Removing the binary and ternary factors from cyclic Weyl rank uncertainty

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Abstract

For a nonzero complex matrix in the cyclic Weyl basis, let $u_d(s)$ be the minimum rank under a support budget of s coefficients. We give exact formulas reducing u_d to the profile in the part of the dimension coprime to 6. Removing a factor 2^k uses a minimum over binary subgroup costs. Removing a factor 3^k uses products of the three costs $1, 2, 3$, with rank multipliers $3, 2, 1$. The lower bounds follow from residue-sector compressions and central Fourier blocks; the ternary two-residue case uses a triangular minor on two sectors of a directed three-cycle. Upper bounds are attained by scalar cyclic projectors and binomials, combined with the Chinese remainder theorem. In particular, in every even dimension whose only prime factors are 2 and 3, the exact minimum is $d/\max\{q : q \mid d,\ q \le s\}$. We obtain sharp uniform coefficient-erasure budgets for arbitrary complex low-rank matrices. Residual profiles with prime factors at least 5, efficient recovery, and numerical stability are not determined.

1 Profiles and the main reductions

Put $\omega_d = e^{2\pi i/d}$ and

$$X_d e_j = e_{j+1 \bmod d}, \quad Z_d e_j = \omega_d^j e_j, \quad W_d(a,b) = X_d^a Z_d^b, \qquad (a,b) \in (\mathbb{Z}/d)^2.$$

These d^2 matrices are orthogonal for the Hilbert–Schmidt inner product. For $C \neq 0$, let $s_W(C)$ count the nonzero coefficients in its unique Weyl expansion. For every integer $s \ge 1$, define

$$u_d(s) = \min\{\operatorname{rank} C : C \in M_d(\mathbb{C}),\ C \neq 0,\ s_W(C) \le s\}.$$

The minimum exists: the admissible set is nonempty and its ranks form a nonempty subset of $\{1, \dots, d\}$. Thus no compactness argument is needed. The profile is nonincreasing, $u_d(1) = d$, and $u_d(s) = 1$ for $s \ge d$, by the diagonal rank-one projector. We use $u_1(s) = 1$.

Theorem 1 (Binary transfer)**.** *For odd $m \ge 1$, $k \ge 0$ and $s \ge 1$,*

$$u_{2^k m}(s) = \min_{\substack{0 \le a \le k \\ 2^a \le s}} 2^{k-a} u_m(\lfloor s/2^a \rfloor)\,.$$

For ternary transfer define, for $3 \nmid m$,

$$F_{k,m}(s) = \min_{\substack{r_1,\dots,r_k \in \{1,2,3\} \\ r_1 \cdots r_k \le s}} \left(\prod_{i=1}^{k} (4 - r_i) \right) u_m(\lfloor s/(r_1 \cdots r_k) \rfloor)\,.$$

The empty tuple gives $F_{0,m}(s) = u_m(s)$.

Theorem 2 (Ternary transfer)**.** *For $3 \nmid m$, $m \geq 1$, $k \geq 0$ and $s \geq 1$,*

$$u_{3^k m}(s) = F_{k,m}(s).$$

The unknown residual profiles are retained as functions, not replaced by an assumed uncertainty conjecture. All coefficients are arbitrary complex numbers, and sharpness means a support budget, not exactly s terms.

Corollary 3 (An even residual formula)**.** *Let $h = 2^a 3^b$ with $a \geq 1$, $b \geq 0$, and let $\gcd(m, 6) = 1$. Then*

$$u_{hm}(s) = \min_{\substack{q \mid h \\ q \leq s}} \frac{h}{q}\, u_m(\lfloor s/q \rfloor)\,.$$

In particular, if $d = 2^a 3^b$ is even and $Q_d(s) = \max\{q : q \mid d,\ q \leq s\}$, then

$$u_d(s) = d/Q_d(s).$$

The scalar uncertainty profile in prime-power dimension is classical [2]; Meshulam's subgroup–quotient argument explicitly uses a smallest occupied coset and a product constraint on support sizes [1, Proposition 1.3 and Section 2]. We credit that architecture. The proposed operator contribution is the exact transfer of the rank profile through the noncommuting cyclic representation, using the block identities below. Earlier work in the author's programme covers low prime-power sparsity and a binary seven-term range [3, 4]. The present argument replaces the binary overlap classification and treats the ternary profile in every sparsity range.

Yin studies the closely related Hermitian traceless rank-two problem in all cyclic dimensions, together with phase-retrieval and equality statements [5]. Our profile allows arbitrary complex matrices and all rank bounds, but does not give the stronger positivity, trace, or phase-retrieval classifications of that setting. The reference's formalization claims have not been replayed here. These comparisons are limited readings, not an exhaustive certificate of priority. The cyclic basis is not being identified with a tensor-qubit Pauli basis.

2 Compression identities and small Fourier bounds

Left multiplication by a Weyl unitary translates the support, up to nonzero phases, and preserves rank. Discrete Fourier conjugation interchanges the two label coordinates, up to a sign and phases.

Lemma 4 (Residue sectors)**.** *Let $d = pn$ for $p = 2$ or 3 and $E_j = \operatorname{span}\{e_{j+pu} : u \in \mathbb{Z}/n\}$, $0 \leq j < p$. For $a = pA + \delta$, $0 \leq \delta < p$, put $j' = (j + \delta) \bmod p$ and $\kappa = \lfloor (j + \delta)/p \rfloor$. Then*

$$W_d(a, b)|_{E_j \to E_{j'}} = \omega_d^{bj} X_n^{A+\kappa} Z_n^b.$$

Within a fixed first-coordinate residue class, distinct labels merge precisely in cosets of $\langle (0, n) \rangle$. An occupied class gives a nonzero block somewhere. A coset with r occupied labels activates at least $p - r + 1$ blocks, for $1 \leq r \leq p$.

Proof. The output index on e_{j+pu} is $j' + p(u + A + \kappa)$ modulo d, and the phase is ω_d^{bj+pbu}. This is the displayed formula. The compressed label is $(A \bmod n, b \bmod n)$ after factoring the common carry. In a merging coset, write $b = \beta + n\ell$; its coefficient, after a nonzero common factor, is $\sum_{\ell=0}^{p-1} c_\ell e^{2\pi i \ell j/p}$. Different cosets give different compressed Weyl labels. Thus an occupied class cannot vanish in all blocks, by invertibility of this transform.

For $p = 2$, one input activates both outputs and two inputs activate at least one. For $p = 3$, one input activates three outputs; two inputs cannot have just one output, because the

inverse of a single Fourier spike has all three entries nonzero; three inputs activate at least one. These are precisely the claimed bounds. No assumption on coefficient signs or algebraicity is used. □

Lemma 5 (Central blocks)**.** *Let $d = p^2 n$, $p = 2$ or 3, and suppose the support lies in $p(\mathbb{Z}/d)^2$. The commuting unitaries X_d^{pn}, Z_d^{pn} are central on this support algebra. Their p^2 joint eigenspaces have dimension n, are invariant under C, and carry cyclic Weyl representations. Distinct cosets of $H = \langle (pn, 0), (0, pn) \rangle$ give distinct Weyl labels in every block. Within an H-coset the coefficients transform by ordinary Fourier transform on $(\mathbb{Z}/p)^2$, apart from nonzero scalar factors.*

Proof. Let $\zeta_p = e^{2\pi i/p}$. For $\alpha, \beta \in \{0, \dots, p-1\}$, the joint eigenspace has the orthogonal basis

$$v_t^{\alpha,\beta} = \omega_d^{-p\beta t} \sum_{\ell=0}^{p-1} \zeta_p^{-\beta\ell} e_{\alpha+pt+pn\ell}, \qquad 0 \le t < n.$$

Its vectors have norm $\sqrt{p}$; normalizing gives orthonormal bases. The two eigenvalues are ζ_p^{β} and ζ_p^{α}. The bases exhaust $\mathbb{C}^d$. Direct action, including wrap-around, gives

$$W_d(pA, pB)|_{E_{\alpha,\beta}} = \omega_d^{p\beta A + p\alpha B} X_n^A Z_n^B. \tag{1}$$

The factor $\omega_d^{p^2}$ is primitive of order n, also for arbitrary n. Representatives $0 \le A, B < n$ label the n^2 distinct compressed Weyls.

The labels in one coset are $(pA + pn\epsilon, pB + pn\eta)$, $0 \le \epsilon, \eta < p$. By (1) its compressed coefficient is

$$\omega_d^{p\beta A + p\alpha B} \sum_{\epsilon,\eta=0}^{p-1} c_{\epsilon,\eta} \zeta_p^{\beta\epsilon + \alpha\eta}.$$

This proves both the Fourier assertion and injectivity between cosets. The formula remains valid when $n = 1$. □

Put $D(r) = 2^{\lfloor \log_2 r \rfloor}$ for $r \ge 1$, and define

$$\phi_9(r) = \min_{\substack{u,v \in \{1,2,3\} \\ uv \le r}} (4-u)(4-v), \qquad 1 \le r \le 9.$$

Lemma 6 (Central Fourier support)**.** *A nonzero vector on $(\mathbb{Z}/2)^2$ with r occupied entries has at least $4/D(r)$ nonzero Fourier coefficients. On $(\mathbb{Z}/3)^2$, the corresponding lower bound is $\phi_9(r)$.*

Proof. For four entries, $r = 1$ gives four outputs. For $r = 2, 3$, a single output is impossible because its inverse has all four entries nonzero. For $r = 4$, invertibility suffices. The bound at $r = 3$ is deliberately not sharpened.

For nine entries, group by one coordinate. Let v be the number of occupied fibers and u the smallest fiber support; then $uv \le r$. Transform that fiber first: at least $4 - u$ coordinate frequencies remain. Each such nonzero vector along the other coordinate has support at most v, and hence at least $4 - v$ Fourier outputs by Lemma 4's three-point argument. Thus there are at least $(4-u)(4-v) \ge \phi_9(r)$ outputs. This is the elementary coset argument credited above. □

Lemma 7 (Two shifts on three sectors)**.** *Suppose a matrix on three equal-sized input/output sectors has exactly two possible block shifts. If one shift has L nonzero blocks, each of rank at least ρ, then the whole matrix has rank at least* $\min(L, 2)\rho$.

Proof. Permute output sectors to place the chosen shift on the diagonal. The other shift is a nonzero translation on $\mathbb{Z}/3$, a directed three-cycle. Any two sectors induce at most one off-diagonal direction, so their block submatrix is triangular after an ordering. Select a nonsingular minor of full rank in each of its two diagonal blocks. Together these give a triangular minor with nonzero determinant, even when the full diagonal blocks are singular. If $L = 1$, use that block alone. □

3 Upper bounds and attainable costs

For coprime r, m, the CRT unitary $e_j \mapsto e_{j \bmod r} \otimes e_{j \bmod m}$ sends

$$X_{rm} \mapsto X_r \otimes X_m, \qquad Z_{rm} \mapsto Z_r^{m^{-1} \bmod r} \otimes Z_m^{r^{-1} \bmod m}.$$

Consequently it bijectively identifies Weyl labels with pairs of labels. Tensor products multiply ranks and support sizes. A factor of dimension one is treated directly, without modular-inverse notation.

For $q \mid N$, let

$$P_q = \sum_{\ell=0}^{q-1} Z_N^{\ell N/q}.$$

It equals q on diagonal positions divisible by q, and zero elsewhere. Thus it has support q and rank N/q. This proves the upper bound in Theorem 1 by taking $N = 2^k$, $q = 2^a$ and tensoring a matrix attaining the residual budget.

For ternary transfer, two entries $r_i = r_j = 2$ in the product defining $F_{k,m}$ may be replaced by $3, 1$: cost drops from 4 to 3, the rank multiplier drops from 4 to 3, and the residual budget increases. Hence the minimum is attained with at most one 2. It is the minimum of the following two families, retaining only positive residual budgets:

$$\begin{aligned} &3^{k-a} u_m(\lfloor s/3^a \rfloor), && 0 \le a \le k, \quad 3^a \le s, \\ &2\,3^{k-a-1} u_m(\lfloor s/(2\,3^a) \rfloor), && 0 \le a < k, \quad 2\,3^a \le s. \end{aligned}$$

The first is attained by P_{3^a} on $N = 3^k$. For the second use

$$P_{3^a}\big(Z_N^{N/(3\cdot 3^a)} - \zeta_3 I\big).$$

The two exponent sets are disjoint modulo N, giving exactly $2\,3^a$ terms. Among the $N/3^a$ nonzero positions of P_{3^a}, one third vanish in the second factor. Its rank is $2N/(3 \cdot 3^a)$. CRT and a residual minimizer now prove the upper bound of Theorem 2.

4 Lower bounds by induction

We prove the two transfer theorems separately by induction on k, holding m fixed. It suffices to treat actual support s_0, because both proposed minima are nonincreasing in the budget.

4.1 Binary transfer

The base $k = 0$ is the definition of u_m. If some coordinate has both parities, exchange coordinates if necessary and take the smaller first-coordinate class. Lemma 4 gives a nonzero block of dimension $d/2$ with support at most $\lfloor s_0/2 \rfloor$. Induction bounds its rank by the minimum over indices $1 \le a' \le k$ with $2^{a'} \le s_0$ of $2^{k-a'} u_m(\lfloor s_0/2^{a'} \rfloor)$. This is at least the desired minimum, which additionally permits $a' = 0$.

Otherwise translate the support to even labels. If $k = 1$, $d = 2m$, the two invariant sectors carry

$$W_d(2A, 2B) \mapsto \omega_m^{Bj} X_m^A Z_m^{2B}, \qquad j = 0, 1.$$

Multiplication by 2 is invertible modulo m. The blocks are equivalent: conjugation by X_m^h with $-2h \equiv 1 \pmod m$ changes the first into the second. Thus $\operatorname{rank} C \geq 2u_m(s_0)$, the $a' = 0$ candidate. For $m = 1$ this case has only one label and is immediate. These are two equivalent representations, not two independently variable algebra blocks.

If $k \geq 2$, put $n = d/4$ and apply Lemma 5. Let t be the number of occupied central cosets and r their minimum occupancy, so $rt \leq s_0$ and $1 \leq r \leq 4$. At least $4/2^\ell$ blocks are nonzero, where $\ell = \lfloor \log_2 r \rfloor$. Each has support at most t. Its rank is at least $u_n(t)$, and ranks add over invariant blocks. For every index a in the inductive formula for $u_n(t)$, put $a' = a + \ell$. Then

$$a' \leq k, \qquad 2^{a'} \leq rt \leq s_0, \qquad \lfloor t/2^a \rfloor \leq \left\lfloor s_0/2^{a'} \right\rfloor.$$

Its scaled rank candidate is therefore at least $2^{k-a'} u_m(\left\lfloor s_0/2^{a'} \right\rfloor)$, and hence at least the desired minimum. All arguments of the profiles are positive. Their definition for every positive budget handles $t \geq n$ without an out-of-domain induction. This proves the lower bound of Theorem 1.

4.2 Ternary transfer

The base is again $k = 0$. Suppose some coordinate varies modulo 3, and make it the first. If all three first-coordinate residues occur, a smallest class has at most $\lfloor s_0/3 \rfloor$ terms and supplies a nonzero block. Thus

$$\operatorname{rank} C \geq F_{k-1,m}(\lfloor s_0/3 \rfloor) \geq F_{k,m}(s_0).$$

The last inequality follows by prepending $r = 3$ to each eligible tuple.

If exactly two residues occur, choose a smaller class of size $M \leq \lfloor s_0/2 \rfloor$. Let it occupy t cosets of $\langle (0, d/3) \rangle$, and let $r \leq 3$ be the smallest coset size. Then $rt \leq M$. At least $4 - r$ blocks are nonzero, each of rank at least $u_{d/3}(t)$. By Lemma 7, $r = 1$ or 2 gives

$$\operatorname{rank} C \geq 2F_{k-1,m}(\lfloor s_0/2 \rfloor) \geq F_{k,m}(s_0),$$

by monotonicity and prepending $r = 2$. If $r = 3$, necessarily $s_0 \geq 6$, and a single nonzero block yields

$$\operatorname{rank} C \geq F_{k-1,m}(\lfloor s_0/6 \rfloor) \geq F_{k-1,m}(\lfloor s_0/3 \rfloor) \geq F_{k,m}(s_0).$$

Thus neither existence nor rank additivity for arbitrary blocks is assumed.

It remains that both coordinates have fixed residues. Translate to $3G$. When $k = 1$, $3 \nmid m$, the three invariant sectors carry

$$W_d(3A, 3B) \mapsto \omega_m^{Bj} X_m^A Z_m^{3B}, \qquad j = 0, 1, 2.$$

These have distinct labels, and are conjugate by X_m^h with $-3h \equiv j \pmod m$. Their equal ranks sum to at least $3u_m(s_0)$, the all-$r_i = 1$ candidate. The case $m = 1$ is immediate.

For $k \geq 2$, put $n = d/9$. Lemmas 5 and 6 give nine invariant blocks. If t cosets are occupied and r is their minimum occupancy, at least $\phi_9(r)$ blocks are nonzero. Thus

$$\operatorname{rank} C \geq \phi_9(r) F_{k-2,m}(t), \qquad rt \leq s_0.$$

Choose u, v attaining $\phi_9(r)$, and a tuple attaining $F_{k-2,m}(t)$ with cost $w \leq t$. The concatenated tuple is eligible for $F_{k,m}(s_0)$, because $uv \leq r$ and $uvw \leq s_0$. Also $\lfloor t/w \rfloor \leq \lfloor s_0/(uvw) \rfloor$. Monotonicity of u_m makes that candidate at most $\phi_9(r) F_{k-2,m}(t)$, proving the required bound. This completes Theorem 2 in its entire stated domain.

5 Removing both factors and the divisor law

Proof of Corollary 3. Apply Theorem 1 with odd residual $3^b m$, then Theorem 2. Floor composition is exact for positive integer costs. Candidates without a ternary cost 2 are

$$2^{a-i}3^{b-j}u_m(\lfloor s/(2^i3^j)\rfloor), \qquad 0 \le i \le a,\ 0 \le j \le b.$$

These are precisely the divisor candidates in the corollary. A candidate with a ternary cost 2 has cost $2^{i+1}3^j$ and multiplier $2^{a-i+1}3^{b-j-1}$, with $j < b$.

If $i < a$, replace it by the divisor candidate at cost $2^{i+1}3^j$: its multiplier is $2^{a-i-1}3^{b-j}$, smaller by a factor $3/4$, and its residual budget is unchanged. If $i = a$, use instead the divisor cost $q' = 2^{a-1}3^{j+1}$, which is $3/4$ of the former cost. Its multiplier is the same, $2\,3^{b-j-1}$, and its residual budget is larger. Thus every additional candidate is dominated. For $m = 1$ the residual profile is constant one, and minimizing d/q chooses the largest eligible divisor. □

For a purely ternary dimension $d = 3^k$, the same proof also gives

$$u_d(s) = (4-c)3^{k-t-1} \quad \text{when} \quad c3^t \le s < (c+1)3^t, \quad c \in \{1,2\},\ s < d,$$

and $u_d(s) = 1$ for $s \ge d$. For a purely binary dimension it gives $d/2^{\min(k,\lfloor \log_2 s \rfloor)}$. The scalar profiles are recovered as operator profiles here; no scalar theorem is silently used to justify the noncommuting lower bound.

d	budget s	product bound $\lceil d/s \rceil$	exact $u_d(s)$	subgroup cost
54	8	7	9	6
72	11	7	8	9
108	8	14	18	6
108	17	7	9	12

The elementary product bound follows from $\|C\|_F^2 = d\sum |c_g|^2$ and $\|C\|_{\text{op}}^2 \le s\sum |c_g|^2$. The divisor law is specific to the stated dimensions. For example, in dimension 10, a degree-three polynomial in Z_{10}^2 with three distinct fifth roots as zeros has at most four Weyl terms and rank four. This is below $10/\max\{q \mid 10 : q \le 4\} = 5$. There is no claim of a divisor-step law for all composite dimensions.

6 Sharp erasure budgets

Let d be even with only prime factors $2, 3$. For an integer $R \in \{1, \dots, d\}$, set

$$q_d(R) = \min\{q : q \mid d,\ qR \ge d\}.$$

Corollary 8. *The minimum support of a nonzero complex matrix of rank at most R is $q_d(R)$. On the set of all complex matrices of rank at most R, the largest universally safe number of omitted Weyl coefficients is $q_d(2R) - 1$ if $2R < d$, and zero if $2R \ge d$.*

Proof. The first statement is the inverse of the divisor profile; the witness $P_{q_d(R)}$ attains it. If two matrices of rank at most R have the same retained coefficients, their difference has rank at most $2R$ and support contained in the omissions. A nonzero difference needs at least $q_d(2R)$ coefficients.

For failure at that number, use P_q, $q = q_d(2R)$, whose rank is $d/q \le 2R$. Split its nonzero diagonal positions into two groups of size at most R. Put $A = P_q$ on the first group and $B = -P_q$ on the second, zero elsewhere. Then $A - B = P_q$, while both ranks are at most R. If $2R \ge d$, the same split of I_d shows failure on erasing its single Weyl label. No omissions retains a basis and is injective. □

For $d = 108$ and $R = 4$, the sharp budget is 17, with failure at 18. The product bound alone guarantees 13. This is exact noiseless identifiability of arbitrary complex matrices. It does not establish a recovery algorithm, conditioning estimate, or a positivity-constrained density-matrix threshold.

7 Checks, dependencies and status

The proof is analytic for all stated dimensions. Exact auxiliary checks compare root-of-unity monomial exponents, CRT label maps and the integer minimum identities. Finite checks do not establish the universal quantifiers. In particular, test residual functions used for minimum bookkeeping are explicitly artificial monotone functions, not asserted physical Weyl rank profiles.

The binary and ternary transfer arguments have received dependent mathematical audits. A full assessment of draft 1 requested two domain clarifications: integer rank budgets in Corollary 8, and admissible positive budgets in the binary induction. Both are explicit in this revision. That assessment and its findings are preserved separately; the revised artifact has not yet received its own assessment. An assessment of the earlier binary-only PDF does not apply to this text.

The residual profile for dimensions with prime factors at least 5 is not determined in general. The two-shift proof uses the special directed three-cycle on the sector labels; it is not asserted for an arbitrary prime number of sectors. All sharpness assertions are budget statements, not classifications of exact-support minimizers or of all equality cases.

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