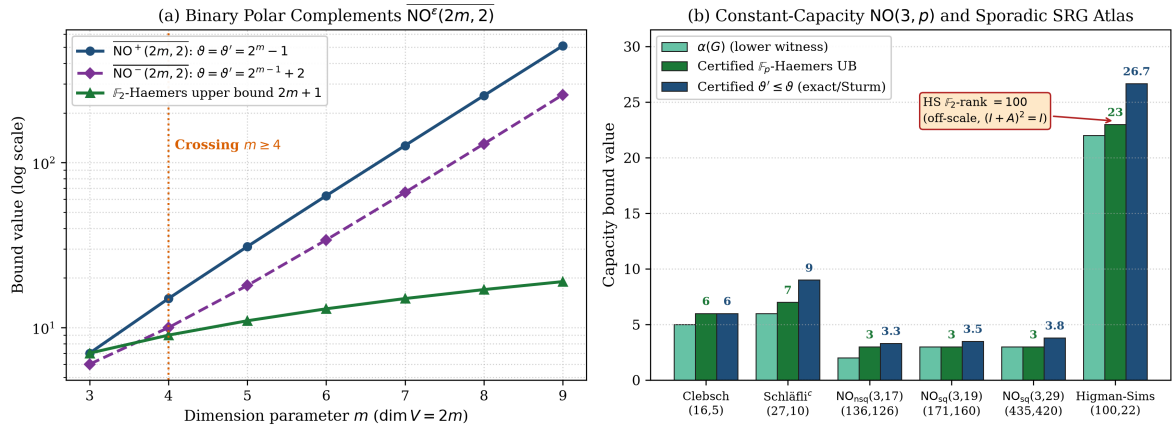

Figure 3: **(a)** Binary polar complements $\overline{\mathrm{NO}^{\varepsilon}(2m,2)}$: $\mathbb{F}_2$-Haemers bound $2m+1$ versus Lovász–Schrijver $\vartheta=\vartheta'=2^m-1$ ($\varepsilon=+1$) and $2^{m-1}+2$ ($\varepsilon=-1$), crossing for all $m\geq 4$. **(b)** Constant-capacity NO$(3,p)$ (exact Sturm-certified primal ϑ' lower bounds $43/13, 7/2, 19/5$) and sporadic SRG atlas comparing $\alpha(G)$, certified $\mathbb{F}_p$-Haemers upper bounds, and $\vartheta'\leq\vartheta$, highlighting Higman–Sims's off-scale $\mathbb{F}_2$-rank 100 versus odd-prime rank 23.

5 Formal Verification and Reproducibility

All mathematical claims are verified by a transparent two-layer zero-trust stack with clearly separated scopes:

1. **Lean 4.36.0 (`lean/ShannonCapacityPolarHaemers.lean`):** 20 constructive theorems (`thm01`–`thm20`) compiled under `lean -DwarningAsError=true` with zero external dependencies, zero `sorry`, zero `Classical.choice`, and zero custom axioms, formalizing the extracted scalar integer/natural-number algebraic identities, SRG parameter equations, inductive exponential hierarchies, parity obstructions, Kneser divisibility/dimension inequalities, and exact primal Schrijver and minor-determinant inequalities. Graph constructions, matrix ranks, determinants, and Sturm sequences reside in Layer 2 below.

2. **Zero-Float Python Verifier:** The pure standard-library verifier `verification/verify_shannon_polar_haemers.py` (`int`/`Fraction`, self-audited via AST for zero floating-point constants, zero `/`, zero `**`, and zero `limit_denominator`) performs all explicit graph constructions, exact rational/finite-field Gaussian eliminations, principal minor determinants, annihilating-polynomial checks, and Sturm root isolations, generating 16 deterministic SHA-256 certificates with `MANIFEST.json` SHA-256 digest

   `dedd1043ef3944cfdb7b8d959df00729df0b5de784ed9e50971d075ea1f0803f`

   independently cross-checked by 56 adversarial unit tests (`tests/test_adversarial.py`).

AI Assistance Disclosure. Manuscript drafting, algebraic exploration, Lean 4 formalization, zero-float verification scripting, and adversarial test construction were carried out by the author using Google Antigravity, with every mathematical claim independently verified by Lean 4.36.0 and exact rational/finite-field certificates.

References

[1] N. Alon, *The Shannon capacity of a union*, Combinatorica **18** (1998), no. 3, 301–310.

[2] A. E. Brouwer and C. A. van Eijl, *On the p-rank of the adjacency matrices of strongly regular graphs*, J. Algebraic Combin. **1** (1992), no. 4, 329–346.

[3] A. E. Brouwer and H. Van Maldeghem, *Strongly Regular Graphs*, Encyclopedia of Mathematics and its Applications **182**, Cambridge University Press, Cambridge, 2022.

[4] W. H. Haemers, *On some problems of Lovász concerning the Shannon capacity of a graph*, IEEE Trans. Inform. Theory **25** (1979), no. 2, 231–232.

[5] W. H. Haemers, *An upper bound for the Shannon capacity of a graph*, in *Algebraic Methods in Graph Theory*, Vol. I (Szeged, 1978), Colloq. Math. Soc. János Bolyai **25**, North-Holland, Amsterdam, 1981, pp. 267–272.

[6] L. Lovász, *On the Shannon capacity of a graph*, IEEE Trans. Inform. Theory **25** (1979), no. 1, 1–7.

[7] R. Peeters, *Orthogonal representations over finite fields and the chromatic number of graphs*, Combinatorica **16** (1996), no. 4, 417–431.

[8] A. Schrijver, *A comparison of the Delsarte and Lovász bounds*, IEEE Trans. Inform. Theory **25** (1979), no. 4, 425–429.

[9] C. E. Shannon, *The zero error capacity of a noisy channel*, IRE Trans. Inform. Theory **2** (1956), no. 3, 8–19.

[10] J. Zuiddam, *The asymptotic spectrum of graphs and the Shannon capacity*, Combinatorica **39** (2019), no. 5, 1173–1184.