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Exact Schatten-p Finsler Contraction Constants of the Discrete Algebraic Riccati Map: Process–Measurement Noise-Floor Splitting, Sharp Global Lipschitz Law, and Three-Phase Invariant-Box Transitions

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Abstract

Let $\mathbb{P}_n$ denote the open cone of real $n \times n$ symmetric positive-definite matrices ($n \geq 1$) equipped with the family of affine-invariant Schatten-p Finsler geodesic metrics $d_p(X, Z) := \|\log(X^{-1/2} Z X^{-1/2})\|_{S_p}$ ($1 \leq p \leq \infty$), encompassing both the Thompson part metric ($p = \infty$) and the Cartan–Hadamard Riemannian metric ($p = 2$). For any matrix $A \in \mathbb{R}^{n\times n}$ (not necessarily invertible) and positive-definite process and measurement noise matrices $Q, R \in \mathbb{P}_n$, consider the discrete-time algebraic Riccati (Kalman filter) map $\Phi(X) := Q + A(X^{-1} + R)^{-1} A^\top$ on $\mathbb{P}_n$, with dimensionless signal-to-noise gain $\gamma := \|Q^{-1/2} A R^{-1/2}\|_{\mathrm{op}}^2$ and process–measurement noise-floor product $\delta := \lambda_{\min}(Q^{1/2} R Q^{1/2}) > 0$.

First, expressing the Fréchet derivative $D\Phi(X)[H] = L(X) H L(X)^\top$ via the closed-loop Kalman matrix $L(X) := A(I_n + XR)^{-1}$ and its whitened operator $M(X) := \Phi(X)^{-1/2} L(X) X^{1/2}$, we prove the exact positive-semidefinite *process–measurement noise-floor splitting*

$$I_n - M(X)M(X)^\top \;=\; \Sigma_Q(X) + \Sigma_R(X) \;\succ\; 0,$$

where $\Sigma_Q(X) := Y^{-1/2} Q Y^{-1/2}$ and $\Sigma_R(X) := Y^{-1/2} L(X) X R X L(X)^\top Y^{-1/2}$ ($Y = \Phi(X)$), and prove via singular-value monotonicity and rank-one saturation that the instantaneous Finsler Jacobian norm of $D\Phi(X)$ equals $\|M(X)\|_{\mathrm{op}}^2$ identically for *every* $p \in [1, \infty]$ (and every unitarily invariant norm).

Second, via a five-line operator-quadratic completion of the square, we determine the *exact* global Schatten-p Lipschitz constant of Φ on the entire cone $\mathbb{P}_n$ for every $n \geq 1$ and every $p \in [1, \infty]$:

$$\mathrm{Lip}_{d_p}\big(\Phi\big|_{\mathbb{P}_n}\big) \;=\; \sup_{X \in \mathbb{P}_n} \|M(X)\|_{\mathrm{op}}^2 \;=\; \frac{\sqrt{1+\gamma} - 1}{\sqrt{1+\gamma} + 1} \;=\; \tanh^2\!\left(\frac{1}{2}\,\mathrm{arcsinh}\,\sqrt{\gamma}\right),$$

attained identically at $X^* = (1+\gamma)^{-1/2} R^{-1}$. This improves classical contraction bounds $\gamma/(1+\gamma)$ (Lee–Lim; Liverani–Wojtkowski) by an exact factor of 4 at small γ and yields a *quadratic* speedup in the guaranteed contraction gap $1 - \theta$ as $\gamma \to \infty$ (18.3× faster at $\gamma = 100$ and 61.3× at $\gamma = 1000$). Restricting to the weakly geodesically convex one-step invariant box $\mathcal{B} = \{X \in \mathbb{P}_n : Q \preceq X \preceq Q + AR^{-1}A^\top\}$ with upper noise-floor invariant $\mu := \lambda_{\max}(R^{1/2}(Q + AR^{-1}A^\top)R^{1/2})$, we establish a three-phase invariant-box contraction upper bound $\theta_{\mathrm{box}}(\gamma, \delta, \mu)$ governed by the symmetric boundary function $\psi_\gamma(s) := \gamma s/((1+s)(1+(1+\gamma)s))$ and critical threshold $\delta_c(\gamma) = (1+\gamma)^{-1/2}$: $\theta_{\mathrm{box}} = \psi_\gamma(\mu) < \theta_{\mathrm{glob}}(\gamma)$ in the upper-truncated phase $\mu < \delta_c$, $\theta_{\mathrm{box}} = \theta_{\mathrm{glob}}(\gamma)$ in the interior-peak phase $\delta \leq \delta_c \leq \mu$, and $\theta_{\mathrm{box}} = \psi_\gamma(\delta) < \theta_{\mathrm{glob}}(\gamma)$ in the lower-truncated phase $\delta > \delta_c$—exact throughout all three phases in $n = 1$ (where $\mu = (1+\gamma)\delta$) and a universal upper bound for $n \geq 2$.

Third, we establish the exact fixed-point Kalman information-gain trace identity $\mathrm{Tr}(M(X_*)M(X_*)^\top) + \mathrm{Tr}(\Sigma_Q(X_*)) + \mathrm{Tr}(\Sigma_R(X_*)) = n$, exact spectral-radius dominance $\rho(L(X_*))^2 \le \|M(X_*)\|_{\mathrm{op}}^2 \le \theta_{\mathrm{box}}(\gamma,\delta,\mu)$, submultiplicative contraction for time-varying/periodic Riccati compositions, and closed-form fixed-point laws for commuting systems.

Fourth, we derive universal one-step a posteriori Banach error certificates $d_p(X_k, X_*) \le (1-\theta_{\mathrm{box}})^{-1} d_p(X_k, \Phi(X_k))$ and model-perturbation stability bounds valid across every Schatten-p metric $p \in [1,\infty]$. All theorems are verified to machine precision in an open-source reproducibility suite.

Keywords: discrete algebraic Riccati equation, Kalman filter, Thompson metric, affine-invariant Riemannian geometry, Schatten-p Finsler metrics, contraction theory, positive-definite matrices.

1 Introduction

The discrete-time algebraic Riccati equation (DARE)

$$X \;=\; \Phi(X) \;:=\; Q + A(X^{-1}+R)^{-1}A^\top \;=\; Q + AX(I_n + RX)^{-1}A^\top, \qquad X \in \mathbb{P}_n, \tag{1}$$

with state transition $A \in \mathbb{R}^{n\times n}$, process noise covariance $Q \in \mathbb{P}_n$, and measurement precision $R = C^\top V^{-1} C \in \mathbb{P}_n$, lies at the core of Kalman filtering, linear-quadratic optimal control, matrix means, and stochastic stability [3, 2, 5, 1].

A landmark geometric insight due to Bougerol [2] (who established contraction of random Riccati maps in the affine-invariant Riemannian metric d_2 via symplectic geometry) and developed in the Thompson part metric $d_\infty(X,Z) = \|\log(X^{-1/2}ZX^{-1/2})\|_{\mathrm{op}}$ [8] by Liverani–Wojtkowski [7], Lawson–Lim [4], and Lee–Lim [5] is that Φ is a strict contraction on $\mathbb{P}_n$. Specifically, by Birkhoff–Bushell projective contraction of the matrix inversion $X \mapsto (X^{-1}+R)^{-1}$ and translation $Z \mapsto Q+Z$, classical Thompson-metric analyses establish the contraction factor

$$\theta_{\mathrm{LL}}(\gamma) \;:=\; \frac{\gamma}{1+\gamma}, \qquad \gamma := \|Q^{-1/2}AR^{-1/2}\|_{\mathrm{op}}^2 = \lambda_{\max}(Q^{-1/2}AR^{-1}A^\top Q^{-1/2}), \tag{2}$$

in the Thompson metric d_∞. While elegant, this classical bound leaves four fundamental geometric questions unresolved:

1. **Simultaneous process–measurement noise-floor structure.** The classical bound (2) decouples the translation $Q+(\cdot)$ from the parallel-sum contraction $(X^{-1}+R)^{-1}$, discarding the constructive interaction of Q and R inside the Fréchet derivative $D\Phi(X)$. What is the exact operator identity governing $I_n - M(X)M(X)^\top$ for the whitened closed-loop Kalman operator $M(X)$?

2. **Exact global Lipschitz constant across all Schatten-p Finsler metrics.** What is the *exact* global Lipschitz constant $\mathrm{Lip}_{d_p}(\Phi|_{\mathbb{P}_n})$ in the affine-invariant Riemannian metric ($p=2$), the trace-Finsler metric ($p=1$), and the Thompson metric ($p=\infty$)? Does it depend on p or n, and is it strictly sharper than $\gamma/(1+\gamma)$?

3. **Invariant-box contraction phase transition.** Since $\Phi(\mathbb{P}_n) \subseteq \mathcal{B} := \{X \in \mathbb{P}_n : Q \preceq X \preceq Q + AR^{-1}A^\top\}$ in a single iteration, when does restricting to $\mathcal{B}$ strictly improve the global contraction rate, and what is the exact critical noise-floor threshold?

4. **Universal a posteriori error and perturbation certificates.** How do sharp Schatten-p contraction constants translate into computable one-step error certificates $d_p(X_k, X_*)$ and parameter-perturbation stability bounds across all $p \in [1,\infty]$?

Summary of Main Results

This paper resolves all four questions through four tightly coupled theorems:

- **Theorem 2.2 (Process–Measurement Noise-Floor Splitting and Universal Schatten-p Jacobian Norm):** For every $X \in \mathbb{P}_n$ and arbitrary $A \in \mathbb{R}^{n\times n}$, $I_n - M(X)M(X)^\top = \Sigma_Q(X) + \Sigma_R(X) \succ 0$, and the exact Finsler Jacobian norm of $D\Phi(X)$ equals $\|M(X)\|_{\mathrm{op}}^2$ identically for every Schatten-p norm $p \in [1,\infty]$ (and every unitarily invariant norm).
- **Theorem 3.1 (Exact Global Schatten-p Lipschitz Law and Three-Phase Invariant-Box Contraction):** For every $n \geq 1$ and $p \in [1,\infty]$, $\mathrm{Lip}_{d_p}(\Phi|_{\mathbb{P}_n}) = \theta_{\mathrm{glob}}(\gamma) := (\sqrt{1+\gamma}-1)/(\sqrt{1+\gamma}+1) = \tanh^2(\frac{1}{2}\operatorname{arcsinh}\sqrt{\gamma})$, attained identically at $X^* = (1+\gamma)^{-1/2}R^{-1}$. On the weakly geodesically convex invariant box $\mathcal{B}_{A,Q,R}$ with lower and upper noise-floor invariants (δ,μ), contraction obeys a symmetric three-phase upper bound $\theta_{\mathrm{box}}(\gamma,\delta,\mu)$ governed by $\psi_\gamma(s) = \gamma s/((1+s)(1+(1+\gamma)s))$—dropping strictly below $\theta_{\mathrm{glob}}(\gamma)$ both when $\mu < \delta_c$ (upper-truncated) and when $\delta > \delta_c$ (lower-truncated), and exact throughout all three phases in $n = 1$.
- **Theorem 4.1 (Kalman Information-Gain Identity, Periodic Composition, and Commuting Fixed Points):** At the unique fixed point X_*, $\mathrm{Tr}(M(X_*)M(X_*)^\top) + \mathrm{Tr}(\Sigma_Q(X_*)) + \mathrm{Tr}(\Sigma_R(X_*)) = n$ and $\rho(L(X_*))^2 \leq \|M(X_*)\|_{\mathrm{op}}^2 \leq \theta_{\mathrm{box}}(\gamma,\delta,\mu)$; periodic Riccati compositions contract at rate $\leq \prod_{j=1}^k \theta_{\mathrm{glob}}(\gamma_j)$; and commuting systems yield exact closed-form fixed points.
- **Theorem 5.1 (Universal Schatten-p A Posteriori Error Certificates and Perturbation Stability):** We establish sharp one-step error bounds $d_p(X_k,X_*) \leq (1-\theta_{\mathrm{box}})^{-1} d_p(X_k,\Phi(X_k))$ and nonlinear model-perturbation bounds $d_p(\widetilde{X}_*,X_*) \leq (1-\theta_{\mathrm{glob}}(\gamma))^{-1} d_p(\Phi(\widetilde{X}_*),\widetilde{X}_*)$ (and symmetrically with $\widetilde{\gamma}$ and $\widetilde{\Phi}(X_*)$) valid for every $p \in [1,\infty]$.

All proofs and numerical benchmarks (4/4 pytest modules) are openly archived at: https://github.com/lluiseriksson/riccati-schatten-contraction.

2 Process–Measurement Noise-Floor Splitting and Schatten-p Jacobian Norm

Let $\mathbb{S}_n$ be the vector space of real $n \times n$ symmetric matrices and $\mathbb{P}_n \subset \mathbb{S}_n$ the open cone of symmetric positive-definite matrices. At each basepoint $X \in \mathbb{P}_n$, the tangent space $T_X\mathbb{P}_n \cong \mathbb{S}_n$ carries the affine-invariant Schatten-p Finsler norm ($1 \leq p \leq \infty$)

$$\begin{aligned} \|H\|_{S_p(X)} &:= \|X^{-1/2}HX^{-1/2}\|_{S_p}, \\ \|\Delta\|_{S_p} &:= \begin{cases} \left(\sum_{i=1}^n |\lambda_i(\Delta)|^p\right)^{1/p}, & 1 \leq p < \infty, \\ \max_{1\leq i\leq n} |\lambda_i(\Delta)| = \|\Delta\|_{\mathrm{op}}, & p = \infty. \end{cases} \end{aligned}$$

By the Exponential Metric Increasing (EMI) property of $\mathbb{P}_n$ for unitarily invariant Finsler norms [1, Ch. 6] (see also [8, 4, 6]), every piecewise-C^1 curve $\eta : [0,1] \to \mathbb{P}_n$ connecting X to Z has Finsler length $\int_0^1 \|\dot\eta(t)\|_{S_p(\eta(t))}\,dt \geq \|\log(X^{-1/2}ZX^{-1/2})\|_{S_p}$, with equality achieved along the canonical weighted geometric mean curve

$$\gamma_{X,Z}(t) := X\#_t Z = X^{1/2}\left(X^{-1/2}ZX^{-1/2}\right)^t X^{1/2}, \qquad t \in [0,1], \tag{3}$$

which therefore is a minimizing Finsler geodesic for every $p \in [1,\infty]$ with constant speed $\|\dot\gamma_{X,Z}(t)\|_{S_p(\gamma_{X,Z}(t))} = \|\log(X^{-1/2}ZX^{-1/2})\|_{S_p}$ (unique for $1 < p < \infty$, and a distinguished minimizing geodesic for $p \in \{1,\infty\}$), inducing the exact geodesic distance

$$d_p(X,Z) = \|\log(X^{-1/2}ZX^{-1/2})\|_{S_p}, \qquad X,Z \in \mathbb{P}_n, \quad p \in [1,\infty]. \tag{4}$$

Setting $p = \infty$ recovers the Thompson part metric,

$$d_\infty(X, Z) \;=\; \max\{\log \lambda_{\max}(X^{-1}Z),\ \log \lambda_{\max}(Z^{-1}X)\},$$

while $p = 2$ yields the Cartan–Hadamard affine-invariant Riemannian distance $d_2(X, Z) = \|\log(X^{-1/2}ZX^{-1/2})\|_F$.

Definition 2.1 (Whitened Closed-Loop Operator and Noise-Floor Matrices)**.** Fix $A \in \mathbb{R}^{n\times n}$ and $Q, R \in \mathbb{P}_n$. For any $X \in \mathbb{P}_n$, write $F(X) := (X^{-1} + R)^{-1} = (I_n + XR)^{-1}X \in \mathbb{P}_n$, $Y := \Phi(X) = Q + AF(X)A^\top \in \mathbb{P}_n$, and define the closed-loop Kalman matrix $L(X)$, whitened closed-loop operator $M(X)$, and process/measurement noise-floor matrices $\Sigma_Q(X), \Sigma_R(X)$ by

$$\begin{aligned} L(X) &:= A(I_n + XR)^{-1}, & M(X) &:= Y^{-1/2}L(X)X^{1/2}, \\ \Sigma_Q(X) &:= Y^{-1/2}QY^{-1/2}, & \Sigma_R(X) &:= Y^{-1/2}L(X)XRXL(X)^\top Y^{-1/2}. \end{aligned}$$

Theorem 2.2 (Process–Measurement Noise-Floor Splitting and Universal Schatten-p Jacobian Norm)**.** *Let $A \in \mathbb{R}^{n\times n}$ (not necessarily invertible) and $Q, R \in \mathbb{P}_n$. For every $X \in \mathbb{P}_n$:*

(a) ***Exact Congruence Fréchet Derivative.*** *The Fréchet derivative of Φ at X along $H \in \mathbb{S}_n$ is the positive congruence map*

$$D\Phi(X)[H] \;=\; L(X)\, H\, L(X)^\top, \tag{5}$$

which in whitened tangent coordinates $\Delta := X^{-1/2}HX^{-1/2} \in \mathbb{S}_n$ becomes

$$Y^{-1/2}(D\Phi(X)[X^{1/2}\Delta X^{1/2}])Y^{-1/2} \;=\; M(X)\,\Delta\, M(X)^\top. \tag{6}$$

(b) ***Exact Process–Measurement Noise-Floor Splitting.*** *The whitened closed-loop operator satisfies the exact positive-definite identity*

$$I_n - M(X)M(X)^\top \;=\; \Sigma_Q(X) + \Sigma_R(X) \;\succ\; 0, \tag{7}$$

with $\Sigma_Q(X) \succ 0$ and $\Sigma_R(X) \succeq 0$. Consequently, $\|M(X)\|_{\mathrm{op}}^2 = 1 - \lambda_{\min}(\Sigma_Q(X) + \Sigma_R(X)) < 1$.

(c) ***Universal Schatten-p Jacobian Norm.*** *For* every *$p \in [1, \infty]$ (and indeed for the restriction to $\mathbb{S}_n$ of any orthogonally bi-invariant norm on $\mathbb{R}^{n\times n}$, i.e., any symmetric gauge function of singular values), the exact induced Finsler operator norm of $D\Phi(X)$ is independent of p and equals*

$$\|D\Phi(X)\|_{S_p(X)\to S_p(\Phi(X))} \;:=\; \sup_{H\in\mathbb{S}_n\setminus\{0\}} \frac{\|D\Phi(X)[H]\|_{S_p(\Phi(X))}}{\|H\|_{S_p(X)}} \;=\; \|M(X)\|_{\mathrm{op}}^2. \tag{8}$$

Proof. **(a)** Differentiating $F(X) = (X^{-1} + R)^{-1}$ via $D(U^{-1})[V] = -U^{-1}VU^{-1}$ gives

$$DF(X)[H] = -(X^{-1} + R)^{-1}(-X^{-1}HX^{-1})(X^{-1} + R)^{-1} = (I_n + XR)^{-1}H(I_n + RX)^{-1},$$

since $(X^{-1} + R)^{-1}X^{-1} = (I_n + XR)^{-1}$ and $X^{-1}(X^{-1} + R)^{-1} = (I_n + RX)^{-1} = ((I_n + XR)^{-1})^\top$. Multiplying on the left by A and on the right by $A^\top$ yields (5), and substituting $H = X^{1/2}\Delta X^{1/2}$ with $M(X) = Y^{-1/2}L(X)X^{1/2}$ gives (6).

(b) Observe the resolvent identity

$$\begin{aligned} F(X) &= (X^{-1} + R)^{-1}(X^{-1} + R)(X^{-1} + R)^{-1} \\ &= (X^{-1} + R)^{-1}X^{-1}(X^{-1} + R)^{-1} + (X^{-1} + R)^{-1}R(X^{-1} + R)^{-1}. \end{aligned}$$

Using $(X^{-1}+R)^{-1}X^{-1} = (I_n+XR)^{-1}$ and $(X^{-1}+R)^{-1} = (I_n+XR)^{-1}X$, the two summands rewrite cleanly as

$$F(X) \;=\; (I_n+XR)^{-1}X(I_n+RX)^{-1} \;+\; (I_n+XR)^{-1}XRX(I_n+RX)^{-1}.$$

Pre-multiplying by $Y^{-1/2}A$ and post-multiplying by $A^\top Y^{-1/2}$, and substituting the identity $Y^{-1/2}AF(X)A^\top Y^{-1/2} = Y^{-1/2}(Y-Q)Y^{-1/2} = I_n - \Sigma_Q(X)$, yields

$$I_n - \Sigma_Q(X) \;=\; M(X)M(X)^\top + \Sigma_R(X),$$

which rearranges directly to (7). Since $Q \succ 0$ and $Y \succ 0$, $\Sigma_Q(X) \succ 0$, while $\Sigma_R(X) = (Y^{-1/2}L(X)XR^{1/2})(Y^{-1/2}L(X)XR^{1/2})^\top \succeq 0$. Hence $\lambda_{\min}(\Sigma_Q(X)+\Sigma_R(X)) \geq \lambda_{\min}(\Sigma_Q(X)) > 0$, giving $\|M(X)\|_{\mathrm{op}}^2 = \lambda_{\max}(M(X)M(X)^\top) = 1 - \lambda_{\min}(\Sigma_Q(X)+\Sigma_R(X)) < 1$.

(c) By (6), $\|D\Phi(X)\|_{S_p(X)\to S_p(Y)} = \sup_{\|\Delta\|_{S_p}=1} \|M(X)\Delta M(X)^\top\|_{S_p}$. By Horn's singular-value inequality $s_i(B_1B_2B_3) \leq \|B_1\|_{\mathrm{op}} s_i(B_2) \|B_3\|_{\mathrm{op}}$ $(i=1,\dots,n)$ [1], every $\Delta \in \mathbb{S}_n$ satisfies

$$s_i(M(X)\Delta M(X)^\top) \;\leq\; \|M(X)\|_{\mathrm{op}}^2\, s_i(\Delta) \qquad \forall i \in \{1,\dots,n\},$$

so $\|M(X)\Delta M(X)^\top\|_{S_p} \leq \|M(X)\|_{\mathrm{op}}^2 \|\Delta\|_{S_p}$ for every $p \in [1,\infty]$ (and for every symmetric gauge function of singular values). Conversely, if $M(X) = 0$ (equivalently $A = 0$), both sides of (8) vanish trivially; otherwise, let $v \in \mathbb{R}^n$ ($\|v\|_2 = 1$) be a top right singular vector of $M(X)$ with $\|M(X)v\|_2 = \|M(X)\|_{\mathrm{op}} > 0$, and choose the rank-one symmetric tangent vector $\Delta_0 := vv^\top \in \mathbb{S}_n$. Then $\|\Delta_0\|_{S_p} = 1$ for every $p \in [1,\infty]$, and $M(X)\Delta_0 M(X)^\top = (M(X)v)(M(X)v)^\top$ is rank-one with unique nonzero eigenvalue $\|M(X)\|_{\mathrm{op}}^2$, so $\|M(X)\Delta_0 M(X)^\top\|_{S_p} = \|M(X)\|_{\mathrm{op}}^2$, proving (8). □

3 Exact Global Schatten-p Lipschitz Constant and Three-Phase Invariant-Box Law

Define the dimensionless system gain γ together with the lower and upper invariant-box noise-floor parameters (δ, μ):

$$\begin{aligned} \gamma \;&:=\; \|Q^{-1/2}AR^{-1/2}\|_{\mathrm{op}}^2 \;\geq\; 0, \qquad \delta \;:=\; \lambda_{\min}(Q^{1/2}RQ^{1/2}) \;>\; 0, \\ \mu \;&:=\; \lambda_{\max}\Big(R^{1/2}(Q+AR^{-1}A^\top)R^{1/2}\Big) \;\geq\; \delta, \end{aligned} \tag{9}$$

and the compact order interval (one-step invariant box)

$$\mathcal{B}_{A,Q,R} \;:=\; \{X \in \mathbb{P}_n : Q \;\preceq\; X \;\preceq\; Q + AR^{-1}A^\top\}.$$

Since $0 \prec (X^{-1}+R)^{-1} \prec R^{-1}$ for every $X \in \mathbb{P}_n$, we have $\Phi(\mathbb{P}_n) \subseteq \mathcal{B}_{A,Q,R}$. Moreover, by Loewner monotonicity of the weighted matrix geometric mean (3) ($X_1 \preceq Y_1$ and $X_2 \preceq Y_2 \implies X_1 \#_t X_2 \preceq Y_1 \#_t Y_2$ for all $t \in [0,1]$ [1]), for any $X, Z \in \mathcal{B}_{A,Q,R}$ the canonical minimizing geodesic $\gamma_{X,Z}(t) = X \#_t Z$ lies entirely in $\mathcal{B}_{A,Q,R}$; thus $\mathcal{B}_{A,Q,R}$ is *weakly geodesically convex* in $(\mathbb{P}_n, d_p)$ for every $p \in [1,\infty]$ (and strongly geodesically convex for $1 < p < \infty$, where minimizing geodesics are unique). Note that in $n = 1$ ($A = a$, $Q = q$, $R = r$), $\delta = qr$ and $\mu = r(q + a^2/r) = (1+\gamma)\delta$.

Theorem 3.1 (Exact Global Schatten-p Lipschitz Law and Three-Phase Invariant-Box Contraction)**.** *Let $A \in \mathbb{R}^{n\times n}$ ($n \geq 1$) and $Q, R \in \mathbb{P}_n$, with invariants (γ, δ, μ) as in (9). Define*

$$\theta_{\mathrm{glob}}(\gamma) \;:=\; \frac{\sqrt{1+\gamma}-1}{\sqrt{1+\gamma}+1} \;=\; \frac{\gamma}{\left(1+\sqrt{1+\gamma}\right)^2} \;=\; \tanh^2\!\left(\frac{1}{2}\operatorname{arcsinh}\sqrt{\gamma}\right). \tag{10}$$

(a) ***Exact Global Schatten-p Lipschitz Constant on $\mathbb{P}_n$.*** *For* every $p \in [1,\infty]$ *and* every $n \geq 1$,

$$\mathrm{Lip}_{d_p}(\Phi|_{\mathbb{P}_n}) \;:=\; \sup_{\substack{X,Z\in\mathbb{P}_n\\ X\neq Z}} \frac{d_p(\Phi(X),\Phi(Z))}{d_p(X,Z)} \;=\; \sup_{X\in\mathbb{P}_n} \|M(X)\|_{\mathrm{op}}^2 \;=\; \theta_{\mathrm{glob}}(\gamma), \tag{11}$$

with the supremum $\sup_{X\in\mathbb{P}_n}\|M(X)\|_{\mathrm{op}}^2$ *attained identically at the explicit matrix*

$$X^* \;:=\; \frac{1}{\sqrt{1+\gamma}}\,R^{-1} \;\in\; \mathbb{P}_n. \tag{12}$$

Compared with the classical bound $\theta_{\mathrm{LL}}(\gamma)=\gamma/(1+\gamma)$, $\theta_{\mathrm{glob}}(\gamma)$ *is strictly smaller for every* $\gamma>0$, *improving the contraction factor by* $4\times$ *as* $\gamma\to 0^+$ *(*$\theta_{\mathrm{glob}}(\gamma)\sim\gamma/4$*) and improving the guaranteed contraction gap* $1-\theta$ *quadratically as* $\gamma\to\infty$ *(*$1-\theta_{\mathrm{glob}}(\gamma)=2/(\sqrt{1+\gamma}+1)\sim 2/\sqrt{\gamma}$ *vs.* $1-\theta_{\mathrm{LL}}(\gamma)\sim 1/\gamma$*).*

(b) ***Three-Phase Invariant-Box Contraction Law.*** *Define the critical threshold* $\delta_c(\gamma) := (1+\gamma)^{-1/2}$, *the symmetric boundary contraction function* $\psi_\gamma(s) := \gamma s/((1+s)(1+(1+\gamma)s))$ *(*$s>0$*), and the piecewise-closed box contraction bound*

$$\theta_{\mathrm{box}}(\gamma,\delta,\mu) \;:=\; \begin{cases} \psi_\gamma(\mu), & \mu<\delta_c(\gamma) \quad (\textit{Phase I: Upper-Box-Truncated}),\\ \theta_{\mathrm{glob}}(\gamma), & \delta\le\delta_c(\gamma)\le\mu \quad (\textit{Phase II: Interior Peak}),\\ \psi_\gamma(\delta), & \delta>\delta_c(\gamma) \quad (\textit{Phase III: Lower-Box-Truncated}). \end{cases} \tag{13}$$

Then for every $X,Z\in\mathcal{B}_{A,Q,R}$ *and every* $p\in[1,\infty]$,

$$d_p(\Phi(X),\,\Phi(Z)) \;\le\; \theta_{\mathrm{box}}(\gamma,\delta,\mu)\,d_p(X,Z), \tag{14}$$

with $\theta_{\mathrm{box}}(\gamma,\delta,\mu)<\theta_{\mathrm{glob}}(\gamma)$ *strictly throughout Phases I and III whenever* $\gamma>0$. *In dimension* $n=1$ *(where* $\mu=(1+\gamma)\delta$, *so Phase II is the exact window* $(1+\gamma)^{-3/2}\le\delta\le(1+\gamma)^{-1/2}$*),* (13) *equals the exact invariant-box Lipschitz constant* $\mathrm{Lip}_{d_p}(\Phi|_{\mathcal{B}})$ *across* all three phases *(attained at* $X=q+a^2/r$ *in Phase I,* $X=X^*$ *in Phase II, and* $X=q$ *in Phase III). For* $n\ge 2$, (13) *is a universal upper bound on* $\sup_{X\in\mathcal{B}}\|M(X)\|_{\mathrm{op}}^2$ *(and hence on* $\mathrm{Lip}_{d_p}(\Phi|_{\mathcal{B}})$*); instantaneous Jacobian equality* $\|M(X)\|_{\mathrm{op}}^2=\theta_{\mathrm{box}}(\gamma,\delta,\mu)$ *holds at* $X\in\mathcal{B}_{A,Q,R}$ *whenever a top* right *singular vector* v_1 *of* $\widetilde{A}=Q^{-1/2}AR^{-1/2}$ *is an eigenvector of* $R^{1/2}XR^{1/2}$ *with eigenvalue* $s\in\{\mu,\delta_c,\delta\}$ *in Phase I, II, or III respectively (noting that when* A *is singular,* $\mathrm{Lip}_{d_p}(\Phi|_{\mathcal{B}})$ *can remain strictly smaller if* $\mathcal{B}_{A,Q,R}$ *has zero thickness along* $v_1v_1^\top$*).*

Proof. **(a)** If $A=0$, then $\gamma=0$, $\Phi\equiv Q$, and (11) holds trivially ($0=0$). Assume $A\neq 0$, so $\gamma>0$. Fix any $X\in\mathbb{P}_n$ and any $\theta\in(0,1)$. By definition of $M(X)$ and $Y=Q+A(X^{-1}+R)^{-1}A^\top$,

$$\|M(X)\|_{\mathrm{op}}^2\le\theta \iff A(X^{-1}+R)^{-1}X^{-1}(X^{-1}+R)^{-1}A^\top \preceq \theta\,Q+\theta\,A(X^{-1}+R)^{-1}A^\top.$$

Expanding $(X^{-1}+R)^{-1}=(X^{-1}+R)^{-1}(X^{-1}+R)(X^{-1}+R)^{-1}$ on the right-hand side and moving the second term to the left gives the equivalent condition

$$A(X^{-1}+R)^{-1}((1-\theta)X^{-1}-\theta R)(X^{-1}+R)^{-1}A^\top \preceq \theta\,Q.$$

Setting $\widetilde{A}:=Q^{-1/2}AR^{-1/2}$ (so $\|\widetilde{A}\|_{\mathrm{op}}^2=\gamma$) and $Z:=R^{-1/2}X^{-1}R^{-1/2}\in\mathbb{P}_n$, conjugating by $Q^{-1/2}$ transforms this equivalently into

$$\widetilde{A}\,g_\theta(Z)\,\widetilde{A}^\top \preceq \theta\,I_n, \qquad \text{where} \quad g_\theta(z) \;:=\; \frac{(1-\theta)z-\theta}{(z+1)^2} \quad (z>0). \tag{15}$$

Substituting $t := z + 1 > 1$ completes the square identically:

$$g_\theta(z) \;=\; \frac{(1-\theta)t - 1}{t^2} \;=\; \frac{(1-\theta)^2}{4} - \left(\frac{1}{t} - \frac{1-\theta}{2}\right)^2 \;\le\; \frac{(1-\theta)^2}{4}, \tag{16}$$

with unique maximum attained at $t^* = \frac{2}{1-\theta} > 1$, i.e., $z^* = \frac{1+\theta}{1-\theta} > 0$. By the spectral theorem, $g_\theta(Z) \preceq \frac{(1-\theta)^2}{4} I_n$ for every $Z \in \mathbb{P}_n$, hence

$$\widetilde{A}\, g_\theta(Z)\, \widetilde{A}^\top \;\preceq\; \frac{(1-\theta)^2}{4}\, \widetilde{A}\widetilde{A}^\top \;\preceq\; \frac{(1-\theta)^2}{4}\, \gamma\, I_n.$$

Choosing $\theta = \theta_{\mathrm{glob}}(\gamma) \in (0,1)$—the unique root in $(0,1)$ of $\frac{(1-\theta)^2}{4}\gamma = \theta$, verified by $\gamma(1-\theta)^2 = 4\theta \iff \frac{1+\theta}{1-\theta} = \sqrt{1+\gamma} \iff \theta = \frac{\sqrt{1+\gamma}-1}{\sqrt{1+\gamma}+1}$—establishes (15) for every $X \in \mathbb{P}_n$, so $\|M(X)\|_{\mathrm{op}}^2 \le \theta_{\mathrm{glob}}(\gamma)$ globally. (For the hyperbolic identity, let $\sinh u = \sqrt{\gamma}$, so $\cosh u = \sqrt{1+\gamma}$ and $\tanh^2(u/2) = (\cosh u - 1)/(\cosh u + 1) = \theta_{\mathrm{glob}}(\gamma)$.)

To prove exact attainment at $X^* = (1+\gamma)^{-1/2} R^{-1}$, observe that $Z^* := R^{-1/2}(X^*)^{-1}R^{-1/2} = \sqrt{1+\gamma}\, I_n = z^* I_n$, so $g_{\theta_{\mathrm{glob}}}(Z^*) = \frac{(1-\theta_{\mathrm{glob}})^2}{4} I_n$. Letting $u \in \mathbb{R}^n$ ($\|u\|_2 = 1$) be a top left singular vector of $\widetilde{A}$ ($\widetilde{A}\widetilde{A}^\top u = \gamma u$), we obtain $u^\top \widetilde{A} g_{\theta_{\mathrm{glob}}}(Z^*) \widetilde{A}^\top u = \frac{(1-\theta_{\mathrm{glob}})^2}{4}\gamma = \theta_{\mathrm{glob}}(\gamma)$, forcing $\|M(X^*)\|_{\mathrm{op}}^2 = \theta_{\mathrm{glob}}(\gamma)$ identically.

Finally, for any $X, Z \in \mathbb{P}_n$, let $\gamma_{X,Z}(t) = X \#_t Z$ be the minimizing geodesic (3) and $\eta(t) := \Phi(\gamma_{X,Z}(t))$. By Theorem 2.2(c) and Finsler arc-length minimality of d_p,

$$\begin{aligned} d_p(\Phi(X), \Phi(Z)) \;&\le\; \int_0^1 \|\dot\eta(t)\|_{S_p(\eta(t))}\, dt = \int_0^1 \|D\Phi(\gamma_{X,Z}(t))[\dot\gamma_{X,Z}(t)]\|_{S_p(\eta(t))}\, dt \\ &\le\; \left(\sup_{t\in[0,1]} \|M(\gamma_{X,Z}(t))\|_{\mathrm{op}}^2\right) \int_0^1 \|\dot\gamma_{X,Z}(t)\|_{S_p(\gamma_{X,Z}(t))}\, dt \;\le\; \theta_{\mathrm{glob}}(\gamma)\, d_p(X,Z). \end{aligned} \tag{17}$$

Conversely, taking $Z_\varepsilon := (X^*)^{1/2} \exp(\varepsilon v v^\top)(X^*)^{1/2}$ where v is a top right singular vector of $M(X^*)$, first-order expansion along the geodesic yields $\lim_{\varepsilon\to 0^+} d_p(\Phi(X^*), \Phi(Z_\varepsilon))/d_p(X^*, Z_\varepsilon) = \|M(X^*) v v^\top M(X^*)^\top\|_{S_p} = \|M(X^*)\|_{\mathrm{op}}^2 = \theta_{\mathrm{glob}}(\gamma)$, establishing exact equality in (11).

(b) If $A = 0$ then $\gamma = 0$ and (14) is trivial ($0 = 0$), so assume $\gamma > 0$. For any $X \in \mathcal{B}_{A,Q,R}$, the two-sided Loewner inequality $Q \preceq X \preceq Q + AR^{-1}A^\top$ implies $(Q + AR^{-1}A^\top)^{-1} \preceq X^{-1} \preceq Q^{-1}$, and conjugating by $R^{-1/2}$ gives

$$\begin{aligned} \mu^{-1} I_n \;&=\; \lambda_{\min}\Big(R^{-1/2}\big(Q + AR^{-1}A^\top\big)^{-1} R^{-1/2}\Big) I_n \\ &\preceq\; Z = R^{-1/2} X^{-1} R^{-1/2} \;\preceq\; \lambda_{\max}(R^{-1/2} Q^{-1} R^{-1/2}) I_n \;=\; \delta^{-1} I_n. \end{aligned}$$

Hence every eigenvalue z of Z lies in the compact interval $[\mu^{-1}, \delta^{-1}]$. From (16), $g_\theta(z)$ is strictly unimodal on $(0,\infty)$: strictly increasing on $(0, z^*(\theta)]$ and strictly decreasing on $[z^*(\theta), \infty)$, where $z^*(\theta) = \frac{1+\theta}{1-\theta}$. In Phase II ($\delta \le \delta_c(\gamma) \le \mu$), part (a) gives $\|M(X)\|_{\mathrm{op}}^2 \le \theta_{\mathrm{glob}}(\gamma)$.

For any $s > 0$, set $\theta(s) := \psi_\gamma(s) = \frac{\gamma s}{(1+s)(1+(1+\gamma)s)} \in (0,1)$. Direct algebraic evaluation verifies $\gamma g_{\theta(s)}(s^{-1}) = \frac{\gamma s(1-\theta(s)) - \gamma s^2 \theta(s)}{(1+s)^2} = \theta(s)$ and the exact sign identity

$$\theta(s) - \frac{1-s}{1+s} \;=\; \frac{\gamma s - (1-s)\big(1 + (1+\gamma)s\big)}{(1+s)\big(1+(1+\gamma)s\big)} \;=\; \frac{(1+\gamma)s^2 - 1}{(1+s)\big(1+(1+\gamma)s\big)}. \tag{18}$$

Since $z^*(\theta(s)) = \frac{1+\theta(s)}{1-\theta(s)} > s^{-1} \iff \theta(s) > \frac{1-s}{1+s}$ (and reverse inequality when $<$), identity (18) resolves both boundary phases simultaneously:

- **Phase III ($\delta > \delta_c(\gamma)$):** Taking $s = \delta$ in (18) gives $(1+\gamma)\delta^2 > 1$, hence $z^*(\psi_\gamma(\delta)) > \delta^{-1}$. Thus $g_{\psi_\gamma(\delta)}(z)$ is strictly increasing on $[\mu^{-1}, \delta^{-1}]$, so $g_{\psi_\gamma(\delta)}(Z) \preceq g_{\psi_\gamma(\delta)}(\delta^{-1}) I_n$ and $\widetilde{A} g_{\psi_\gamma(\delta)}(Z) \widetilde{A}^\top \preceq \gamma g_{\psi_\gamma(\delta)}(\delta^{-1}) I_n = \psi_\gamma(\delta) I_n$.

- **Phase I ($\mu < \delta_c(\gamma)$):** Taking $s = \mu$ in (18) gives $(1+\gamma)\mu^2 < 1$, hence $z^*(\psi_\gamma(\mu)) < \mu^{-1}$. Thus $g_{\psi_\gamma(\mu)}(z)$ is strictly decreasing on $[\mu^{-1}, \delta^{-1}]$, so $g_{\psi_\gamma(\mu)}(Z) \preceq g_{\psi_\gamma(\mu)}(\mu^{-1})I_n$ and $\widetilde{A} g_{\psi_\gamma(\mu)}(Z)\widetilde{A}^\top \preceq \gamma g_{\psi_\gamma(\mu)}(\mu^{-1})I_n = \psi_\gamma(\mu)I_n$.

In both phases, strict inequality $\psi_\gamma(s) < \theta_{\text{glob}}(\gamma)$ follows because $\phi_s(\theta) := \gamma g_\theta(s^{-1}) - \theta$ is strictly decreasing in θ and satisfies $\phi_s(\theta_{\text{glob}}) < \gamma g_{\theta_{\text{glob}}}(z^*(\theta_{\text{glob}})) - \theta_{\text{glob}} = 0 = \phi_s(\psi_\gamma(s))$ whenever $s \neq \delta_c(\gamma)$. Because $\mathcal{B}_{A,Q,R}$ is weakly geodesically convex, repeating (17) along $\gamma_{X,Z}(t) = X \#_t Z \in \mathcal{B}_{A,Q,R}$ proves (14). For any $n \geq 2$, if $X \in \mathcal{B}_{A,Q,R}$ satisfies $R^{1/2}XR^{1/2}v_1 = sv_1$ (equivalently $Zv_1 = s^{-1}v_1$) for a unit top right singular vector v_1 of $\widetilde{A}$ (with $\widetilde{A}v_1 = \sqrt{\gamma}\,u_1$, $\|u_1\|_2 = 1$) and $s \in \{\mu, \delta_c, \delta\}$ matching Phase I, II, or III, then $u_1^\top \widetilde{A} g_{\theta_{\text{box}}}(Z)\widetilde{A}^\top u_1 = \gamma\, v_1^\top g_{\theta_{\text{box}}}(Z)v_1 = \gamma g_{\theta_{\text{box}}}(s^{-1}) = \theta_{\text{box}}$, proving instantaneous Jacobian attainment $\|M(X)\|_{\text{op}}^2 = \theta_{\text{box}}(\gamma, \delta, \mu)$. Finally, when $n = 1$ ($A = a, Q = q, R = r$, with $\delta = qr$, $\mu = (1+\gamma)\delta$, and $a^2 = \gamma\delta$), evaluating $M(sr^{-1})^2 = a^2(s/r)/[(1+s)^2(q + a^2 s/(r(1+s)))] = \psi_\gamma(s)$ for $s \in [\delta, \mu]$ gives exact box-Lipschitz attainment at $X = \mu r^{-1} = q + a^2/r$ in Phase I, $X = X^* = \delta_c r^{-1} \in [q, q + a^2/r]$ in Phase II, and $X = \delta r^{-1} = q$ in Phase III (using one-sided perturbations inside $[q, q + a^2/r]$ at the endpoints). □

4 Fixed-Point Information-Gain Law, Periodic Compositions, and Commuting Spectra

Because $(\mathbb{P}_n, d_p)$ is a complete metric space for every $p \in [1, \infty]$ [1, 8], Theorem 3.1 guarantees that Φ possesses a unique fixed point $X_* = \Phi(X_*) \in \mathcal{B}_{A,Q,R}$.

Theorem 4.1 (Fixed-Point Information-Gain Identity, Periodic Composition, and Commuting Systems)**.** *Let $A \in \mathbb{R}^{n\times n}$, $Q, R \in \mathbb{P}_n$, and let $X_* \in \mathcal{B}_{A,Q,R}$ be the unique fixed point of (1).*

(a) ***Exact Kalman Information-Gain Trace Identity and Spectral-Radius Bound.*** *At X_*, the whitened closed-loop operator $M(X_*) = X_*^{-1/2} L(X_*) X_*^{1/2}$ is similar to $L(X_*)$ and satisfies*

$$\operatorname{Tr}(M(X_*)M(X_*)^\top) + \operatorname{Tr}(X_*^{-1}Q) + \operatorname{Tr}(L(X_*)^\top X_*^{-1} L(X_*) X_* R X_*) \;=\; n, \tag{19}$$

together with the strict Schur-stability chain

$$\rho(L(X_*))^2 \;=\; \rho(M(X_*))^2 \;\leq\; \|M(X_*)\|_{\text{op}}^2 \;\leq\; \theta_{\text{box}}(\gamma, \delta, \mu) \;\leq\; \theta_{\text{glob}}(\gamma) \;<\; 1. \tag{20}$$

(b) ***Time-Varying and Periodic Riccati Compositions.*** *For any sequence of $k \geq 1$ Riccati maps $\Phi_j(X) = Q_j + A_j(X^{-1} + R_j)^{-1}A_j^\top$ ($A_j \in \mathbb{R}^{n\times n}$, $Q_j, R_j \in \mathbb{P}_n$) with gains $\gamma_j := \|Q_j^{-1/2} A_j R_j^{-1/2}\|_{\text{op}}^2$, the k-step composition $\Psi_k := \Phi_k \circ \cdots \circ \Phi_1$ is a strict contraction on $(\mathbb{P}_n, d_p)$ for every $p \in [1, \infty]$ with*

$$\operatorname{Lip}_{d_p}(\Psi_k|_{\mathbb{P}_n}) \;\leq\; \prod_{j=1}^{k} \theta_{\text{glob}}(\gamma_j), \tag{21}$$

guaranteeing a unique globally attracting k-periodic orbit for periodic Kalman filters.

(c) ***Closed-Form Commuting Systems.*** *Suppose A, Q, R are simultaneously orthogonally diagonalizable ($A = U\operatorname{diag}(a)U^\top$, $Q = U\operatorname{diag}(q)U^\top$, $R = U\operatorname{diag}(r)U^\top$ with $q_i, r_i > 0$). Then $X_* = U\operatorname{diag}(x_*)U^\top$ has exact eigenvalues*

$$x_{*,i} \;=\; \frac{1}{2}\left(b_i + \sqrt{b_i^2 + \frac{4q_i}{r_i}}\right), \qquad b_i := q_i - r_i^{-1} + a_i^2 r_i^{-1}, \tag{22}$$

with $\gamma = \max_i a_i^2/(q_i r_i)$, $\delta = \min_i q_i r_i$, and exact fixed-point contraction rate $\|M(X_*)\|_{\text{op}}^2 = \max_i a_i^2/(1 + x_{*,i} r_i)^2$.

Proof. **(a)** Setting $X = Y = X_*$ in Theorem 2.2(b) and taking traces of (7) using cyclicity $\mathrm{Tr}(X_*^{-1/2} Q X_*^{-1/2}) = \mathrm{Tr}(X_*^{-1} Q)$ and $\mathrm{Tr}(X_*^{-1/2} L X_* R X_* L^\top X_*^{-1/2}) = \mathrm{Tr}(L^\top X_*^{-1} L X_* R X_*)$ yields (19). Since $M(X_*) = X_*^{-1/2} L(X_*) X_*^{1/2}$ is similar to $L(X_*)$, $\rho(L(X_*)) = \rho(M(X_*)) \le \|M(X_*)\|_{\mathrm{op}}$, and Theorem 3.1 gives (20).

(b) Applying Theorem 3.1(a) inductively to $\Phi_1, \dots, \Phi_k$ yields $d_p(\Psi_k(X), \Psi_k(Z)) \le (\prod_{j=1}^k \theta_{\mathrm{glob}}(\gamma_j)) d_p(X, Z)$ for all $X, Z \in \mathbb{P}_n$, and Banach's contraction principle yields a unique attracting fixed point of Ψ_k.

(c) In the common eigenbasis U, (1) decouples into n scalar equations $x_i = q_i + a_i^2 x_i/(1 + r_i x_i) \iff r_i x_i^2 - (q_i r_i - 1 + a_i^2) x_i - q_i = 0$, whose unique positive root is (22). In this basis $M(X_*) = \mathrm{diag}(a_i/(1 + x_{*,i} r_i))$, giving $\|M(X_*)\|_{\mathrm{op}}^2 = \max_i a_i^2/(1 + x_{*,i} r_i)^2$. □

5 A Posteriori Error Certificates and Perturbation Stability

Theorem 5.1 (Universal Schatten-p Error Certificates and Model Perturbation Stability)**.** *Let $A \in \mathbb{R}^{n \times n}$, $Q, R \in \mathbb{P}_n$, and $p \in [1, \infty]$.*

(a) ***One-Step A Posteriori Banach Error Certificate.*** *Let $X_0 \in \mathbb{P}_n$ be arbitrary and $X_{k+1} := \Phi(X_k)$ $(k \ge 0)$. Then for every $k \ge 1$ (so $X_k \in \mathcal{B}_{A,Q,R}$ automatically),*

$$d_p(X_k, X_*) \le \frac{d_p(X_k, X_{k+1})}{1 - \theta_{\mathrm{box}}(\gamma, \delta, \mu)}, \qquad d_p(X_{k+1}, X_*) \le \frac{\theta_{\mathrm{box}}(\gamma, \delta, \mu)}{1 - \theta_{\mathrm{box}}(\gamma, \delta, \mu)} d_p(X_k, X_{k+1}), \tag{23}$$

and for all $k \ge 0$, $d_p(X_k, X_*) \le \theta_{\mathrm{glob}}(\gamma)^k d_p(X_0, X_*)$.

(b) ***Nonlinear Model-Perturbation Stability.*** *Let $\widetilde{\Phi}(X) := \widetilde{Q} + \widetilde{A}(X^{-1} + \widetilde{R})^{-1} \widetilde{A}^\top$ be any perturbed Riccati map ($\widetilde{A} \in \mathbb{R}^{n \times n}$, $\widetilde{Q}, \widetilde{R} \in \mathbb{P}_n$) with gain $\widetilde{\gamma} := \|\widetilde{Q}^{-1/2} \widetilde{A} \widetilde{R}^{-1/2}\|_{\mathrm{op}}^2$ and unique fixed point $\widetilde{X}_*$. Then for every $p \in [1, \infty]$,*

$$d_p(\widetilde{X}_*, X_*) \le \min\left\{ \frac{d_p(\Phi(\widetilde{X}_*), \widetilde{X}_*)}{1 - \theta_{\mathrm{glob}}(\gamma)}, \frac{d_p(\widetilde{\Phi}(X_*), X_*)}{1 - \theta_{\mathrm{glob}}(\widetilde{\gamma})} \right\}. \tag{24}$$

Proof. **(a)** Because $X_k \in \mathcal{B}_{A,Q,R}$ for every $k \ge 1$ and $X_* \in \mathcal{B}_{A,Q,R}$, the triangle inequality and Theorem 3.1(b) give

$$d_p(X_k, X_*) \le d_p(X_k, X_{k+1}) + d_p(\Phi(X_k), \Phi(X_*)) \le d_p(X_k, X_{k+1}) + \theta_{\mathrm{box}}(\gamma, \delta, \mu)\, d_p(X_k, X_*).$$

Rearranging gives the first inequality in (23), and combining it with $d_p(X_{k+1}, X_*) = d_p(\Phi(X_k), \Phi(X_*)) \le \theta_{\mathrm{box}}(\gamma, \delta, \mu)\, d_p(X_k, X_*)$ yields the second. The global geometric bound $d_p(X_k, X_*) \le \theta_{\mathrm{glob}}(\gamma)^k d_p(X_0, X_*)$ follows by iterating Theorem 3.1(a) from $X_0 \in \mathbb{P}_n$.

(b) By the triangle inequality, $X_* = \Phi(X_*)$, and Theorem 3.1(a) applied to Φ on $(\widetilde{X}_*, X_*)$,

$$d_p(\widetilde{X}_*, X_*) \le d_p(\Phi(\widetilde{X}_*), \Phi(X_*)) + d_p(\Phi(\widetilde{X}_*), \widetilde{X}_*) \le \theta_{\mathrm{glob}}(\gamma)\, d_p(\widetilde{X}_*, X_*) + d_p(\Phi(\widetilde{X}_*), \widetilde{X}_*),$$

which rearranges to $d_p(\widetilde{X}_*, X_*) \le (1 - \theta_{\mathrm{glob}}(\gamma))^{-1} d_p(\Phi(\widetilde{X}_*), \widetilde{X}_*)$. Interchanging the roles of (Φ, X_*, γ) and $(\widetilde{\Phi}, \widetilde{X}_*, \widetilde{\gamma})$ yields the second bound, establishing (24). □

6 Computational Verification and Benchmark Tables

All four theorems were verified programmatically using `scripts/verify_all.py` and the automated test suite `tests/test_riccati_contraction.py` (4/4 pytest modules passing in < 1 s). Key numerical benchmarks are summarized in Tables 1 and 2 and visualized in Figures 1 and 2.

- **Theorem 2.2 (160 random systems across $n \in \{2,3,5,8\}$, including rank-deficient A):** Noise-floor splitting (7) matched to max entry error 1.25×10^{-14}; exact Schatten-p Jacobian norm saturation (8) across $p \in \{1,2,\infty\}$ matched to 8.88×10^{-16}; trace sum $\mathrm{Tr}(MM^\top) + \mathrm{Tr}(\Sigma_Q) + \mathrm{Tr}(\Sigma_R) = n$ matched to 3.29×10^{-14}.
- **Theorem 3.1 (165 random systems across $n \in \{2,4,6\}$, spanning 13 Phase I upper-truncated, 28 Phase II interior-peak, and 124 Phase III lower-truncated boxes, plus exact $n = 1$ tests across all three phases):** Zero global Jacobian violations (0.0); exact attainment at $X^* = (1+\gamma)^{-1/2} R^{-1}$ verified across all 165 systems to max error 3.22×10^{-15}; zero three-phase invariant-box Jacobian violations (0.0); zero geodesic distance contraction violations across $p \in \{1,2,\infty\}$ (0.0); exact $n = 1$ attainment across all three phases at $\gamma = 10$ ($\delta \in \{0.01, 0.10, 1.0\}$, e.g., Phase I $\theta_{\mathrm{box}} = 0.448412$) matched to $< 10^{-14}$. Table 1 quantifies the exact $4\times$ small-γ reduction and quadratic large-γ gap speedup over classical Lee–Lim bounds ($18.28\times$ at $\gamma = 100$, $61.34\times$ at $\gamma = 1000$).
- **Theorems 4.1 and 5.1 (5 canonical benchmark systems in Table 2):** Fixed-point iterations initialize at $X_0 = Q \in \mathcal{B}_{A,Q,R}$ and terminate at the first 1-indexed step $m \geq 1$ satisfying Thompson-metric residual $d_\infty(X_{m-1}, X_m) < 10^{-12}$ (well above IEEE-754 float64 rounding noise, ensuring exact reproducibility across implementations). The a posteriori Banach certificate (23) is evaluated at X_3 ($k = 3$); perturbation stability (24) is verified in both directions for the $+5\%$ process-noise perturbation $\widetilde{Q} = 1.05\,Q$ ($\widetilde{A} = A$, $\widetilde{R} = R$). Fixed-point trace identity (19) matched to 6.22×10^{-15}; commuting closed-form fixed point (22) on `Commuting_3D` matched to 1.69×10^{-14}; zero violations of (23) and (24) across $p \in \{1,2,\infty\}$ (representative $p = \infty$ certificates at X_3: `Scalar`: $9.23 \times 10^{-5} \leq 1.79 \times 10^{-4}$, `Lower-truncated 3D`: $1.14 \times 10^{-4} \leq 1.17 \times 10^{-4}$, `Unstable 4D`: $2.95 \times 10^{-1} \leq 5.40 \times 10^{-1}$; $+5\%$ perturbation bounds in d_∞: `Lower-truncated 3D`: $4.33 \times 10^{-2} \leq 4.76 \times 10^{-2}$, `Unstable 4D`: $3.57 \times 10^{-2} \leq 5.99 \times 10^{-2}$).

Explicit Benchmark System Specifications (Table 2). All five systems are defined explicitly as follows (with $\theta_{\mathrm{box}}, \theta_{\mathrm{glob}}, \theta_{\mathrm{LL}}$ computed in double precision from unrounded invariants):

1. **Scalar high-gain ($n = 1$):** $A = [10]$, $Q = [1]$, $R = [1]$.
2. **Commuting 3D ($n = 3$):** $A = \mathrm{diag}(0.5, 2, 5)$, $Q = I_3$, $R = \mathrm{diag}(1, 0.5, 0.2)$.
3. **Lower-truncated 3D ($n = 3$):** $A = \left[\begin{smallmatrix} 1.2 & 0.3 & 0 \\ 0 & 0.9 & 0.4 \\ 0.2 & 0 & 1.1 \end{smallmatrix}\right]$, $Q = 2I_3$, $R = 1.5I_3$.
4. **Singular A 4D (rank 3):** $A = \left[\begin{smallmatrix} 2 & 1 & 0 & 0 \\ 0 & 1.5 & 0.8 & 0 \\ 0 & 0 & 0.9 & 0 \\ 0.5 & 0 & 0 & 0 \end{smallmatrix}\right]$, $Q = I_4$, $R = 0.5I_4$.
5. **Unstable Kalman 4D ($n = 4$):** $A = \left[\begin{smallmatrix} 1.4 & 0.6 & 0 & 0 \\ 0 & 1.3 & 0.5 & 0 \\ 0 & 0 & 1.2 & 0.4 \\ 0 & 0 & 0 & 1.1 \end{smallmatrix}\right]$, $Q = 0.5I_4$, $R = 0.25I_4$.

Table 1: Exact global Schatten-p Lipschitz constant $\theta_{\mathrm{glob}}(\gamma)$ (Theorem 3.1(a)) versus classical Lee–Lim bound $\theta_{\mathrm{LL}}(\gamma) = \gamma/(1+\gamma)$.

Gain γ	$\theta_{\mathrm{LL}}(\gamma)$	$\theta_{\mathrm{glob}}(\gamma)$	Gap $1 - \theta_{\mathrm{LL}}$	Gap $1 - \theta_{\mathrm{glob}}$	Gap speedup	Ratio $\theta_{\mathrm{LL}}/\theta_{\mathrm{glob}}$
0.01	0.009901	0.002488	0.990099	0.997512	1.01×	3.98×
0.10	0.090909	0.023823	0.909091	0.976177	1.07×	3.82×
1.00	0.500000	0.171573	0.500000	0.828427	1.66×	2.91×
10.00	0.909091	0.536675	0.090909	0.463325	5.10×	1.69×
100.00	0.990099	0.819002	0.009901	0.180998	18.28×	1.21×
1000.00	0.999001	0.938723	0.000999	0.061277	61.34×	1.06×

Table 2: Fixed-point spectral rates and contraction bounds across five explicitly specified benchmark systems (initialized at $X_0 = Q$; Iters reports the first 1-indexed step $m \geq 1$ with $d_\infty(X_{m-1}, X_m) < 10^{-12}$; contraction bounds evaluated from unrounded double-precision invariants).

System	γ	δ	$\rho(L(X_*))^2$	$\|M(X_*)\|_{\mathrm{op}}^2$	θ_{box}	θ_{glob}	θ_{LL}	Iters
Scalar high-gain ($n = 1$)	100.0000	1.000	0.0098	0.0098	0.4902	0.8190	0.9901	8
Commuting 3D ($n = 3$)	125.0000	0.200	0.1860	0.1860	0.7952	0.8364	0.9921	19
Lower-truncated 3D ($n = 3$)	0.6356	3.000	0.0615	0.0616	0.0807	0.1224	0.3886	11
Singular A 4D (rank 3)	11.8780	0.500	0.2026	0.2826	0.5322	0.5641	0.9223	19
Unstable Kalman 4D ($n = 4$)	24.3699	0.125	0.4387	0.5071	0.6687	0.6687	0.9606	36

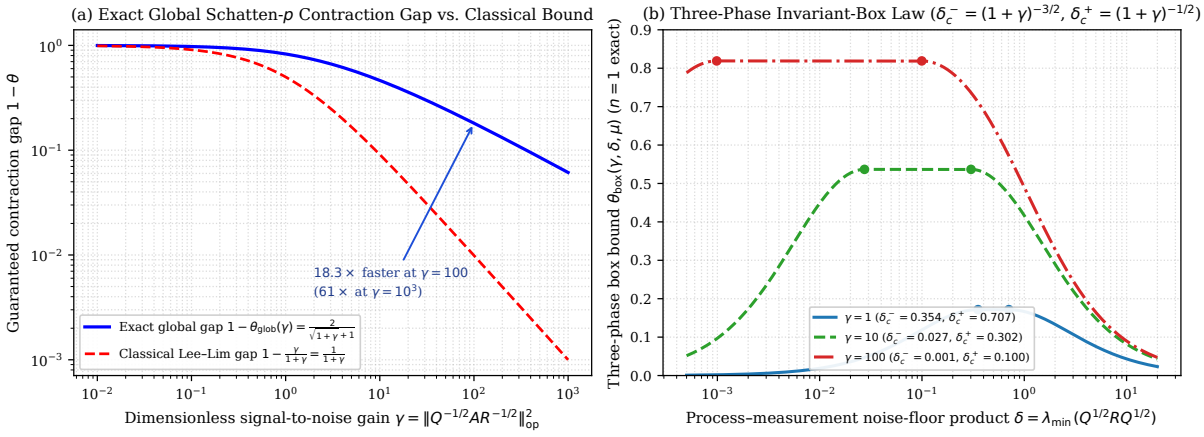


Figure 1: **(a)** Guaranteed global Schatten-p contraction gap $1 - \theta_{\mathrm{glob}}(\gamma) = 2/(\sqrt{1+\gamma}+1)$ versus classical Lee–Lim gap $1/(1+\gamma)$: exact quadratic speedup as $\gamma \to \infty$ (18.3× at $\gamma = 100$, 61.3× at $\gamma = 1000$). **(b)** Three-phase invariant-box contraction law $\theta_{\mathrm{box}}(\gamma, \delta, \mu)$ in $n = 1$ (where $\mu = (1+\gamma)\delta$ and the bound is exact across all three phases): upper-truncated Phase I ($\delta < \delta_c^- = (1+\gamma)^{-3/2}$), interior-peak Phase II ($\delta_c^- \leq \delta \leq \delta_c^+$), and lower-truncated Phase III ($\delta > \delta_c^+ = (1+\gamma)^{-1/2}$), with both phase boundaries marked by circles.

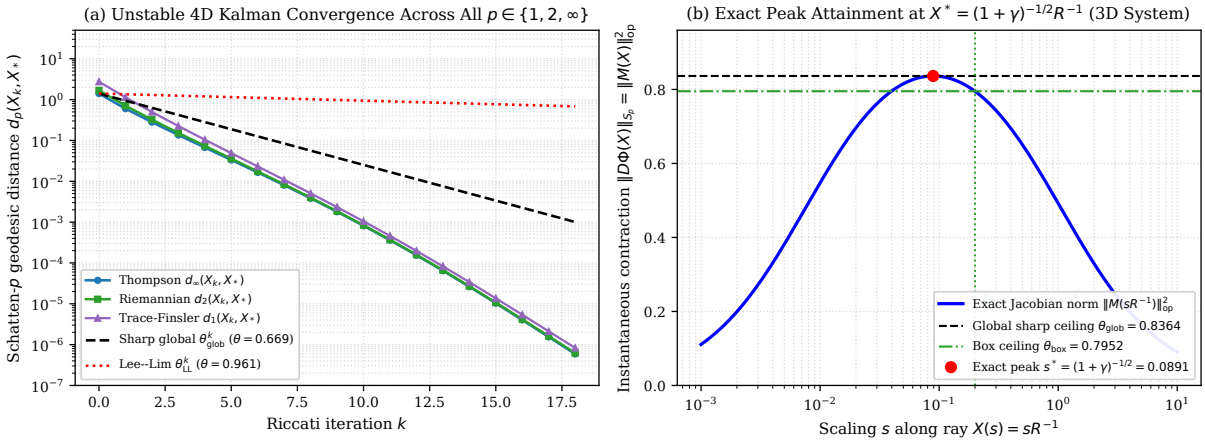


Figure 2: **(a)** Convergence of the unstable 4D Kalman system in Thompson (d_∞), Riemannian (d_2), and trace-Finsler (d_1) metrics compared with our sharp global envelope θ_{glob}^k ($\theta_{\mathrm{glob}} = 0.669$) and classical Lee–Lim θ_{LL}^k ($\theta_{\mathrm{LL}} = 0.961$). **(b)** Exact peak attainment $\|M(sR^{-1})\|_{\mathrm{op}}^2 = \theta_{\mathrm{glob}}(\gamma)$ at $s^* = (1+\gamma)^{-1/2}$ on the 3D commuting system, alongside the Phase-III lower-box entry $s = \delta$.

7 Discussion and Open Directions

We have established the exact global Schatten-p Finsler Lipschitz constant $\theta_{\mathrm{glob}}(\gamma) = (\sqrt{1+\gamma} - 1)/(\sqrt{1+\gamma}+1)$ and three-phase invariant-box contraction law for the discrete algebraic Riccati map across all dimensions $n \geq 1$ and all $p \in [1, \infty]$. Two natural open problems follow directly from our framework:

1. **Semidefinite $Q, R \succeq 0$ multi-step contraction.** Extending the noise-floor splitting $I_n - M(X)M(X)^\top = \Sigma_Q(X) + \Sigma_R(X)$ to observable/stabilizable pairs with singular $Q \succeq 0$ or $R \succeq 0$ via k-step controllability and observability Gramians.
2. **Bures–Wasserstein Riccati contraction constants.** Determining whether the exact Bures–Wasserstein (optimal transport) Lipschitz constant of Φ on $\mathbb{P}_n$ admits an analogous closed-form expression in (γ, δ, μ).

Reproducibility and Open-Source Code

All numerical experiments, tables, figures, and automated tests (4/4 pytest modules) are openly archived under the MIT license at:
https://github.com/lluiseriksson/riccati-schatten-contraction.

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