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THE HURWITZ STABILITY DICHOTOMY OF IMMANANT POLYNOMIALS OVER THE HERMITIAN POSITIVE SEMIDEFINITE CONE

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Abstract. For a partition $\lambda \vdash n$ and an $n \times n$ Hermitian positive semidefinite matrix $M \succeq 0$, we study the multivariate immanant polynomial $P_\lambda(X; M) := \mathrm{imm}_\lambda(X+M)$ on the open right poly-half-plane $\mathcal{H}_n^+ := \{\mathrm{diag}(x_1, \ldots, x_n) : \mathrm{Re}(x_k) > 0 \ \forall k\}$ and its univariate specialization $X = xI_n$. While classical results of Schur (1918) and Lieb (1966) establish pointwise inequalities on the Hermitian positive semidefinite cone, the location of complex zeros of $\mathrm{imm}_\lambda(X+M)$ exhibits a sharp structural dichotomy governed by two partition invariants: the non-first-column box count $b(\lambda) := |\lambda| - \ell(\lambda)$ and the first row length λ_1.

We prove four main theorems. First, at $b(\lambda) \le 1$, $\mathrm{imm}_{(2,1^{n-2})}(X+M)$ is universally multivariate Hurwitz stable over $\mathcal{H}_n^+$, satisfying the sharp half-plane bound $\mathrm{Re}(\mathrm{imm}_{(2,1^{n-2})}(X+M)/\det(X+M)) > \frac{3}{4}n - 1$ saturated as $y \to 0^+$ along $X = (y^3 + iy)I_{2k}$ on $M = J_2^{\oplus k}$, together with a univariate quadrant Kantorovich refinement. Second, in small dimensions, $\mathrm{per}(X+M)/\mathrm{imm}_{(2,1)}(X+M)$ is universally multivariate positive-real at $n = 3$, and both $\mathrm{per}(xI_4 + R)$ and $\mathrm{imm}_{(2,2)}(xI_4 + R)$ are Hurwitz stable for every 4×4 correlation matrix $R \succeq 0$. Third, for rank-one perturbations $M = vv^*$, we derive an exact Frobenius–Pieri master formula, prove via an explicit four-column Vandermonde–Frobenius certificate and Newton's inequalities that universal rank-one stability across all shapes of width λ_1 holds if and only if $\lambda_1 \le 4$, obtain an explicit Laguerre integral formula for rectangular partitions, establish universal rank-one positive-real separation at $b(\lambda) = 2$, and resolve the complete hook trichotomy on J_n ($r \le 4$ always stable; $r = 5$ stable iff $n \ge 7$; $r \ge 6$ always unstable via Hermite–Biehler and Newton's real-root discriminant). Fourth, even at small fixed first row $\lambda_1 \in \{2, 3, 4\}$, multi-block PSD matrices destabilize: for $M = J_2^{\oplus k}$, $\mathrm{imm}_{(2^k)}(xI_{2k} + J_2^{\oplus k})$ is a shifted Catalan polynomial that is Hurwitz stable if and only if $k \le 4$, failing via $\Delta_4 < 0$ for every $k \ge 5$, and explicit negative Hurwitz certificates destabilize $(2^{15}, 1, 1)$, (3^3), and (4^2). We conclude with exact symbolic/rational verification and two open conjectures on $b = 2$ multivariate separation and the $b \le 3$ stability boundary.

1. Introduction and Main Dichotomy

Let S_n be the symmetric group on $[n] := \{1, \ldots, n\}$. For an integer partition $\lambda \vdash n$ with irreducible character $\chi_\lambda : S_n \to \mathbb{Z}$ and degree $f^\lambda := \chi_\lambda(\mathrm{id})$, the λ-*immanant* of an $n \times n$ complex matrix $A = (a_{ij})$ is

$$\mathrm{imm}_\lambda(A) := \sum_{\sigma \in S_n} \chi_\lambda(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)},$$

introduced by Littlewood [9] to interpolate between the determinant ($\lambda = (1^n)$) and the permanent ($\lambda = (n)$). Over the cone of $n \times n$ Hermitian positive semidefinite (PSD) matrices $M \succeq 0$, Schur [12] proved the fundamental inequality $\mathrm{imm}_\lambda(M) \ge f^\lambda \det(M) \ge 0$, later refined by Lieb's permanental inequality [7], Heyfron's hook dominance theorem [4], and Stembridge's positivity theory [14].

In contrast to real positivity on $M \succeq 0$, the complex zero locus of the *immanant polynomial* $P_\lambda(X; M) := \mathrm{imm}_\lambda(X + M)$ for $X = \mathrm{diag}(x_1, \ldots, x_n) \in \mathbb{C}^n$ on the open right poly-half-plane $\mathcal{H}_n^+ := \{\mathrm{diag}(x_1, \ldots, x_n) : \mathrm{Re}(x_k) > 0 \text{ for all } k \in [n]\}$ (and its univariate diagonal restriction $P_\lambda(x; M) :=$

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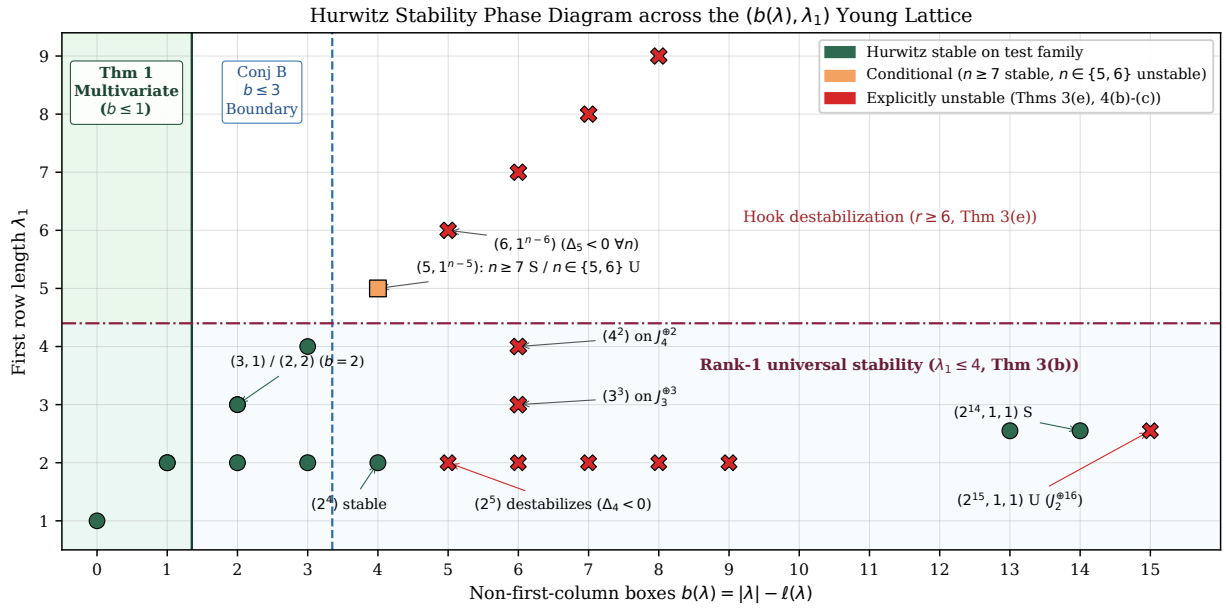


FIGURE 1. Hurwitz stability phase diagram across the $(b(\lambda), \lambda_1)$ Young lattice. Universal multivariate stability holds at $b(\lambda) \le 1$ (Theorem 1.1); rank-one stability across all widths holds iff $\lambda_1 \le 4$ (Theorem 1.3); multi-block matrices $J_r^{\oplus k}$ destabilize (2^k) at $k \ge 5$, $(2^{15}, 1, 1)$ on $J_2^{\oplus 16}$ (plotted with $(2^{13}, 1, 1)$, $(2^{14}, 1, 1)$ slightly vertically offset above $\lambda_1 = 2$ for readability), (3^3), and (4^2) (Theorem 1.4).

$\mathrm{imm}_\lambda(xI_n + M)$) is far subtler. A polynomial $P(x_1, \ldots, x_n)$ is *multivariate Hurwitz stable* (or possesses the *half-plane property* [2, 1]) if $P(X) \neq 0$ for all $X \in \mathcal{H}_n^+$; a univariate polynomial $P(x)$ is *Hurwitz stable* (resp. *strictly Hurwitz stable*) if all its roots satisfy $\mathrm{Re}(x) \le 0$ (resp. $\mathrm{Re}(x) < 0$).

For $\lambda = (1^n)$, $P_{(1^n)}(X; M) = \det(X + M)$ is trivially non-vanishing on $\mathcal{H}_n^+$ because $\mathrm{Re}(X + M) = \mathrm{Re}(X) + M \succeq \mathrm{Re}(X) \succ 0$. At the opposite extreme $\lambda = (n)$, the permanent $\mathrm{per}(xI_5 + J_5)$ of the 5×5 all-ones matrix J_5 already has roots in $\mathrm{Re}(x) > 0$ ($\max \mathrm{Re}(x) \approx 0.2398$). This paper identifies the exact structural boundary separating universal Hurwitz stability from complex-plane destabilization across the Young lattice.

The governing coordinates are the *first row length* λ_1 and the *non-first-column box count*

$$b(\lambda) := |\lambda| - \ell(\lambda) = |\lambda| - \lambda_1' = \sum_{i=1}^{\ell(\lambda)} (\lambda_i - 1),$$

where $\ell(\lambda)$ denotes the number of nonzero parts of λ and λ' is the conjugate partition. Notice that $b((1^n)) = 0$, $b((2, 1^{n-2})) = 1$, and every partition λ satisfies $\lambda_1 \le b(\lambda) + 1$. Figure 1 summarizes the complete phase diagram established by Theorems 1.1–1.4.

Main Results. Our first theorem resolves the $b(\lambda) \le 1$ regime completely for arbitrary $n \ge 2$ and arbitrary PSD $M \succeq 0$.

Theorem 1.1 (Universal Multivariate Hurwitz Stability and Sharp Half-Plane Law at $b(\lambda) \le 1$)**.** *Let $n \ge 2$, let $M \succeq 0$ be any $n \times n$ Hermitian positive semidefinite matrix, and let $X = \mathrm{diag}(x_1, \ldots, x_n) \in \mathcal{H}_n^+$. Set $B := X + M$ and $R := B^{-1}$.*

(a) (Schur-Complement Disk Bound) *For every $k \in [n]$,*

$$\mathrm{Re}(B_{kk} R_{kk}) > \frac{3}{4}.$$

(b) (Universal Positive-Real Ratio) *The subdominant hook immanant satisfies*

$$\mathrm{Re}\left(\frac{\mathrm{imm}_{(2,1^{n-2})}(X+M)}{\det(X+M)}\right) = \mathrm{Re}\left(\sum_{k=1}^{n} B_{kk}R_{kk} - 1\right) > \frac{3}{4}n - 1 \geq \frac{1}{2}.$$

In particular, $\mathrm{imm}_{(2,1^{n-2})}(X+M)$ *is universally multivariate Hurwitz stable over* $\mathcal{H}_n^+$.

(c) (Sharpness) *For every even* $n = 2k \geq 2$*, the bound in* (a) *and* (b) *is sharp: taking* $M = J_2^{\oplus k}$ *(where* $J_m = \mathbf{1}_m\mathbf{1}_m^*$*) and* $X = (y^3 + iy)I_{2k}$ *with* $y > 0$*, one has* $\lim_{y\to 0^+} \mathrm{Re}(B_{kk}R_{kk}) = \frac{3}{4}$ *and hence*

$$\lim_{y\to 0^+} \mathrm{Re}\left(\frac{\mathrm{imm}_{(2,1^{n-2})}(X + J_2^{\oplus k})}{\det(X + J_2^{\oplus k})}\right) = \frac{3}{4}n - 1.$$

(d) (Quadrant Kantorovich Refinement) *For univariate* $X = xI_n$ *with* $\mathrm{Re}(x) > 0$*, if* $M = U\Lambda U^*$ *is a spectral decomposition with* $p_{ka} := |U_{ka}|^2$*, then for every* $k \in [n]$*,*

$$\mathrm{Re}(B_{kk}R_{kk}) \geq \frac{1}{2} + \frac{1}{2}\sum_{a=1}^{n} p_{ka}^2 \geq \frac{1}{2} + \frac{1}{2n}.$$

At small dimensions $n \leq 4$, stability extends beyond $b(\lambda) = 1$, including full multivariate positive-real separation at $n = 3$ and equal-diagonal quartet stability at $n = 4$.

Theorem 1.2 (Small-n Positive-Real Separation and Equal-Diagonal Quartet Stability)**.**

(a) (Complete $n = 3$ Multivariate Positive-Real Separation) *For every* 3×3 *Hermitian PSD matrix* $M \succeq 0$ *and every* $X = \mathrm{diag}(x_1, x_2, x_3) \in \mathcal{H}_3^+$*,*

$$\mathrm{Re}\left(\frac{\mathrm{per}(X+M)}{\mathrm{imm}_{(2,1)}(X+M)}\right) > 0.$$

In particular, $\mathrm{per}(X+M)$ *is universally multivariate Hurwitz stable over* $\mathcal{H}_3^+$.

(b) (Exact Equal-Diagonal $n = 4$ Quartet Stability) *Let* $R \succeq 0$ *be any* 4×4 *Hermitian correlation matrix (*$R_{kk} = 1$ *for* $k = 1, \ldots, 4$*). Then all Hurwitz minors of* $\mathrm{per}(xI_4 + R)$ *are strictly positive (*$\Delta_3(\mathrm{per}(xI_4 + R)) \geq 64$*), so* $\mathrm{per}(xI_4 + R)$ *is strictly Hurwitz stable. Furthermore,* $\frac{1}{2}\mathrm{imm}_{(2,2)}(xI_4 + R)$ *has* $\Delta_1 = 4$*,* $\Delta_2 = 20 + \frac{1}{2}C_3 > 0$*,* $\Delta_3 = \frac{1}{4}(256 - C_3^2 - 64S_{2,2}) \geq 0$*, and all roots in the closed left half-plane* $\mathrm{Re}(x) \leq 0$*.*

Restricting $M = vv^*$ to the rank-one cone yields an exact algebraic theory that characterizes first-row stability, connects rectangular immanants to Laguerre moments, proves universal $b = 2$ rank-one separation, and settles every hook partition on J_n.

Theorem 1.3 (Rank-One Master Formula, $\lambda_1 \leq 4$ Threshold, Laguerre Law, $b = 2$ Separation, and Hook Trichotomy)**.** *Let* $v \in \mathbb{C}^n$ *with* $w_i := |v_i|^2 \geq 0$ *for* $i \in [n]$*, and let* $\lambda \vdash n$*.*

(a) (Multivariate Rank-One Master Formula) *For every* $X = \mathrm{diag}(x_1, \ldots, x_n)$*,*

$$\mathrm{imm}_\lambda(X + vv^*) = \sum_{k=0}^{\lambda_1} c_k(\lambda) \sum_{\substack{S\subseteq[n]\\|S|=k}} \left(\prod_{j\in S} w_j\right)\left(\prod_{i\notin S} x_i\right), \qquad c_k(\lambda) := k! \sum_{\substack{\mu\vdash n-k\\ \lambda/\mu\in\mathcal{H}_k}} f^\mu,$$

where $\lambda/\mu \in \mathcal{H}_k$ *means that* λ/μ *is a horizontal* k*-strip.*

(b) (Width $\lambda_1 \leq 4$ Threshold) *For every* $n \geq 1$*, every partition* $\lambda \vdash n$ *with* $\lambda_1 \leq 4$*, and every* $v \in \mathbb{C}^n$*, the univariate polynomial* $\mathrm{imm}_\lambda(xI_n + vv^*)$ *has all roots in* $\mathrm{Re}(x) \leq 0$ *(and strictly in* $\mathrm{Re}(x) < 0$ *outside zero roots when* $k_0 := |\{i : w_i > 0\}| \geq 1$ *is accounted for). Conversely, for every* $r \geq 5$*,* $\lambda = (r)$ *fails Hurwitz stability on* J_r*; hence universal rank-one Hurwitz stability over all shapes of width* $\lambda_1 = r$ *holds if and only if* $r \leq 4$*.*

(c) (Exact Rank-One Rectangle Laguerre Law) *For any rectangular partition* $\lambda = (w^k) \vdash n = wk$,

$$\operatorname{imm}_{(w^k)}(xI_{wk} + J_{wk}) = f^{(w^k)} x^{wk-w} \sum_{j=0}^{w} \binom{w}{j} (k)_j \, x^{w-j} = \frac{f^{(w^k)} x^{wk-w}}{\Gamma(k)} \int_0^\infty e^{-t} t^{k-1} (x+t)^w \, dt,$$

where $(k)_j := k(k+1)\cdots(k+j-1)$ *is the rising factorial.*

(d) (Rank-One $b = 2$ Positive-Real Law) *For every* $n \geq 4$ *and every* $v \in \mathbb{C}^n$ *with* $e_2(w) > 0$, *both ratios*

$$\frac{\operatorname{imm}_{(3,1^{n-3})}(xI_n + vv^*)}{\operatorname{imm}_{(2,1^{n-2})}(xI_n + vv^*)} \qquad \text{and} \qquad \frac{\operatorname{imm}_{(2,2,1^{n-4})}(xI_n + vv^*)}{\operatorname{imm}_{(2,1^{n-2})}(xI_n + vv^*)}$$

are strictly positive-real on $\operatorname{Re}(x) > 0$ *(and equal to the positive constant* $\frac{n-2}{2}$ *or* $\frac{n(n-3)}{2(n-1)}$ *when* $e_2(w) = 0$*).*

(e) (Complete Hook Trichotomy on J_n) *For every hook* $\lambda = (r, 1^{n-r})$ *with* $1 \leq r \leq n$,

$$\operatorname{imm}_{(r,1^{n-r})}(xI_n + J_n) = \binom{n-1}{r-1} x^{n-r} Q_{r,n}(x), \qquad Q_{r,n}(x) := x^r + n(r-1)! \sum_{m=0}^{r-1} \frac{x^m}{m!}.$$

The factor $Q_{r,n}(x)$ *obeys an exact trichotomy:*

(i) For $r \leq 4$*:* $Q_{r,n}$ *is strictly Hurwitz stable for all* $n \geq r$.

(ii) For $r = 5$*:* $Q_{5,n}$ *is unstable for* $n \in \{5, 6\}$ *and strictly Hurwitz stable for all* $n \geq 7$.

(iii) For $r \geq 6$*:* $Q_{r,n}$ *has at least one root in* $\operatorname{Re}(x) > 0$ *for every* $n \geq r$.

Crucially, bounded first row $\lambda_1 \leq 4$ does *not* suffice for multi-block PSD matrices $M \succeq 0$. Our fourth theorem exhibits the exact multi-block Catalan–Chebyshev mechanism through which small-width shapes destabilize as soon as $b(\lambda) \geq 5$.

Theorem 1.4 (Multi-Block Catalan–Chebyshev Destabilization)**.**

(a) (Two-Column Generating Function on $J_2^{\oplus k}$) *For all* $k \geq 1$ *and* $0 \leq r \leq k$, *the immanant* $\operatorname{imm}_{(2^r,1^{2k-2r})}(xI_{2k} + J_2^{\oplus k})$ *equals the coefficient* $[z^r]$ *of* z^r *in*

$$G_k(x, z) := (1-z)\Big(x(x+2)(1+z)^2 + 2z\Big)^k.$$

(b) (Exact Catalan Dichotomy at $\lambda_1 = 2$) *Specializing to* $r = k$ *gives the Catalan–Chebyshev polynomial*

$$P_{(2^k)}(x; J_2^{\oplus k}) = \sum_{m=0}^{k} \binom{k}{m} 2^{k-m} C_m \, x^m (x+2)^m, \qquad C_m := \frac{1}{m+1}\binom{2m}{m},$$

which is strictly Hurwitz stable if and only if $k \leq 4$. *For* $k \in \{1, 2, 3, 4\}$, *all* $2k$ *Hurwitz minors of* $P_{(2^k)}$ *are strictly positive. For every* $k \geq 5$, *the reciprocal polynomial* $R_k(t) := t^{2k} P_{(2^k)}(1/t; J_2^{\oplus k})$ *has strictly negative* 4×4 *Hurwitz determinant*

$$\Delta_4(R_k) = -\frac{2^{4k} k^2 (k-1)}{302400} g_7(k) < 0,$$

where $g_7(k) := 4654k^7 - 10136k^6 - 79166k^5 - 16796k^4 + 539221k^3 + 402511k^2 - 2051184k + 806796$ *satisfies* $g_7(5+u) > 0$ *for all* $u \geq 0$*; moreover* $\Delta_3(R_k) < 0$ *for all* $k \geq 7$.

(c) (Explicit Negative Integer Certificates at $\lambda_1 \in \{2, 3, 4\}$) *The polynomials* $\operatorname{imm}_{(2^{15},1,1)}(xI_{32} + J_2^{\oplus 16})$ *(after factoring out* x*),* $\operatorname{imm}_{(3^3)}(xI_9 + J_3^{\oplus 3})$, *and* $\operatorname{imm}_{(4^2)}(xI_8 + J_4^{\oplus 2})$ *all possess roots in* $\operatorname{Re}(x) > 0$, *certified by strictly negative exact integer Hurwitz determinants* Δ_{31}, Δ_9, *and* Δ_8.

The remainder of the paper is organized as follows. Section 2 establishes the representation-theoretic and polynomial-stability preliminaries. Sections 3–6 contain complete, self-contained proofs of Theorems 1.1–1.4. Section 7 summarizes the exact symbolic/rational verification suite and formulates Conjectures 7.1 and 7.2.

2. Preliminaries and Representation-Theoretic Reductions

We collect the symmetric-group character identities and complex-polynomial criteria invoked throughout the proofs. For a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0) \vdash n$, its conjugate λ' has parts $\lambda'_j = |\{i : \lambda_i \geq j\}|$, so $\ell(\lambda) = \lambda'_1$ and $b(\lambda) = n - \lambda'_1$. The degree $f^\lambda = \chi_\lambda(\mathrm{id})$ is given by the Frame–Robinson–Thrall hook-length formula [3]

$$f^\lambda = \frac{n!}{\prod_{(i,j)\in\lambda} h_\lambda(i,j)}, \qquad h_\lambda(i,j) := (\lambda_i - j) + (\lambda'_j - i) + 1. \tag{1}$$

Tensoring with the sign representation conjugates the partition: $\chi_{\lambda'}(\sigma) = \mathrm{sgn}(\sigma)\chi_\lambda(\sigma)$ for all $\sigma \in S_n$ [10]. Moreover, for any Hermitian matrix $M = M^*$, conjugating every permutation σ to σ^{-1} yields $\overline{\prod_i M_{i,\sigma(i)}} = \prod_i M_{i,\sigma^{-1}(i)}$; since $\chi_\lambda(\sigma^{-1}) = \chi_\lambda(\sigma) \in \mathbb{Z}$, the polynomial $\mathrm{imm}_\lambda(xI_n + M)$ always has real coefficients.

Let $p(x) = a_0x^N + a_1x^{N-1} + \cdots + a_N \in \mathbb{R}[x]$ with $a_0 > 0$. Its $N \times N$ *Hurwitz matrix* $H(p)$ has entries $H(p)_{ij} = a_{2j-i}$ (with $1 \leq i, j \leq N$ and $a_m := 0$ outside $0 \leq m \leq N$), and its leading principal minors are $\Delta_m(p) := \det H(p)_{[m],[m]}$ for $1 \leq m \leq N$ [6, 11]. By the Routh–Hurwitz theorem, p is strictly Hurwitz stable if and only if $\Delta_m(p) > 0$ for all $1 \leq m \leq N$. Under Liénard–Chipart [8], if $a_m > 0$ for all $m \geq 0$, then strict Hurwitz stability is equivalent to $\Delta_m(p) > 0$ for odd m (or equivalently even m). When $a_N > 0$, the reciprocal polynomial $R(x) := x^N p(1/x)$ has roots $\{1/\alpha : p(\alpha) = 0\}$; since $z \mapsto 1/z$ preserves the open left half-plane, p is strictly Hurwitz stable if and only if R is strictly Hurwitz stable.

For boundary arguments, we repeatedly combine two classical complex-analytic principles:

Lemma 2.1 (Hurwitz Continuity and Hermite–Biehler Criterion [11])**.**

(i) (Root Continuity) *If a sequence of monic polynomials $p_\varepsilon \to p$ coefficient-wise and every p_ε has all roots in $\mathrm{Re}(x) < 0$, then every root of p satisfies $\mathrm{Re}(x) \leq 0$.*

(ii) (Hermite–Biehler) *Write $p(iy) = U(y) + iV(y)$ for $y \in \mathbb{R}$ with real polynomials U, V. Suppose $p(x)$ has all roots in $\mathrm{Re}(x) \leq 0$ and $p(0) \neq 0$. Writing $p(x) = \widetilde{W}(x)\widetilde{p}(x)$ where $\widetilde{W}(x) = \prod_{j=1}^s (x^2 + y_j^2)$ $(y_j > 0)$ gathers all imaginary-axis roots and $\widetilde{p}(x)$ is strictly Hurwitz stable, we have $U(y) = W(y)\widetilde{U}(y)$ and $V(y) = W(y)\widetilde{V}(y)$ where $W(y) = \prod_{j=1}^s (y_j^2 - y^2)$ and both $\widetilde{U}, \widetilde{V}$ have only real simple roots. Consequently, every root of $U(y)$ and every root of $V(y)$ must be real.*

3. Universal Multivariate Stability at $b(\lambda) \leq 1$

We prove Theorem 1.1 via a Schur-complement reduction on $\mathrm{range}(M_k) \oplus \ker(M_k)$, the operator Cayley equivalence, and an exact rational sharpness family.

Proof of Theorem 1.1. **Part (a).** Fix $k \in [n]$. Without loss of generality (by permuting indices), take $k = 1$ and partition $B = X + M$ as

$$B = \begin{pmatrix} x_1 + m_{11} & v^* \\ v & X_1 + M_1 \end{pmatrix}, \qquad M = \begin{pmatrix} m_{11} & v^* \\ v & M_1 \end{pmatrix} \succeq 0,$$

where $X_1 := \mathrm{diag}(x_2, \ldots, x_n)$ satisfies $\mathrm{Re}(X_1) \succ 0$ and $M_1 \succeq 0$. Notice that $B_1 := X_1 + M_1$ is strictly accretive ($\mathrm{Re}(B_1) = \mathrm{Re}(X_1) + M_1 \succ 0$), hence invertible, and by block inversion the $(1,1)$-entry of $R = B^{-1}$ is $R_{11} = \left(B_{11} - v^* B_1^{-1} v\right)^{-1}$.

If $m_{11} = 0$, then $M \succeq 0$ forces $v = 0$, so $B_{11}R_{11} = 1$ and $\mathrm{Re}(B_{11}R_{11}) = 1 > \frac{3}{4}$. Assume henceforth that $m_{11} > 0$. Because $M \succeq 0$, the vector v lies in $\mathrm{range}(M_1)$ and satisfies $m_{11} - v^* M_1^+ v \geq 0$, where M_1^+ is the Moore–Penrose pseudoinverse of M_1 [5]. Let $\mathbb{C}^{n-1} = \mathrm{range}(M_1) \oplus \ker(M_1)$ be the orthogonal decomposition along M_1; in an adapted orthonormal basis, $M_1 = M_{11} \oplus 0$ with $M_{11} \succ 0$ on $\mathrm{range}(M_1)$, while $X_1 = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}$ has $\mathrm{Re}(X_1) \succ 0$ and thus $\mathrm{Re}(X_{22}) \succ 0$.

By block inversion of $X_1 + (M_{11} \oplus 0)$, the upper-left $(\mathrm{range}(M_1) \times \mathrm{range}(M_1))$ block of B_1^{-1} is $(M_{11} + \widetilde{X}_1)^{-1}$, where

$$\widetilde{X}_1 := X_{11} - X_{12} X_{22}^{-1} X_{21}.$$

We claim that $\mathrm{Re}(\widetilde{X}_1) \succ 0$ on $\mathrm{range}(M_1)$. Indeed, for any nonzero $u_1 \in \mathrm{range}(M_1)$, set $u_2 := -X_{22}^{-1} X_{21} u_1$ and $u := (u_1, u_2)^T \neq 0$. Because $X_{21}u_1 + X_{22}u_2 = 0$, we have

$$u^* X_1 u = u_1^*(X_{11}u_1 + X_{12}u_2) + u_2^*(X_{21}u_1 + X_{22}u_2) = u_1^* \widetilde{X}_1 u_1,$$

and taking real parts gives $\mathrm{Re}(u_1^* \widetilde{X}_1 u_1) = u^* \,\mathrm{Re}(X_1) u > 0$.

Write $v_1 := M_{11}^{-1/2} v|_{\mathrm{range}(M_1)}$, so $\|v_1\|_2^2 = v^* M_1^+ v \leq m_{11}$. If $v_1 = 0$, then $v = 0$ and $B_{11}R_{11} = 1$. Assume $v_1 \neq 0$ and put $\alpha := \|v_1\|_2^2 / m_{11} \in (0, 1]$. Defining $T := I + M_{11}^{-1/2} \widetilde{X}_1 M_{11}^{-1/2}$ on $\mathrm{range}(M_1)$, we have $\mathrm{Re}(T) = I + M_{11}^{-1/2} \,\mathrm{Re}(\widetilde{X}_1) M_{11}^{-1/2} \succ I$. By the matrix Cayley equivalence, $\mathrm{Re}(T) \succ I$ holds if and only if

$$0 \prec T^* + T - 2I \iff 2Z^*Z \prec Z^* + Z \iff \left(Z - \tfrac{1}{2}I\right)^* \left(Z - \tfrac{1}{2}I\right) \prec \tfrac{1}{4}I \iff \left\|Z - \tfrac{1}{2}I\right\|_{\mathrm{op}} < \frac{1}{2},$$

where $Z := T^{-1}$. Consequently, the normalized Rayleigh quotient $w := (v_1^* T^{-1} v_1)/\|v_1\|_2^2$ satisfies $|w - \frac{1}{2}| < \frac{1}{2}$, which is equivalent to $\mathrm{Re}(1/w) > 1$. Writing $V := 1/w$ (so $\mathrm{Re}(V) > 1$) and $U := B_{11}/m_{11} = 1 + x_1/m_{11}$ (so $\mathrm{Re}(U) > 1$), we obtain

$$v^* B_1^{-1} v = v_1^* T^{-1} v_1 = \|v_1\|_2^2 w = \frac{\alpha m_{11}}{V}, \qquad B_{11}R_{11} = \frac{m_{11}U}{m_{11}U - \alpha m_{11}/V} = \frac{UV}{UV - \alpha}.$$

Setting $Z_0 := UV$, a direct calculation gives

$$\begin{aligned} 4\,\mathrm{Re}\left(\frac{Z_0}{Z_0 - \alpha}\right) - 3 &= \frac{4\,\mathrm{Re}\left(Z_0(\overline{Z_0} - \alpha)\right) - 3|Z_0 - \alpha|^2}{|Z_0 - \alpha|^2} \\ &= \frac{|Z_0 + \alpha|^2 - 4\alpha^2}{|Z_0 - \alpha|^2} = \frac{\alpha^2\left(|(U/\alpha)V + 1|^2 - 4\right)}{|Z_0 - \alpha|^2}. \end{aligned}$$

Put $U/\alpha = c + is$ and $V = d + it$; since $0 < \alpha \leq 1$, we have $c = \mathrm{Re}(U)/\alpha > 1$ and $d = \mathrm{Re}(V) > 1$. Expanding $|(c+is)(d+it) + 1|^2 - 4 = (cd - st + 1)^2 + (ct + ds)^2 - 4$ yields the exact identity

$$|(c+is)(d+it) + 1|^2 - 4 = \left((cd+1)^2 - 4\right) + (s-t)^2 + (c^2-1)t^2 + (d^2-1)s^2 + s^2t^2 > 0, \quad (2)$$

because $cd > 1$ and $c^2 - 1, d^2 - 1 > 0$. Thus $\mathrm{Re}(B_{11}R_{11}) > \frac{3}{4}$ (and simultaneously $Z_0 \neq \alpha$).

Part (b). The irreducible representation of S_n indexed by $(n-1, 1)$ is the $(n-1)$-dimensional standard representation V_{std}, whose character is $\chi_{(n-1,1)}(\sigma) = c_1(\sigma) - 1$, where $c_1(\sigma) = |\{i \in [n] : \sigma(i) = i\}|$. Since $(2, 1^{n-2}) = (n-1, 1)'$, conjugation gives $\chi_{(2,1^{n-2})}(\sigma) = \mathrm{sgn}(\sigma)(c_1(\sigma) - 1)$. Grouping permutations by their fixed points yields

$$\mathrm{imm}_{(2,1^{n-2})}(B) = \sum_{\sigma \in S_n} c_1(\sigma)\,\mathrm{sgn}(\sigma) \prod_{i=1}^{n} B_{i,\sigma(i)} - \det(B) = \sum_{k=1}^{n} B_{kk} \det B[k^c \mid k^c] - \det(B),$$

where $B[k^c \mid k^c]$ is the $(n-1) \times (n-1)$ principal submatrix obtained by deleting row and column k. Since $\mathrm{Re}(B) \succeq \mathrm{Re}(X) \succ 0$, $\det(B) \neq 0$ everywhere on $\mathcal{H}_n^+$, and Jacobi's cofactor identity gives

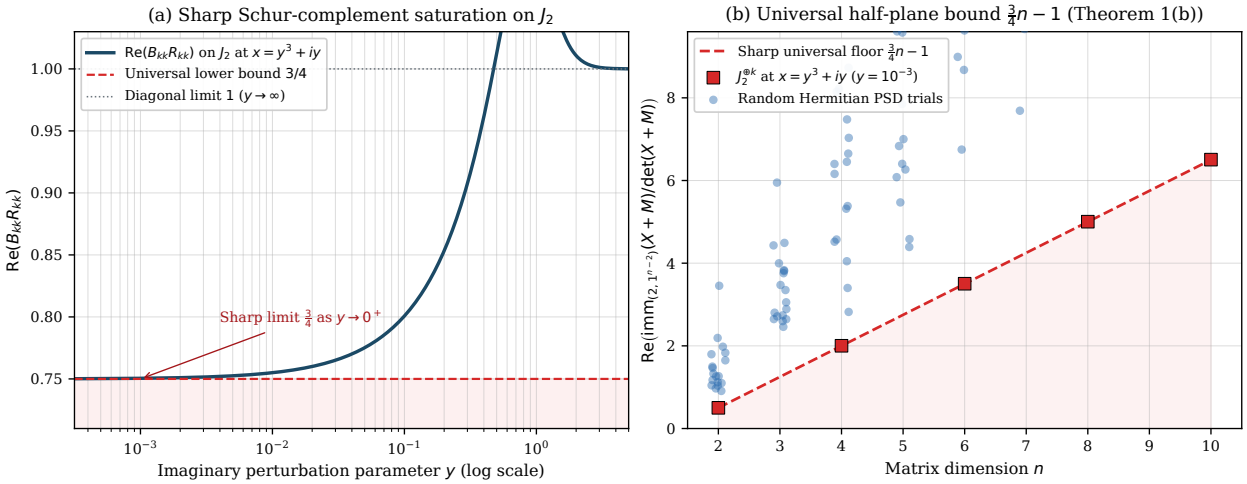


FIGURE 2. Sharpness of Theorem 1.1. (a) Exact convergence $\mathrm{Re}(B_{kk}R_{kk}) \to \frac{3}{4}$ on J_2 along $x = y^3 + iy$. (b) Universal lower bound $\frac{3}{4}n - 1$ saturated by $J_2^{\oplus k}$ across even $n = 2k$, with random Hermitian PSD trials strictly above the floor.

$\det B[k^c \mid k^c]/\det(B) = R_{kk}$. Applying Part (a) term by term gives

$$\mathrm{Re}\left(\frac{\mathrm{imm}_{(2,1^{n-2})}(B)}{\det(B)}\right) = \sum_{k=1}^{n} \mathrm{Re}(B_{kk}R_{kk}) - 1 > \frac{3}{4}n - 1 \geq \frac{1}{2}.$$

Because $\det(B) \neq 0$ and the real part of the ratio is strictly positive, $\mathrm{imm}_{(2,1^{n-2})}(X+M) \neq 0$ on all of $\mathcal{H}_n^+$.

Part (c). Let $n = 2k \geq 2$, $M = J_2^{\oplus k}$, and $X = xI_{2k}$ with $x = y^3 + iy$ ($y > 0$). By block-diagonality, each 2×2 block $B_{(2)} = \begin{pmatrix} x+1 & 1 \\ 1 & x+1 \end{pmatrix}$ has inverse entry $R_{kk} = \frac{x+1}{x(x+2)}$, so $B_{kk}R_{kk} = \frac{(x+1)^2}{x(x+2)}$ and

$$\mathrm{Re}(B_{kk}R_{kk}) = \frac{y^{10} + 4y^7 + 2y^6 + 5y^4 + 4y^3 + y^2 + 2y + 3}{y^{10} + 4y^7 + 2y^6 + 4y^4 + 4y^3 + y^2 + 4} = \frac{3}{4} + \frac{1}{2}y + O(y^2)$$

as $y \to 0^+$. Hence $\lim_{y \to 0^+} \mathrm{Re}(B_{kk}R_{kk}) = \frac{3}{4}$. By Part (b), the real part of the ratio $\mathrm{imm}_{(2,1^{n-2})}/\det$ equals $n\ \mathrm{Re}(B_{kk}R_{kk}) - 1 \to \frac{3}{4}n - 1$ (Figure 2).

Part (d). Let $X = xI_n$ with $x = r + it$ ($r > 0, t \in \mathbb{R}$) and diagonalize $M = U\Lambda U^*$ with eigenvalues $\lambda_1, \ldots, \lambda_n \geq 0$. Set $Z_a := x + \lambda_a = r_a + it$, where $r_a := r + \lambda_a > 0$, and $p_{ka} := |U_{ka}|^2 \geq 0$ ($\sum_{a=1}^n p_{ka} = 1$). Then $B_{kk} = \sum_{a=1}^n p_{ka} Z_a$ and $R_{kk} = \sum_{b=1}^n p_{kb} Z_b^{-1}$, so

$$B_{kk}R_{kk} = \sum_{a=1}^{n} p_{ka}^2 + \sum_{1 \leq a < b \leq n} p_{ka}p_{kb}\left(\frac{Z_a}{Z_b} + \frac{Z_b}{Z_a}\right).$$

Because Z_a and Z_b share the exact imaginary part t, expanding the real part gives the exact rational identity

$$\mathrm{Re}\left(\frac{Z_a}{Z_b} + \frac{Z_b}{Z_a}\right) - 1 = \frac{r_a r_b\big((r_a - r_b)^2 + r_a r_b\big) + 2r_a r_b t^2 + t^4}{(r_a^2 + t^2)(r_b^2 + t^2)} > 0.$$

Thus $\mathrm{Re}(Z_a/Z_b + Z_b/Z_a) > 1$ for all $a, b \in [n]$, whence

$$\mathrm{Re}(B_{kk}R_{kk}) \geq \sum_{a=1}^{n} p_{ka}^2 + \sum_{1 \leq a < b \leq n} p_{ka}p_{kb} = \sum_{a=1}^{n} p_{ka}^2 + \frac{1 - \sum_{a=1}^n p_{ka}^2}{2} = \frac{1}{2} + \frac{1}{2}\sum_{a=1}^{n} p_{ka}^2 \geq \frac{1}{2} + \frac{1}{2n},$$

the final step following from Cauchy–Schwarz ($\sum_{a=1}^n p_{ka}^2 \geq \frac{1}{n}(\sum_a p_{ka})^2 = \frac{1}{n}$). □

4. Small-n Separation and Equal-Diagonal Quartet Stability

We next prove Theorem 1.2, handling boundary degeneracies via correlation regularization and proving the $n = 3$ and $n = 4$ bounds algebraically.

Proof of Theorem 1.2. **Part (a).** Let $M = (m_{ij}) \succeq 0$ and $X = \operatorname{diag}(x_1, x_2, x_3) \in \mathcal{H}_3^+$. Recall the character values on S_3: $\chi_{(3)} \equiv 1$, whereas $\chi_{(2,1)}(\mathrm{id}) = 2$, $\chi_{(2,1)}((ij)) = 0$, and $\chi_{(2,1)}((ijk)) = -1$.

Case 1: Some diagonal entry $m_{kk} = 0$. By permutation symmetry, assume $m_{33} = 0$. Since $M \succeq 0$, the third row and column of M vanish, so $X + M = (X_2 + M_2) \oplus (x_3)$. Only permutations fixing 3 contribute, giving

$$\operatorname{per}(X + M) = \big(B_{11}B_{22} + |m_{12}|^2\big)x_3, \qquad \operatorname{imm}_{(2,1)}(X + M) = 2B_{11}B_{22}x_3,$$

since $\chi_{(2,1)}((12)) = 0$. If $m_{11} = 0$ or $m_{22} = 0$, then $m_{12} = 0$ and the ratio is $\frac{1}{2} > 0$. If $m_{11}, m_{22} > 0$, set $z_j := B_{jj}/m_{jj} = 1 + x_j/m_{jj}$ ($\operatorname{Re}(z_j) > 1$) and $\rho := |m_{12}|^2/(m_{11}m_{22}) \in [0, 1]$, so

$$\frac{\operatorname{per}(X + M)}{\operatorname{imm}_{(2,1)}(X + M)} = \frac{1}{2}\left(1 + \frac{\rho}{z_1 z_2}\right).$$

Write $z_j = r_j e^{i\theta_j}$ with $|\theta_j| < \frac{\pi}{2}$ and $r_j > \sec\theta_j$, and set $u_j := \tan\theta_j \in \mathbb{R}$. We claim $\operatorname{Re}(1/(z_1z_2)) = \cos(\theta_1+\theta_2)/(r_1r_2) > -\frac{1}{8}$. Indeed, if $\cos(\theta_1+\theta_2) \geq 0$, then $\operatorname{Re}(1/(z_1z_2)) \geq 0 > -\frac{1}{8}$ trivially. If instead $\cos(\theta_1 + \theta_2) < 0$, then multiplying $0 < (r_1r_2)^{-1} < \cos\theta_1\cos\theta_2$ by the negative factor $8\cos(\theta_1 + \theta_2)$ and using $\cos(\theta_1 + \theta_2)\cos\theta_1\cos\theta_2 = \frac{1-u_1u_2}{(1+u_1^2)(1+u_2^2)}$ gives

$$8\operatorname{Re}\left(\frac{1}{z_1z_2}\right) + 1 = \frac{8\cos(\theta_1 + \theta_2)}{r_1r_2} + 1 > \frac{8(1 - u_1u_2)}{(1 + u_1^2)(1 + u_2^2)} + 1 = \frac{(u_1u_2 - 3)^2 + (u_1 - u_2)^2}{(1 + u_1^2)(1 + u_2^2)} \geq 0.$$

Hence in all cases $\operatorname{Re}(1/(z_1z_2)) > -\frac{1}{8}$ and $\operatorname{Re}(\operatorname{per}/\operatorname{imm}_{(2,1)}) = \frac{1}{2}(1 + \rho\operatorname{Re}(1/(z_1z_2))) > \frac{1}{2}(1 - \frac{1}{8}) = \frac{7}{16} > 0$.

Case 2: $m_{11}, m_{22}, m_{33} > 0$. Define $D := \operatorname{diag}(m_{11}, m_{22}, m_{33}) \succ 0$. Setting $W := D^{-1}X = \operatorname{diag}(w_1, w_2, w_3) \in \mathcal{H}_3^+$ and $R := D^{-1/2}MD^{-1/2} \succeq 0$, we have $R_{kk} = 1$ and $X + M = D^{1/2}(W + R)D^{1/2}$. Because both per and $\operatorname{imm}_{(2,1)}$ scale by $\det(D) > 0$ under congruence by $D^{1/2}$, the ratio $f(W; R) := \operatorname{per}(W + R)/\operatorname{imm}_{(2,1)}(W + R)$ agrees with $\operatorname{per}(X + M)/\operatorname{imm}_{(2,1)}(X + M)$.

To avoid any boundary pole when $\operatorname{rank}(R) = 1$, fix $\delta \in (0, 1)$ and pass to the strictly positive definite correlation matrix $R_\delta := (1 - \delta)R + \delta I_3 \succ 0$. Letting $a := |(R_\delta)_{23}|^2$, $b := |(R_\delta)_{13}|^2$, $d := |(R_\delta)_{12}|^2$, and $c := \operatorname{Re}\big((R_\delta)_{12}(R_\delta)_{23}(R_\delta)_{31}\big)$ (equivalently $\operatorname{Re}\big((R_\delta)_{12}(R_\delta)_{23}\overline{(R_\delta)_{13}}\big)$ since $R_\delta = R_\delta^*$), we have $0 \leq a, b, d \leq (1 - \delta)^2 < 1$, $|c| \leq \sqrt{abd} < 1$, and $\det(R_\delta) = 1 - a - b - d + 2c > 0$. Direct expansion of S_3 gives

$$\begin{aligned}\operatorname{per}(W + R_\delta) &= w_1w_2w_3 + (w_1w_2 + w_1w_3 + w_2w_3)\\ &\quad + (1 + a)w_1 + (1 + b)w_2 + (1 + d)w_3 + (1 + a + b + d + 2c),\\ \operatorname{imm}_{(2,1)}(W + R_\delta) &= 2w_1w_2w_3 + 2(w_1w_2 + w_1w_3 + w_2w_3) + 2(w_1 + w_2 + w_3) + 2(1 - c).\end{aligned}$$

Crucially, for *every* $W = \operatorname{diag}(w_1, w_2, w_3) \in \overline{\mathcal{H}_3^+}$ (including the pure imaginary boundary $i\mathbb{R}^3$ and every mixed boundary face where one or two $\operatorname{Re}(w_k) = 0$), the shifted matrix $\widetilde{W} := W + \delta I_3$ lies strictly in the open poly-half-plane $\mathcal{H}_3^+$ ($\operatorname{Re}(w_k + \delta) \geq \delta > 0$ for $k = 1, 2, 3$), while $(1 - \delta)R \succeq 0$ is Hermitian PSD. Writing $W + R_\delta = \widetilde{W} + (1 - \delta)R$, Theorem 1.1(b) guarantees $\operatorname{imm}_{(2,1)}(W + R_\delta) \neq 0$ on all of $\overline{\mathcal{H}_3^+}$ (in fact, on the open neighborhood $\{\operatorname{Re}(w_k) > -\delta\}$). Consequently, $f_\delta(W) := f(W; R_\delta)$ is holomorphic on an open neighborhood of the entire closed orthant $\overline{\mathcal{H}_3^+}$.

Evaluating on the boundary $W = iT$ ($t = (t_1, t_2, t_3)^T \in \mathbb{R}^3$) and writing $y := (t_2t_3, t_1t_3, t_1t_2)^T$, the numerator of $\operatorname{Re}(f_\delta(iT))$ decomposes into homogeneous even-degree forms:

$$\operatorname{Re}\Big(\operatorname{per}(iT + R_\delta)\,\overline{\operatorname{imm}_{(2,1)}(iT + R_\delta)}\Big) = G_0 + t^TG_2t + y^TG_4y + 2(t_1t_2t_3)^2, \tag{3}$$

where $G_0 = 2(1-c)(1+a+b+d+2c)$ and

$$G_2 = \begin{pmatrix} 2(a+1) & -(c+d) & -(b+c) \\ -(c+d) & 2(b+1) & -(a+c) \\ -(b+c) & -(a+c) & 2(d+1) \end{pmatrix}, \qquad G_4 = \begin{pmatrix} 2 & -d & -b \\ -d & 2 & -a \\ -b & -a & 2 \end{pmatrix}.$$

Clearly $G_0 > 0$ (as $c < 1$ and $1+a+b+d+2c = \det(R_\delta) + 2(a+b+d) > 0$) and $G_4 \succ 0$ by strict diagonal dominance ($2 > d+b$ since $b, d < 1$). We prove that G_2 is also diagonally dominant, verifying the first row (the remaining two follow symmetrically): we must show $2(a+1) \geq |c+d| + |b+c|$.

- If $c < 0$: when $c+d \geq 0$ and $b+c \geq 0$, $|c+d|+|b+c| = b+d+2c < b+d < 2 \leq 2(a+1)$; when $c+d < 0$ and $b+c < 0$, $\det(R_\delta) > 0$ gives $-2c < 1-a-b-d$, so $|c+d|+|b+c| = -2c-b-d < 1-a-2b-2d \leq 1 < 2(a+1)$; when $b+c$ and $c+d$ have opposite signs (say $b+c \geq 0 > c+d$), $|c+d|+|b+c| = b-d \leq 1 < 2(a+1)$.
- If $c \geq 0$: then $|c+d|+|b+c| = b+d+2c$. Rotating column phases of R_δ so $(R_\delta)_{12} = s := \sqrt{d} \in [0,1)$ and $(R_\delta)_{13} = \tau := \sqrt{b} \in [0,1)$, write $(R_\delta)_{23} = \xi + i\eta$ with $a = \xi^2 + \eta^2$ and $c = s\tau\xi$. Here $\det(R_\delta) > 0$ is the disk $(\xi - s\tau)^2 + \eta^2 < \rho_0^2$, where $\rho_0 := \sqrt{(1-s^2)(1-\tau^2)}$. Rewriting the gap yields

$$2(a+1) - (b+d+2c) = 2\Big[\Big(\xi - \frac{s\tau}{2}\Big)^2 + \eta^2\Big] + 2 - s^2 - \tau^2 - \frac{1}{2}s^2\tau^2.$$

  If $\frac{1}{2}s\tau \leq \rho_0$, then $(1-s\tau)^2 \geq \rho_0^2 \geq \frac{1}{4}s^2\tau^2$ gives $s\tau \leq \frac{2}{3}$, so $2 - s^2 - \tau^2 - \frac{1}{2}s^2\tau^2 = \big(1 - s^2 - \tau^2 + \frac{3}{4}s^2\tau^2\big) + \big(1 - \frac{5}{4}s^2\tau^2\big) \geq (\rho_0^2 - \frac{1}{4}s^2\tau^2) + \frac{4}{9} > 0$. If $\frac{1}{2}s\tau > \rho_0$, the triangle inequality on the disk gives $\sqrt{(\xi - \frac{1}{2}s\tau)^2 + \eta^2} > \frac{1}{2}s\tau - \rho_0$, so

$$\begin{aligned} 2(a+1) - (b+d+2c) &> 2\Big(\frac{s\tau}{2} - \rho_0\Big)^2 + 2 - s^2 - \tau^2 - \frac{1}{2}s^2\tau^2 \\ &= 4 - 3(s^2+\tau^2) + 2s^2\tau^2 - 2\big(s\sqrt{1-\tau^2}\big)\big(\tau\sqrt{1-s^2}\big) \\ &\geq 4 - 4(s^2+\tau^2) + 4s^2\tau^2 = 4(1-s^2)(1-\tau^2) > 0, \end{aligned}$$

  by AM–GM.

Thus $G_2 \succeq 0$ and $\mathrm{Re}(f_\delta(iT)) > 0$ for all $T \in \mathbb{R}^3$. For fixed $w_2, w_3 \in \overline{\mathcal{H}_1^+}$, $w_1 \mapsto f_\delta(w_1, w_2, w_3)$ is holomorphic on $\overline{\mathcal{H}_1^+}$ with finite limit $\frac{(w_2+1)(w_3+1)+a}{2(w_2+1)(w_3+1)}$ as $|w_1| \to \infty$ (since $\mathrm{Re}(w_j+1) \geq 1$). Applying the one-variable Phragmén–Lindelöf / harmonic minimum principle successively in w_1, w_2, and w_3 shows $\mathrm{Re}(f_\delta(W)) \geq 0$ on $\mathcal{H}_3^+$. Letting $\delta \to 0^+$ yields $\mathrm{Re}(f(W;R)) \geq 0$ on $\mathcal{H}_3^+$; because $f(xI_3;R) \to \frac{1}{2} > 0$ as $x \to +\infty$, the strong minimum principle for harmonic functions guarantees $\mathrm{Re}(f(W;R)) > 0$ everywhere on $\mathcal{H}_3^+$.

Part (b). For any 4×4 Hermitian correlation matrix $R = (r_{ij}) \succeq 0$ ($r_{ii} = 1$), define the invariants

$$S_2 := \sum_{0 \leq i < j \leq 3} |r_{ij}|^2, \qquad S_{2,2} := |r_{01}|^2|r_{23}|^2 + |r_{02}|^2|r_{13}|^2 + |r_{03}|^2|r_{12}|^2,$$

$$C_3 := 2 \sum_{0 \leq i < j < \ell \leq 3} \mathrm{Re}(r_{ij} r_{j\ell} r_{\ell i}), \qquad C_4 := 2 \sum_{\text{4-cycles}} \mathrm{Re}(r_{i_1 i_2} r_{i_2 i_3} r_{i_3 i_4} r_{i_4 i_1}),$$

where the C_4 sum runs over the three undirected 4-cycles of K_4. Expanding $\chi_{(4)} \equiv 1$ and $\chi_{(2,2)}$ ($\chi_{(2,2)}(1^4) = 2$, $\chi_{(2,2)}(2,1^2) = 0$, $\chi_{(2,2)}(2^2) = 2$, $\chi_{(2,2)}(3,1) = -1$, $\chi_{(2,2)}(4) = 0$) gives

$$\mathrm{per}(xI_4 + R) = x^4 + 4x^3 + (6 + S_2)x^2 + (4 + 2S_2 + C_3)x + (1 + S_2 + S_{2,2} + C_3 + C_4),$$

$$\tfrac{1}{2}\mathrm{imm}_{(2,2)}(xI_4 + R) = x^4 + 4x^3 + 6x^2 + \big(4 - \tfrac{1}{2}C_3\big)x + \big(1 + S_{2,2} - \tfrac{1}{2}C_3\big).$$

Since $R \succeq 0$ with $R_{ii} = 1$, every entry satisfies $|r_{ij}| \leq 1$, whence $S_2 \in [0,6]$, $S_{2,2} \in [0,3]$, $|C_3| \leq 8$, and $|C_4| \leq 6$. Moreover, all immanants of a PSD matrix are nonnegative by Schur's theorem [12], so $\frac{1}{2}\mathrm{imm}_{(2,2)}(R) = 1 + S_{2,2} - \frac{1}{2}C_3 \geq 0$; for the permanent, iterating Lieb's permanental inequality [7]

on 1×1 and complementary principal blocks yields strict positivity of both the constant and linear coefficients of $\operatorname{per}(xI_4+R)$:

$$a_4=\operatorname{per}(R)\geq\prod_{k=0}^{3}R_{kk}=1>0,\qquad a_3=4+2S_2+C_3=\sum_{j=0}^{3}\operatorname{per}\big(R[\{j\}^c]\big)\geq\sum_{j=0}^{3}1=4>0.$$

For $\operatorname{per}(xI_4+R)$, we have $\Delta_1=4>0$, $\Delta_2=20+2S_2-C_3\geq 12>0$, and

$$\Delta_3(\operatorname{per}(xI_4+R))=64+32S_2+4S_2^2-C_3^2-16S_{2,2}-16C_4.$$

Writing $x_e:=|r_{ij}|^2\in[0,1]$ for the six edges e of K_4 and m_1,m_2,m_3 for the products of opposite edge moduli ($S_{2,2}=m_1^2+m_2^2+m_3^2$), we have $|C_4|\leq 2(m_1m_2+m_1m_3+m_2m_3)\leq 2S_{2,2}$ and $C_3^2\leq 16\sum_T\prod_{e\in T}x_e\leq\frac{16}{3}\sum_T\sum_{e\in T}x_e=\frac{32}{3}S_2$ by Cauchy–Schwarz over the four triangles T of K_4 (since $x_e\in[0,1]$ gives $x_ax_bx_c\leq\frac{x_a+x_b+x_c}{3}$). Expanding $S_2^2=\sum_e x_e^2+2\sum_{e\sim f}x_ex_f+2S_{2,2}$ yields $4S_2^2-48S_{2,2}\geq -40S_{2,2}\geq -20S_2$ (as $2x_ax_b\leq x_a+x_b$ on $[0,1]$), so

$$\Delta_3(\operatorname{per}(xI_4+R))\geq 64+\frac{4}{3}S_2+4\sum_e x_e^2+8\sum_{e\sim f}x_ex_f\geq 64>0.$$

Since all coefficients and $\Delta_1,\Delta_2,\Delta_3$ are strictly positive, $\operatorname{per}(xI_4+R)$ is strictly Hurwitz stable by Liénard–Chipart.

For $\frac{1}{2}\operatorname{imm}_{(2,2)}(xI_4+R)$, we compute directly: $\Delta_1=4>0$, $\Delta_2=20+\frac{1}{2}C_3\geq 16>0$, and

$$\Delta_3\big(\tfrac{1}{2}\operatorname{imm}_{(2,2)}(xI_4+R)\big)=\frac{256-C_3^2-64S_{2,2}}{4}\geq\frac{256-64-192}{4}=0,$$

using only $C_3^2\leq 64$ and $S_{2,2}\leq 3$. Whenever $R\succ 0$ strictly (in particular, along the perturbation $R_\varepsilon:=(1-\varepsilon)R+\varepsilon I_4$ for $\varepsilon\in(0,1)$), every off-diagonal entry satisfies $|(R_\varepsilon)_{ij}|\leq 1-\varepsilon<1$, so $C_3(R_\varepsilon)^2+64S_{2,2}(R_\varepsilon)<256$ strictly, giving $\Delta_3>0$ and strict positivity of the constant term $c_4(R_\varepsilon)\geq(1-\varepsilon)^4c_4(R)+\varepsilon^4>0$ (indeed, $\operatorname{imm}_{(2,2)}(R_\varepsilon)\geq 2\det(R_\varepsilon)>0$ by Schur's theorem). Hence all four roots of $\operatorname{imm}_{(2,2)}(xI_4+R_\varepsilon)$ lie in $\operatorname{Re}(x)<0$; letting $\varepsilon\to 0^+$ via Lemma 2.1(i) places all roots of $\operatorname{imm}_{(2,2)}(xI_4+R)$ in $\operatorname{Re}(x)\leq 0$. (Equality $\Delta_3=0$ is attained at $R=J_4$, where $\frac{1}{2}\operatorname{imm}_{(2,2)}(xI_4+J_4)=x^2(x^2+4x+6)$.) □

5. Rank-One Theory, Laguerre Rectangles, and Hook Trichotomy

We now prove Theorem 1.3, establishing the complete rank-one picture.

Proof of Theorem 1.3. **Part (a).** Write $v_i=\sqrt{w_i}e^{i\theta_i}$. Along any cycle $(i_1\dots i_m)$ of $\sigma\in S_n$ of length $m\geq 2$, the phase factors telescope: $\prod_{s=1}^m(v_{i_s}\overline{v_{i_{s+1}}})=\prod_{s=1}^m\sqrt{w_{i_s}w_{i_{s+1}}}=\prod_{s=1}^m w_{i_s}$. Expanding $\prod_{i=1}^n(x_i\delta_{i,\sigma(i)}+v_i\overline{v_{\sigma(i)}})$ over the subset $S\subseteq[n]$ of indices where we select $v_i\overline{v_{\sigma(i)}}$ (so $\sigma(i)=i$ on $S^c:=[n]\setminus S$, i.e., $\sigma=\tau\times\mathrm{id}_{S^c}$ for $\tau\in S_S$) gives

$$\operatorname{imm}_\lambda(X+vv^*)=\sum_{S\subseteq[n]}\left(\prod_{j\in S}w_j\right)\left(\prod_{i\notin S}x_i\right)\sum_{\tau\in S_S}\chi_\lambda(\tau\times\mathrm{id}_{S^c}).$$

If $|S|=k$, then $\sum_{\tau\in S_k}\chi_\lambda(\tau\times\mathrm{id}_{n-k})=k!\sum_{\mu\vdash n-k}m_\mu\chi_\mu(\mathrm{id}_{n-k})$, where $m_\mu=\langle\chi_\lambda\downarrow^{S_n}_{S_k\times S_{n-k}},1_{S_k}\otimes\chi_\mu\rangle$. By Frobenius reciprocity and Pieri's rule [10, 13], $m_\mu=\langle\chi_\lambda,(1_{S_k}\otimes\chi_\mu)\uparrow^{S_n}\rangle$ equals 1 if λ/μ is a horizontal k-strip and 0 otherwise. Since any horizontal strip has at most one box per column, $k\leq\lambda_1$, proving Part (a).

Part (b). Assume $\lambda_1\leq 4$. Let $h_1\geq h_2\geq h_3\geq h_4\geq 0$ be the column lengths of λ (so $\lambda'=(h_1,h_2,h_3,h_4)$ and $n=\sum_{j=1}^4 h_j$), and define the gaps $d_4:=h_4$, $d_3:=h_3-h_4$, $d_2:=h_2-h_3$,

$d_1 := h_1 - h_2 \in \mathbb{Z}_{\geq 0}$. Setting $x_j := h_j + 4 - j$ gives $x_1 > x_2 > x_3 > x_4 = d_4 \geq 0$ and, by the Frobenius determinant formula for $f^{\lambda'} = f^{\lambda}$ [10],

$$f^{\lambda} = \frac{n!}{\prod_{j=1}^{4} x_j!}\Delta(x), \qquad \Delta(x) := \prod_{1 \leq i < j \leq 4}(x_i - x_j).$$

A horizontal k-strip λ/μ removes at most one box from the bottom of each of the four columns, corresponding to subsets $S \subseteq \{1,2,3,4\}$ with $|S| = k$ such that $x_j - 1 \geq x_{j+1} + 1$ whenever $j \in S, j + 1 \notin S$ and $x_4 \geq 1$ if $4 \in S$. Remarkably, if $4 \in S$ with $x_4 = 0$ then $\prod_{j \in S} x_j = 0$, while if $j \in S, j+1 \notin S$ with $x_j - 1 = x_{j+1}$ then $\Delta(x - \mathbf{1}_S) = 0$. Therefore, setting $\tilde{c}_k(\lambda) := \binom{n}{k} c_k(\lambda) = \frac{n!}{(n-k)!}\sum_{\lambda/\mu \in \mathcal{H}_k} f^{\mu}$, every invalid subset drops out automatically and the normalized ratios $R_k := \tilde{c}_k(\lambda)/f^{\lambda}$ are exact polynomials in (d_1, d_2, d_3, d_4):

$$R_k(d_1,d_2,d_3,d_4) = \sum_{\substack{S \subseteq \{1,2,3,4\} \\ |S| = k}} \left(\prod_{j \in S} x_j\right) \frac{\Delta(x - \mathbf{1}_S)}{\Delta(x)} \qquad (0 \leq k \leq 4).$$

Explicitly, $R_0 = 1$, $R_1 = d_1 + 2d_2 + 3d_3 + 4d_4 = n$, $R_4 = d_4(d_3 + d_4 + 1)(d_2 + d_3 + d_4 + 2)(d_1 + d_2 + d_3 + d_4 + 3)$, and

$$\begin{aligned} R_2 &= d_1d_2 + 2d_1d_3 + 3d_1d_4 + d_2^2 + 4d_2d_3 + 6d_2d_4 + d_2 + 3d_3^2 + 9d_3d_4 + 3d_3 + 6d_4^2 + 6d_4, \\ R_3 &= d_1d_2d_3 + 2d_1d_2d_4 + d_1d_3^2 + 4d_1d_3d_4 + d_1d_3 + 3d_1d_4^2 + 3d_1d_4 + d_2^2d_3 + 2d_2^2d_4 + 2d_2d_3^2 \\ &\quad + 8d_2d_3d_4 + 3d_2d_3 + 6d_2d_4^2 + 8d_2d_4 + d_3^3 + 6d_3^2d_4 + 3d_3^2 + 9d_3d_4^2 + 15d_3d_4 + 2d_3 \\ &\quad + 4d_4^3 + 12d_4^2 + 8d_4. \end{aligned}$$

Let $D_2 := R_1R_2 - R_0R_3$ and $D_3 := R_1R_2R_3 - R_0R_3^2 - R_1^2R_4$. Substituting $d_4 = 1 + u_4$ (valid whenever $\lambda_1 = 4$, with $d_1, d_2, d_3, u_4 \geq 0$) gives polynomials in (d_1, d_2, d_3, u_4) every single monomial coefficient of which is at least $+1$, with constant terms $D_2(d_4 = 1) \geq 24$ and $D_3(d_4 = 1) \geq 192$. Likewise, when $\lambda_1 = 3$ ($d_4 = 0, d_3 = 1 + u_3$), every monomial coefficient of $D_2(d_1, d_2, 1 + u_3, 0)$ is at least $+1$ with constant term 12.

Now fix $w \in \mathbb{R}^n_{\geq 0}$ and let $k_0 := |\{i \in [n] : w_i > 0\}|$. If $k_0 = 0$, then $P_{\lambda}(x; 0) = f^{\lambda}x^n$ has only zero roots. Assume henceforth that $k_0 \geq 1$, and set $m := \min(\lambda_1, k_0) \in \{1,2,3,4\}$. Then the normalized elementary symmetric averages $\hat{e}_k := e_k(w)/\binom{n}{k}$ satisfy $\hat{e}_k > 0$ for $0 \leq k \leq m$ and $\hat{e}_k = 0$ for $k > m$. Because every $d_j \geq 0$ and $R_k > 0$ for $0 \leq k \leq \lambda_1$, we factor

$$P_{\lambda}(x; vv^*) = f^{\lambda}x^{n-m}Q_0(x), \qquad Q_0(x) := \sum_{k=0}^{m} R_k\hat{e}_k x^{m-k},$$

where all $m + 1$ coefficients of $Q_0(x)$ are strictly positive. By Newton's inequalities [5] on $w \in \mathbb{R}^n_{\geq 0}$, $\hat{e}_k^2 \geq \hat{e}_{k-1}\hat{e}_{k+1}$ holds for every $1 \leq k \leq m-1$, yielding $\hat{e}_3 \leq \hat{e}_1\hat{e}_2$ (when $m \geq 3$) and $\hat{e}_1\hat{e}_4 \leq \hat{e}_2\hat{e}_3$, hence $\hat{e}_3^2 \leq \hat{e}_1\hat{e}_2\hat{e}_3$ and $\hat{e}_1^2\hat{e}_4 \leq \hat{e}_1\hat{e}_2\hat{e}_3$ (when $m = 4$):

- If $m \leq 2$, Q_0 is linear or quadratic with strictly positive coefficients, hence strictly Hurwitz stable.
- If $m = 3$ (so $\lambda_1 \in \{3,4\}$ and $D_2 \geq 12 > 0$), $\Delta_2(Q_0) = R_1R_2\hat{e}_1\hat{e}_2 - R_0R_3\hat{e}_3 \geq D_2\hat{e}_3 > 0$.
- If $m = 4$ (so $\lambda_1 = 4$, $D_2 \geq 24 > 0$, $D_3 \geq 192 > 0$), $\Delta_2(Q_0) \geq D_2\hat{e}_3 > 0$ and $\Delta_3(Q_0) = R_1R_2R_3\hat{e}_1\hat{e}_2\hat{e}_3 - R_0R_3^2\hat{e}_3^2 - R_1^2R_4\hat{e}_1^2\hat{e}_4 \geq D_3\hat{e}_1\hat{e}_2\hat{e}_3 > 0$.

By Liénard–Chipart, every root of Q_0 lies strictly in $\mathrm{Re}(x) < 0$, so all nonzero roots of $\mathrm{imm}_{\lambda}(xI_n + vv^*)$ lie strictly in $\mathrm{Re}(x) < 0$ (and all roots lie in $\mathrm{Re}(x) \leq 0$). Conversely, if $r \geq 5$, Part (e)(ii)–(iii) below shows $\mathrm{imm}_{(r)}(xI_r + J_r) = Q_{r,r}(x)$ has roots in $\mathrm{Re}(x) > 0$.

Part (c). For the rectangle $\lambda = (w^k) \vdash n = wk$, the unique horizontal j-strip ($0 \le j \le w$) removes j boxes from the bottom row, leaving $\mu = (w^{k-1}, w-j)$. Comparing the hook products (1) of (w^k) and $(w^{k-1}, w-j)$:

- In rows $1, \dots, k-1$ and columns $1, \dots, w-j$, every hook length $(w-c)+(k-r)+1$ is unchanged.
- In rows $1, \dots, k-1$ and columns $c = w-j+1, \dots, w$, each column height drops from k to $k-1$, so the column-c hook product telescopes by $\prod_{r=1}^{k-1} \frac{w-c+k-r+1}{w-c+k-r} = \frac{w-c+k}{w-c+1}$; across all j columns this gives $\frac{(k)_j}{j!}$.
- Row k shrinks from hook product $w!$ to $(w-j)!$, contributing $\frac{w!}{(w-j)!} = \binom{w}{j} j!$.

Multiplying these contributions shows that the total hook-product ratio equals $\binom{w}{j}(k)_j$, whence

$$\binom{wk}{j} c_j((w^k)) = \frac{(wk)!}{(wk-j)!} f^{(w^{k-1}, w-j)} = f^{(w^k)} \binom{w}{j} (k)_j.$$

Summing $j = 0, \dots, w$ in Part (a) with $w_i = 1$ yields the polynomial identity

$$\operatorname{imm}_{(w^k)}(xI_{wk} + J_{wk}) = f^{(w^k)} x^{wk-w} \sum_{j=0}^{w} \binom{w}{j} (k)_j x^{w-j}.$$

Since $\frac{1}{\Gamma(k)} \int_0^\infty e^{-t} t^{k-1+j}\, dt = \frac{\Gamma(k+j)}{\Gamma(k)} = (k)_j$, expanding $(x+t)^w$ under the integral yields the Laguerre representation.

Part (d). Let $n \ge 4$. Direct evaluation of Part (a) gives $c((2,1^{n-2})) = [n-1,\, n-1,\, 2]$, $c((3,1^{n-3})) = [\binom{n-1}{2}, \binom{n-1}{2}, 2(n-2), 6]$, and $c((2,2,1^{n-4})) = [\frac{n(n-3)}{2}, \frac{n(n-3)}{2}, 2(n-3)]$. Let $e_j := e_j(w) \ge 0$. If $e_2 = 0$, then w has at most one nonzero coordinate so $e_3 = 0$ and both ratios reduce to positive constants $\frac{n-2}{2}$ and $\frac{n(n-3)}{2(n-1)}$. Assume $e_2 > 0$. Then $P_{(2,1^{n-2})}(x; vv^*) = x^{n-3} q_0(x)$ with $q_0(x) = (n-1)x^3 + (n-1)e_1 x^2 + 2e_2 x$ having all roots in $\operatorname{Re}(x) \le 0$ and no poles in $\operatorname{Re}(x) > 0$. On the boundary $x = iy$ ($y \in \mathbb{R} \setminus \{0\}$), dividing out y^2 gives quadratics in $t := y^2$:

$$\frac{\operatorname{Re}\left(P_{(3,1^{n-3})}(iy)\, \overline{P_{(2,1^{n-2})}(iy)}\right)}{y^{2n-4}} = \frac{(n-1)^2(n-2)}{2} t^2 + \frac{(n-1)(n-2)\big[(n-1)e_1^2 - 6e_2\big]}{2} t + \big[4(n-2)e_2^2 - 6(n-1)e_1 e_3\big],$$

$$\frac{\operatorname{Re}\left(P_{(2,2,1^{n-4})}(iy)\, \overline{P_{(2,1^{n-2})}(iy)}\right)}{y^{2n-4}} = \frac{n(n-1)(n-3)}{2} t^2 + \frac{(n-3)\big[n(n-1)e_1^2 - 2(3n-2)e_2\big]}{2} t + 4(n-3)e_2^2.$$

By Newton's inequalities on $w_1, \dots, w_n \ge 0$, we have $e_1^2 \ge \frac{2n}{n-1} e_2$ and $e_2^2 \ge \frac{3(n-1)}{2(n-2)} e_1 e_3$. The constant term of the first quadratic is therefore ≥ 0 (and $4(n-3)e_2^2 > 0$ for the second). Writing $e_1^2 = \frac{2n}{n-1} e_2 + \delta$ ($\delta \ge 0$), the coefficient of t in the first quadratic becomes $\frac{1}{2}(n-1)(n-2)\big[(n-1)\delta + 2(n-3)e_2\big] > 0$ for all $n \ge 4$, and in the second becomes $\frac{1}{2}(n-1)(n-3)\big[n\delta + 2(n-2)e_2\big] > 0$ for all $n \ge 4$. Thus both boundary real parts are strictly positive on $i\mathbb{R} \setminus \{0\}$; since $q_0(x)$ has at most a simple zero at $x = 0$ (where $P_{(3,1^{n-3})}/q_0$ has positive residue $\frac{6e_3}{2e_2} \ge 0$ and $P_{(2,2,1^{n-4})}/q_0$ is regular with value $n-3 > 0$), the harmonic minimum principle on $\operatorname{Re}(x) > 0$ (with positive limit at ∞) completes the proof.

Part (e). For $\lambda = (r, 1^{n-r})$, $f^\lambda = \binom{n-1}{r-1}$. A horizontal j-strip ($1 \le j \le r$) either leaves the first column intact ($j \le r-1$, giving $(r-j, 1^{n-r})$ of dimension $\binom{n-j-1}{r-j-1}$) or removes its bottom box ($n > r$, giving $(r-j+1, 1^{n-r-1})$ of dimension $\binom{n-j-1}{r-j}$, or 1 if $j = r$). By Pascal's identity, $\binom{n}{j} c_j(\lambda) = \frac{n!}{(r-j)!(n-r)!} = f^\lambda n \frac{(r-1)!}{(r-j)!}$, which gives $Q_{r,n}(x) = x^r + n(r-1)! \sum_{m=0}^{r-1} \frac{x^m}{m!}$ upon setting $m = r-j$ (including $n = r$).

For $r \le 4$: at $r = 1, 2, 3$ all Hurwitz minors are strictly positive ($Q_{3,n} = x^3 + nx^2 + 2nx + 2n$ has $\Delta_2 = 2n(n-1) > 0$ for $n \ge 3$), and at $r = 4$ we obtain $\Delta_1 = n$, $\Delta_2 = 3n(n-2)$, $\Delta_3 = 12n^2(n-3)$, $\Delta_4 = 72n^3(n-3) > 0$ for all $n \ge 4$.

For $r = 5$: direct evaluation gives

$$\Delta_1 = n, \qquad \Delta_2 = 4n(n-3), \qquad \Delta_3 = 24n^2(n-5),$$
$$\Delta_4 = 192n^2(n^2-6n-3), \qquad \Delta_5 = 4608n^3(n^2-6n-3).$$

At $n \in \{5,6\}$, $n^2-6n-3 \in \{-8,-3\} < 0$, so $\Delta_4 < 0$ and $Q_{5,n}$ is unstable; for every $n \geq 7$, $n^2-6n-3 = (n-3)^2 - 12 \geq 4 > 0$, so all five Hurwitz minors are strictly positive (Figure 3(b)).

For $r = 6$: $\Delta_5(Q_{6,n}) = -38400n^3\big((n-5)^2+20\big) < 0$ for all $n \geq 6$.

For arbitrary $r \geq 6$ and $n \geq r$, suppose for contradiction that $Q_{r,n}(x)$ has all roots in $\mathrm{Re}(x) \leq 0$. Since $Q_{r,n}(0) = n(r-1)! > 0$, Lemma 2.1(ii) implies that both $U(y)$ and $V(y)$ in $Q_{r,n}(iy) = U(y) + iV(y)$ have only real roots.

- If $r = 2K+1 \geq 7$ ($K \geq 3$), then $U(y) = n(2K)!\, C_K(y^2)$, where $C_K(t) := \sum_{j=0}^{K}(-1)^j \frac{t^j}{(2j)!}$ of degree K must have all K roots $t_1, \ldots, t_K \in \mathbb{R}_{>0}$ (if any $t_j \leq 0$, $y = \pm\sqrt{t_j}$ would be non-real or zero). Yet the elementary symmetric sums of the roots of $C_K(t)$ are $e_1 = 2K(2K-1)$ and $e_2 = 2K(2K-1)(2K-2)(2K-3)$, giving
$$\sum_{1 \leq i < j \leq K} (t_i - t_j)^2 = (K-1)e_1^2 - 2Ke_2 = 4K^2(2K-1)(K-1)(5-2K) < 0,$$
a contradiction for every $K \geq 3$.
- If $r = 2M \geq 6$ ($M \geq 3$), then $V(y) = n(2M-1)!\, y\, S_M(y^2)$, where $S_M(t) := \sum_{j=0}^{M-1}(-1)^j \frac{t^j}{(2j+1)!}$ of degree $d := M-1 \geq 2$ must have all d roots $t_1, \ldots, t_d \in \mathbb{R}_{>0}$. Yet $e_1 = (2M-1)(2M-2)$ and $e_2 = (2M-1)(2M-2)(2M-3)(2M-4)$, giving
$$\sum_{1 \leq i < j \leq d} (t_i - t_j)^2 = (d-1)e_1^2 - 2de_2 = 4(2M-1)(M-1)^2(M-2)(5-2M) < 0,$$
a contradiction for every $M \geq 3$.

Therefore $Q_{r,n}(x)$ has at least one root in $\mathrm{Re}(x) > 0$ for every $r \geq 6$ and $n \geq r$. □

Remark 5.1 (Link to Szegő Curves, Sturm Sequences, and Rouché–Vieta Tails). The polynomial $Q_{r,n}(x)$ pairs x^r with the truncated exponential partial sum $e_{r-1}(x) = \sum_{m=0}^{r-1} x^m/m!$ studied by Szegő [15], whose zeros asymptotically trace $|ze^{1-z}| = 1$. As an independent check alongside the exact two-line Newton real-root discriminant proof above, the verification script computes exact integer Sturm sequences on $C_K(t)$ ($K \in \{3,4,5,6,9,10\}$) and $S_M(t)$ ($M \in \{3,\ldots,8\}$) as well as exact rational Rouché–Vieta tail bounds comparing $C_K(y^2)$ with $\cos(y)$ on $|y| = m\pi$ ($K \in \{7, 11..25\}$) and $S_M(y^2)$ with $\frac{\sin y}{y}$ on $|y| = (m+\frac{1}{2})\pi$ ($M \in \{9..20\}$), where $K-m$ roots forced outside $|y| = m\pi$ violate $\sum t_i = 2K(2K-1)$.

6. Multi-Block Catalan–Chebyshev Destabilization

We now prove Theorem 1.4, demonstrating how direct sums $J_b^{\oplus k}$ destabilize partitions of small constant width $\lambda_1 \in \{2,3,4\}$.

Proof of Theorem 1.4. **Part (a).** Since $M = J_2^{\oplus k}$ is block diagonal with 2×2 blocks J_2, only permutations $\sigma \in S_2^k \leq S_{2k}$ contribute to $\mathrm{imm}_{(2^r,1^{2k-2r})}(xI_{2k} + J_2^{\oplus k})$. If σ consists of j transpositions and $k-j$ identities across the k blocks, its cycle type is $(2^j, 1^{2k-2j})$, its matrix-entry product is $(x+1)^{2k-2j}$, and there are $\binom{k}{j}$ such permutations.

Conjugating $(2^r, 1^{2k-2r})$ yields $(2k-r, r)$, so $\chi_{(2^r,1^{2k-2r})}(\sigma) = (-1)^j \chi_{(2k-r,r)}(2^j, 1^{2k-2j})$. By the Frobenius two-row character formula [10], $\chi_{(2k-r,r)}(\mu) = [z^r]\,(1-z)\prod_{c\in\mu}(1+z^c)$ for $0 \leq r \leq k$, giving the polynomial congruence modulo z^{k+1}:

$$\sum_{r=0}^{k} \chi_{(2^r,1^{2k-2r})}(2^j, 1^{2k-2j}) z^r \equiv (-1)^j(1-z)(1+z^2)^j(1+z)^{2k-2j} \pmod{z^{k+1}}.$$

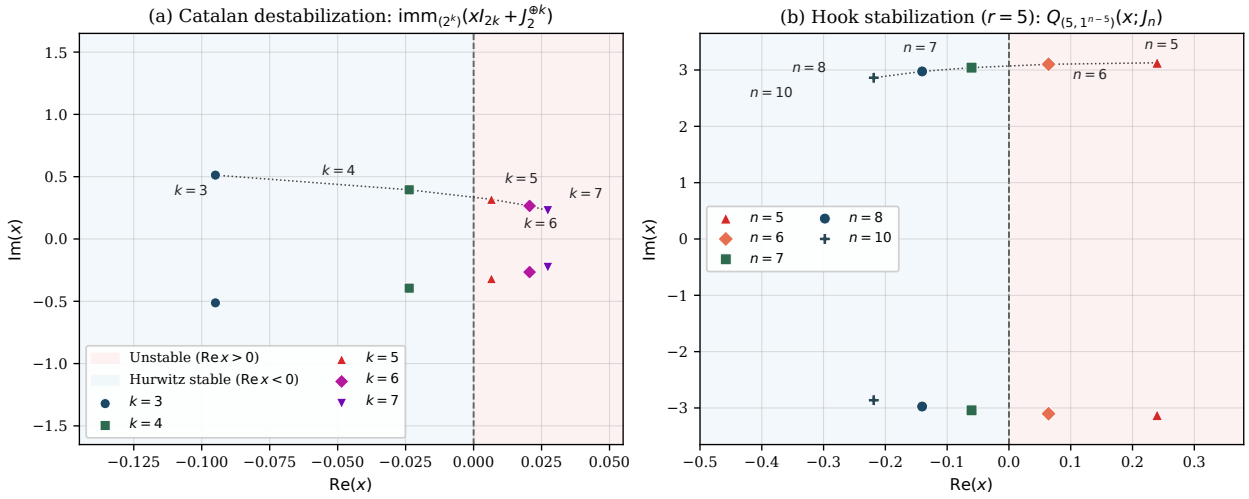


FIGURE 3. Complex zero migration across $\mathrm{Re}(x)=0$. (a) Rightmost zero pair of the Catalan polynomial $\mathrm{imm}_{(2^k)}(xI_{2k}+J_2^{\oplus k})$ crosses from $-0.0237\pm0.395i$ ($k=4$) into $+0.00657\pm0.318i$ ($k=5$). (b) Rightmost zero pair of the $r=5$ hook $Q_{5,n}(x)$ stabilizes across $\mathrm{Re}(x)=0$ from $+0.0641\pm3.102i$ ($n=6$) to $-0.0605\pm3.042i$ ($n=7$).

Multiplying by $\binom{k}{j}(x+1)^{2k-2j}$ and summing over $j=0,\dots,k$ yields

$$\begin{aligned}\sum_{r=0}^{k}\mathrm{imm}_{(2^r,1^{2k-2r})}(xI_{2k}+J_2^{\oplus k})z^r &\equiv (1-z)\sum_{j=0}^{k}\binom{k}{j}\big(-(1+z^2)\big)^j\big((x+1)(1+z)\big)^{2k-2j}\\ &=(1-z)\Big((x+1)^2(1+z)^2-(1+z^2)\Big)^k\\ &=(1-z)\Big(x(x+2)(1+z)^2+2z\Big)^k=G_k(x,z)\pmod{z^{k+1}},\end{aligned}$$

so $\mathrm{imm}_{(2^r,1^{2k-2r})}(xI_{2k}+J_2^{\oplus k})=[z^r]G_k(x,z)$ for every $0\le r\le k$.

Part (b). Expanding $G_k(x,z)$ by the binomial theorem gives

$$G_k(x,z)=(1-z)\sum_{m=0}^{k}\binom{k}{m}2^{k-m}z^{k-m}(1+z)^{2m}x^m(x+2)^m.$$

Extracting the coefficient $[z^k]G_k(x,z)$ uses $[z^m](1-z)(1+z)^{2m}=\binom{2m}{m}-\binom{2m}{m-1}=C_m$, yielding $P_{(2^k)}(x;J_2^{\oplus k})=\sum_{m=0}^{k}\binom{k}{m}2^{k-m}C_mx^m(x+2)^m$.

For $k\in\{1,2,3,4\}$, direct exact evaluation of the Hurwitz matrix $H(P_{(2^k)})$ gives strictly positive minors across all $2k$ orders (Table 1). Notice that $P_{(2^k)}(0;J_2^{\oplus k})=2^k>0$, so $P_{(2^k)}$ is strictly Hurwitz stable if and only if its reciprocal polynomial $R_k(t):=t^{2k}P_{(2^k)}(1/t;J_2^{\oplus k})=2^k\sum_{j=0}^{2k}d_j(k)t^{2k-j}$ is strictly Hurwitz stable. Since $t^{2k-2m}(1+2t)^m=\sum_{s=0}^{m}\binom{m}{s}2^st^{2k-2m+s}$, setting $j=2m-s$ ($s=2m-j$) yields

$$d_j(k)=\sum_{m=\lceil j/2\rceil}^{j}\binom{k}{m}C_m\binom{m}{j-m}2^{m-j}\qquad(0\le j\le 7),$$

namely $d_0=1$, $d_1=k$, $d_2=\frac{1}{2}k+k(k-1)$, $d_3=k(k-1)+\frac{5}{6}k(k-1)(k-2)$, and so on through $d_7(k)$. Forming the 4×4 leading Hurwitz minor $\Delta_4(R_k)=2^{4k}\det\begin{pmatrix}d_1&d_3&d_5&d_7\\d_0&d_2&d_4&d_6\\0&d_1&d_3&d_5\\0&d_0&d_2&d_4\end{pmatrix}$ and factoring symbolically

TABLE 1. Exact Hurwitz minor diagnostics for $P_{(2^k)}(x; J_2^{\oplus k})$ and $R_k(t) = t^{2k} P_{(2^k)}(1/t; J_2^{\oplus k})$.

k	$\min_{1\le m\le 2k} \Delta_m(P_{(2^k)})$	$\Delta_3(R_k)$	$\Delta_4(R_k)$	$\max \mathrm{Re}(x)$	Status
1	2	—	—	-1.0000	Stable
2	8	384	768	-0.2929	Stable
3	30	29056	1212672	-0.0950	Stable
4	112	770048	177668096	-0.0237	Stable
5	-7.4468×10^{26}	10567680	-5320474624	$+0.0066$	Unstable
6	-5.1348×10^{41}	24117248	-3115931664384	$+0.0206$	Unstable
7	-1.7756×10^{59}	-2954362880	-381168331522048	$+0.0273$	Unstable

TABLE 2. Exact integer Hurwitz determinant destabilization certificates at constant $\lambda_1 \in \{2, 3, 4\}$.

(λ, M)	$\deg Q$	Neg. Minors	Exact Δ_{last}	$\max \mathrm{Re}(x)$
$(2^{15}, 1, 1),\ J_2^{\oplus 16}$	31	Δ_{30}, Δ_{31}	365-digit neg. int. ($\approx -4.917 \times 10^{364}$)	$+0.001478$
$(3^3),\ J_3^{\oplus 3}$	9	Δ_8, Δ_9	$-1107433047093800394848993 28$	$+0.101840$
$(4^2),\ J_4^{\oplus 2}$	8	Δ_7, Δ_8	-336793649845226176512	$+0.158835$

gives

$$\Delta_4(R_k) = -\frac{2^{4k} k^2 (k-1)}{302400}\, g_7(k),$$

where the degree-7 factor shifted to $k = 5 + u$ expands with strictly positive integer coefficients:

$$\begin{aligned} g_7(5+u) = {} & 4654u^7 + 152754u^6 + 2060104u^5 + 14564304u^4 \\ & + 56878051u^3 + 117407676u^2 + 105605001u + 15343776. \end{aligned}$$

Hence $\Delta_4(R_k) < 0$ for every integer $k \ge 5$, proving that R_k and $P_{(2^k)}$ are unstable for all $k \ge 5$ (Figure 3(a)). Note that the 3×3 minor of R_k (including the d_5 entry in $H(R_k)_{1,3}$) is $\Delta_3(R_k) = -2^{3k} k^2 (k-1)(34k^3 - 146k^2 - 511k + 794)/360$, which remains positive at $k \in \{5, 6\}$ before turning negative for all $k \ge 7$; thus $\Delta_4(R_k)$ is the sharp leading obstruction beginning at $k = 5$.

Part (c). Using the conjugacy-class decomposition on $S_b^{\times m}$, exact integer arithmetic produces the certificates reported in Table 2. Notice that the polynomial $\mathrm{imm}_{(2^{k-1},1,1)}(xI_{2k} + J_2^{\oplus k})$ remains strictly Hurwitz stable (after dividing out the simple root $x = 0$) for all $k \le 15$ ($\max \mathrm{Re}(x) \approx -0.000278$ at $k = 15$) before crossing into $\mathrm{Re}(x) > 0$ at $k = 16$ ($\lambda = (2^{15}, 1, 1)$, $\max \mathrm{Re}(x) \approx +0.001478$). □

7. Computational Verification and Open Frontier

Every algebraic identity, symbolic Hurwitz determinant, Sturm sequence, Rouché–Vieta tail bound, and random PSD trial in Theorems 1.1–1.4 is reproduced deterministically by the verification script `code/verify_immanant_hurwitz.py` and verified by `pytest` via `tests/test_immanant_hurwitz.py` (all 6 test suites passing). In particular, exact integer Hurwitz checks on skewed full-rank and singular PSD matrices verify universal stability for $(2, 2, 1)$ and $(2, 2, 2)$, while extensive numerical trials across $n \le 6$ reveal two structural open problems that frame the remaining frontier.

Conjecture 7.1 (Universal $b = 2$ Positive-Real Separation)**.** *For every $n \ge 4$, every Hermitian PSD matrix $M \succeq 0$, and every $X \in \mathcal{H}_n^+$,*

$$\mathrm{Re}\left(\frac{\mathrm{imm}_{(3,1^{n-3})}(X+M)}{\mathrm{imm}_{(2,1^{n-2})}(X+M)}\right) > 0 \qquad \textit{and} \qquad \mathrm{Re}\left(\frac{\mathrm{imm}_{(2,2,1^{n-4})}(X+M)}{\mathrm{imm}_{(2,1^{n-2})}(X+M)}\right) > 0,$$

and along $X = iyI_n$ *(*$y \in \mathbb{R} \setminus \{0\}$*) every even coefficient of* y^{2m} *in the boundary real part*

$$\mathrm{Re}\Big(\mathrm{imm}_\lambda(iyI_n + M)\,\overline{\mathrm{imm}_{(2,1^{n-2})}(iyI_n + M)}\Big)$$

is nonnegative for both $b = 2$ *shapes* $\lambda \in \{(3,1^{n-3}),(2,2,1^{n-4})\}$*. In contrast,* $b = 3$ *over* $b = 2$ *analogues fail already on* J_n*: e.g., at* $n = 4$*,* $\mathrm{Re}\left(\mathrm{imm}_{(4)}(0.5i\,I_4 + J_4)/\mathrm{imm}_{(3,1)}(0.5i\,I_4 + J_4)\right) \approx -0.0116 < 0$ *and* $\mathrm{Re}\left(\mathrm{imm}_{(4)}(0.01i\,I_4 + J_4)/\mathrm{imm}_{(2,2)}(0.01i\,I_4 + J_4)\right) \approx -2.0 \times 10^4 < 0$*.*

Theorem 1.1(b) establishes positivity for $b = 1$ over $b = 0$, Theorem 1.2(a) resolves the $n = 3$ boundary case $\mathrm{imm}_{(3)}/\mathrm{imm}_{(2,1)}$, and Theorem 1.3(d) proves Conjecture 7.1 on the entire rank-one cone.

Conjecture 7.2 (Skew Row-Removal Ultra-Log-Concavity and the $b \le 3$ Stability Boundary)**.**

(a) For every partition $\lambda \vdash n$*, setting* $b_k(\lambda) := \binom{n}{k} \sum_{\mu \vdash n-k,\, \lambda/\mu \in \mathcal{H}_k} f^\mu$*, the sequence* $\{b_k(\lambda)\}_{k=0}^{\lambda_1}$ *is ultra-log-concave of order* λ_1*:*

$$\left(\frac{b_k(\lambda)}{\binom{\lambda_1}{k}}\right)^2 \ge \left(\frac{b_{k-1}(\lambda)}{\binom{\lambda_1}{k-1}}\right)\left(\frac{b_{k+1}(\lambda)}{\binom{\lambda_1}{k+1}}\right) \qquad (1 \le k \le \lambda_1 - 1).$$

(Verified across all 249 *inner triples for* $1 \le n \le 9$*, and holding with equality at* $n = \lambda_1$ *and strictly for every rectangle* (w^m) *via* $\binom{m+k-1}{k}^2 > \binom{m+k-2}{k-1}\binom{m+k}{k+1}$*.)*

(b) Every partition $\lambda \vdash n$ *with* $b(\lambda) \le 3$ *(namely* (1^n)*,* $(2,1^{n-2})$*,* $(3,1^{n-3})$*,* $(2,2,1^{n-4})$*,* $(4,1^{n-4})$*,* $(3,2,1^{n-5})$*, and* $(2,2,2,1^{n-6})$*) makes* $\mathrm{imm}_\lambda(X + M)$ *universally multivariate Hurwitz stable over* $\mathcal{H}_n^+$ *for every* $M \succeq 0$*. Since* (5) *and* $(5,1)$ *destabilize at* $b(\lambda) = 4$ *(Theorem 1.3(e)) and* (2^5) *destabilizes at* $b(\lambda) = 5$ *(Theorem 1.4(b)),* $b(\lambda) \le 3$ *is the sharp universal stability boundary in* $b(\lambda)$*.*

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