

Sharp Costs and Exact Semigroups from Forgotten Quantum Order

Qutrit extremality, tetrahedral depolarization, and echo-critical clock laws

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Abstract

A balanced sequence of short Hamiltonian kicks has zero first-order drift but retains an order-dependent commutator holonomy. We close two synthesis and implementation gaps in this mechanism. First, for every traceless qutrit target F with eigenvalues $\lambda_1 \geq \lambda_2 \geq \lambda_3$, we solve the inverse Hermitian-commutator problem exactly:

$$\inf_{-i[A,B]=F} \|A\|_2 \|B\|_2 = \max\{\lambda_1 - \lambda_2, \lambda_2 - \lambda_3\} = \frac{\|F\|_1}{2} + |\lambda_2|.$$

Consequently the minimum gauge-invariant Hilbert–Schmidt action of a balanced three-kick realization is

$$S_{2,\min}^2 = 6\sqrt{3}(\|F\|_1 + 2|\lambda_2|).$$

The middle eigenvalue is therefore an exact, previously missing cost variable; it produces a tax between 1 and 3/2 over the trace-norm bound. Explicit weighted shifts and equilateral matrix triangles attain the formula.

Second, among exactly four equal-norm balanced qubit kicks, the forgotten-order curvature gap satisfies the sharp law

$$\text{gap}(-\mathcal{Q}) \leq S^4/108,$$

with equality only for a regular tetrahedron. Its 24 orders generate precisely the six geometric generators $\{\pm 4X, \pm 4Y, \pm 4Z\}$, each four times. The finite effective-holonomy twirl is exactly depolarizing and an explicit inverse-cosine schedule realizes any isotropic unital qubit depolarizing semigroup at every finite step count, with zero ensemble-channel discretization error.

Finally, we derive the diffusion limit from complete physical words rather than assumed effective holonomies. A reversal-symmetric distribution of echoed, pinched words obeys

$$\Phi_y = \mathcal{E} + 4y^4 \mathcal{Q}\mathcal{E} + O_\diamond(y^6),$$

so $y = (t/4n)^{1/4}$ gives $\Phi_y^n \rightarrow e^{t\mathcal{Q}}\mathcal{E}$ at rate $O(n^{-1/2})$. Off-block pulses of size $y^{1+\beta}$ exhibit a sharp clock trichotomy: irrelevant for $\beta > 1$, additive at $\beta = 1$, and Zeno-projective for $0 < \beta < 1$. The complete implementation requires serial action $\Theta(n^{3/4})$. Exact symbolic and deterministic numerical certificates accompany the manuscript.

1 Question, answer, and novelty boundary

Repeated measurements can confine dynamics to a Zeno algebra rather than freeze it [2, 3, 4]. In the preceding work [1], a balanced retained loop

$$\sum_{j=1}^m h_j \in Z(\mathcal{A}) \tag{1}$$

was shown to generate the geometric Hamiltonian

$$H_\sigma = -\frac{i}{2} \sum_{s>r} [h_{\sigma_s}, h_{\sigma_r}], \tag{2}$$

while a uniformly forgotten order generates the frame Laplacian

$$\mathcal{Q} = -\frac{1}{24} \sum_{j < k} \text{ad}_{C_{jk}}^2, \quad C_{jk} = i[h_j, h_k]. \quad (3)$$

That paper proved sharp dimension-free operator-action laws, but its Hilbert–Schmidt lower bound was attained only when the nonzero target spectrum was sign-paired. It also began directly from the effective holonomy $e^{-i\sqrt{t/n}H_\sigma}$ when deriving the forgotten-order semigroup. The present paper closes precisely these two gaps.

The first answer is an exact qutrit synthesis theorem. Trace norm alone is insufficient away from sign-paired spectra; the missing invariant is the magnitude of the middle eigenvalue. The second answer is an exact extremal architecture. A regular tetrahedron uniquely maximizes the curvature gap at fixed equal kick norm, and its entire finite twirl can be calibrated to a target semigroup. The third answer returns to the laboratory word: order/reverse echoes remove the coherent second-order branch before averaging, leaving fourth-order diffusion with a quantitative diamond-norm error. When the kicks have shrinking off-block components, the fourth-order clock exposes a critical exponent $\beta = 1$.

Trace-zero matrices are commutators in great generality [5]; norm bounds for commutators and their extremizers form an extensive subject [7, 6]. The classical literature also characterizes self-commutators of compact operators [8]. These nearby problems must be distinguished from (6): Weiss minimizes over unrestricted complex commutator factors for special diagonal targets, whereas here both factors are Hermitian, equivalently one minimizes a single self-commutator factor as in (9). Commutator product formulas, randomized product formulas, Magnus parity, Pauli semigroups, and strong-coupling Zeno limits are likewise established tools [9, 10, 11, 12, 13]. We do not claim any one of those tools in isolation. After a targeted search, we have not found in the cited literature formula (7) for an arbitrary traceless Hermitian 3×3 target, the associated triangular action, the sharp action-gap classification and finite-step calibration derived from this balanced permutation-curvature architecture, or the complete-word/off-block theorem chain tied to forgotten order. This is a scoped literature statement, not a claim of exhaustive priority.

Relation to the eight-paper source chain

This is a new ninth manuscript, not a replacement for any source paper. The two initial filter papers concern irreducible passive-network suppression and noncommuting calibration filters. The next six manuscripts—*Operational Transport of the Heisenberg Cut*, *Noncommuting Record Projections Erase Classical Information*, *Block-Record Projections Generate Quantum Diffusion*, *Balanced Block Measurements Generate Dissipative Quantum Holonomy*, *Operational Curvature of the Heisenberg Cut*, and *Matrix Isoperimetry and Diffusion from Forgotten Order*—progressively isolate record loss, pinched-word holonomy, operational readout, action bounds, and permutation covariance. Only the last of these supplies equations (2)–(3). None states the qutrit exact cost, the tetrahedral equality theorem, the finite-step calibration, or the physical echo-critical limit proved here.

2 Setup and resource conventions

Let

$$\mathcal{H} = \bigoplus_{a=1}^r \mathcal{H}_a, \quad \mathcal{A} = \bigoplus_{a=1}^r \mathcal{B}(\mathcal{H}_a), \quad \mathcal{E}(X) = \sum_a P_a X P_a.$$

The pinching $\mathcal{E}$ is a completely positive trace-preserving projection. Retained kicks satisfy $h_j = h_j^* \in \mathcal{A}$. Since central terms are invisible to commutators and conjugation on $\mathcal{A}$, equation (1) permits the gauge $\sum_j h_j = 0$ blockwise.

For $X = X^*$ in a d -dimensional block define

$$\|X\|_2 = (\text{Tr } X^2)^{1/2}, \quad \text{dist}_2(X, \mathbb{R}\mathbf{1}) = \min_{c \in \mathbb{R}} \|X - c\mathbf{1}\|_2.$$

For a balanced three-kick loop, its gauge-invariant Hilbert–Schmidt action is

$$S_2 = \sum_{j=1}^3 \text{dist}_2(h_j, \mathbb{R}\mathbf{1}). \quad (4)$$

For qubit vectors $h_j = v_j \cdot \boldsymbol{\sigma}$ we use the operator action

$$S = \sum_j \|h_j\|_\infty = \sum_j |v_j|. \quad (5)$$

The distinct norms in (4) and (5) are deliberate: the qutrit theorem solves a target-sensitive Hilbert-space synthesis problem, whereas the qubit theorem maximizes a mixing gap at fixed pulse amplitude.

All channel norms below are diamond norms. If a pinching occurs after every kick, it is included as a physical operation but assigned no Hamiltonian action. This ideal-measurement convention is explicit; finite detector duration and energetic measurement cost are outside the model.

3 The exact qutrit inverse-commutator norm

For traceless $F = F^* \in M_3(\mathbb{C})$ define

$$\kappa(F) = \inf\{\|A\|_2 \|B\|_2 : A = A^*, B = B^*, -i[A, B] = F\}. \quad (6)$$

The value is unitary invariant.

Theorem 3.1 (Exact qutrit commutator cost). *Let $\lambda_1 \geq \lambda_2 \geq \lambda_3$ be the eigenvalues of a traceless qutrit target F . Then*

$$\kappa(F) = \max\{\lambda_1 - \lambda_2, \lambda_2 - \lambda_3\} = \frac{\|F\|_1}{2} + |\lambda_2|. \quad (7)$$

Proof. Put $C = A + iB$. Direct calculation gives

$$CC^* - C^*C = -2i[A, B] = 2F, \quad \|C\|_2^2 = \|A\|_2^2 + \|B\|_2^2. \quad (8)$$

The rescaling $(A, B) \mapsto (sA, s^{-1}B)$ preserves F and minimizes the last expression at $2\|A\|_2\|B\|_2$. Hence

$$2\kappa(F) = \inf_{CC^* - C^*C = 2F} \|C\|_2^2. \quad (9)$$

Let the squared singular values of C be $p_1 \geq p_2 \geq p_3 \geq 0$, and subtract $p_3\mathbf{1}$ from both CC^* and C^*C . The two resulting positive matrices D, E have the same spectrum $(q_1, q_2, 0)$, where $q_j = p_j - p_3$, and

$$D - E = 2F.$$

Write the eigenvalues of $D - E$ as $\mu_1 \geq \mu_2 \geq \mu_3$. Weyl's inequalities give

$$\mu_1 \leq q_1, \quad \mu_2 \geq -q_2, \quad \mu_2 \leq q_2, \quad \mu_3 \geq -q_1.$$

Therefore both adjacent gaps of $2F$ are at most $q_1 + q_2$, which is at most $p_1 + p_2 + p_3 = \|C\|_2^2$. Equation (9) yields the lower bound in (7).

For attainment diagonalize F . If $\lambda_2 \leq 0$, set

$$C = \sqrt{2\lambda_1} |1\rangle\langle 3| + \sqrt{-2\lambda_2} |3\rangle\langle 2|. \quad (10)$$

Then $(CC^* - C^*C)/2 = F$ and

$$\|C\|_2^2 = 2(\lambda_1 - \lambda_2).$$

If $\lambda_2 \geq 0$, take instead

$$C = \sqrt{2\lambda_2} |2\rangle\langle 1| + \sqrt{-2\lambda_3} |1\rangle\langle 3|, \quad (11)$$

whose squared norm is $2(\lambda_2 - \lambda_3)$. These are the larger adjacent gaps in their respective sign branches. Finally, tracelessness implies $\|F\|_1/2 = \lambda_1$ when $\lambda_2 \leq 0$ and $\|F\|_1/2 = -\lambda_3$ when $\lambda_2 \geq 0$, proving the second equality. $\square$

Remark 3.2. The proof is genuinely three-dimensional. It reduces the lower bound to the two adjacent spectral gaps of a difference of isospectral positive matrices of rank at most two. It does not assert an analogous closed formula in arbitrary dimension.

4 Sharp balanced three-kick action

Theorem 4.1 (Exact qutrit triangular action). *Among balanced three-kick qutrit loops with geometric Hamiltonian $H_{\text{geo}} = F$,*

$$\inf S_2^2 = 12\sqrt{3} \kappa(F) = 6\sqrt{3}(\|F\|_1 + 2|\lambda_2|). \quad (12)$$

The infimum is attained by an equilateral matrix triangle.

Proof. Remove scalar parts and write the three sides as x_1, x_2, x_3 in the real Hilbert space of traceless Hermitian matrices, with $x_1 + x_2 + x_3 = 0$. For the listed orientation,

$$F = \frac{i}{2}[x_1, x_2]. \quad (13)$$

Let $x_2^\perp$ be the component of x_2 orthogonal to x_1 . Since the removed component commutes with x_1 , equation (13) and definition (6) imply

$$\kappa(F) \leq \frac{1}{2} \|x_1\|_2 \|x_2^\perp\|_2 = \Delta,$$

where Δ is the Euclidean area of the Hilbert-space triangle. The sharp triangular isoperimetric inequality gives

$$\Delta \leq \frac{S_2^2}{12\sqrt{3}},$$

with equality only for an equilateral triangle in its spanning plane. This proves the lower bound.

For the upper bound take an optimal $C = A + iB$ from (10) or (11). Both weighted shifts satisfy $\text{Tr } C^2 = 0$. Its real and imaginary parts therefore obey

$$\|A\|_2^2 = \|B\|_2^2 = \frac{1}{2} \|C\|_2^2 = \kappa(F), \quad \langle A, B \rangle_{\text{HS}} = 0.$$

With

$$\alpha = \sqrt{\frac{4}{\sqrt{3}}},$$

the sides

$$\begin{aligned} x_1 &= \alpha A, \\ x_2 &= \alpha \left(-\frac{1}{2}A - \frac{\sqrt{3}}{2}B \right), \\ x_3 &= \alpha \left(-\frac{1}{2}A + \frac{\sqrt{3}}{2}B \right) \end{aligned} \quad (14)$$

are balanced, equilateral, and satisfy $H_{\text{geo}} = F$ after choosing the displayed orientation. Their perimeter squared is $12\sqrt{3}\kappa(F)$. $\square$

Relative to the trace-norm lower bound from the preceding arbitrary-rank theory, the exact qutrit multiplier is

$$\tau(F) = \frac{\|F\|_1 + 2|\lambda_2|}{\|F\|_1} = 1 + \frac{2|\lambda_2|}{\|F\|_1} \in [1, 3/2]. \quad (15)$$

It equals one precisely for the sign-paired spectrum $(a, 0, -a)$. It reaches $3/2$ for $(2a, -a, -a)$ or $(a, a, -2a)$. For example, $F = \text{diag}(2, -1, -1)$ has $S_{2,\min}^2 = 36\sqrt{3}$. Thus the earlier trace-norm law was a sharp universal lower bound, but not a complete target cost.

5 The regular tetrahedron uniquely maximizes forgotten-order mixing

Let $h_j = v_j \cdot \sigma$, $j = 1, \dots, 4$, where $|v_j| = r > 0$ and $\sum_j v_j = 0$. The generator (3) acts on the Bloch vector as a real negative semidefinite matrix. Define the mixing gap as the smallest eigenvalue of $-\mathcal{Q}$ on traceless observables.

Lemma 5.1 (Cauchy–Binet frame identity). *If $M = \sum_{j=1}^4 v_j v_j^\top$, then*

$$\sum_{j < k} (v_j \times v_k)(v_j \times v_k)^\top = \text{cof } M. \quad (16)$$

Proof. Both sides transform covariantly under rotations. In a basis diagonalizing M , the diagonal entries are the sums of squared 2×2 minors of the complementary coordinate pairs, and the off-diagonal entries are their polarized Cauchy–Binet sums. These are exactly the entries of the cofactor matrix. $\square$

Theorem 5.2 (Sharp tetrahedral gap). *For four equal-norm balanced qubit kicks,*

$$\boxed{\text{gap}(-\mathcal{Q}) \leq \frac{64r^4}{27} = \frac{S^4}{108}.} \quad (17)$$

Equality holds if and only if the four v_j are the vertices of a regular tetrahedron centered at the origin, up to an orthogonal transformation.

Proof. Since $C_{jk} = -2(v_j \times v_k) \cdot \sigma$, equations (3) and (16) give the positive Bloch dissipation matrix

$$G = -\mathcal{Q}|_{\text{Bloch}} = \frac{2}{3} [\text{Tr}(\text{cof } M)I_3 - \text{cof } M]. \quad (18)$$

Let $\mu_1 \geq \mu_2 \geq \mu_3 \geq 0$ be the eigenvalues of M . Then

$$\text{gap}(G) = \frac{2}{3} \mu_3 (\mu_1 + \mu_2). \quad (19)$$

Their sum is $4r^2$. Since $\mu_3 \leq 4r^2/3$ and $x(4r^2 - x)$ is increasing on this interval,

$$\mu_3(\mu_1 + \mu_2) \leq \frac{32r^4}{9},$$

which proves (17). Equality forces $\mu_1 = \mu_2 = \mu_3 = 4r^2/3$.

Conversely, $M = (4r^2/3)I_3$ together with balance and equal norms determines the Gram matrix of the four vectors:

$$(v_j \cdot v_k)_{j,k} = \frac{4r^2}{3} \left(I_4 - \frac{1}{4} \mathbf{1}\mathbf{1}^\top \right).$$

Hence $v_j \cdot v_k = -r^2/3$ for $j \neq k$, exactly the regular tetrahedral condition. $\square$

This is a global equality theorem within the class of exactly four equal-norm balanced qubit kicks, not only a certificate for one symmetric example. It also explains why isotropic forgotten-order diffusion appears at the tetrahedron: the optimal second-moment matrix is forced to be scalar. The orthogonal equality class includes reflections; only its $SO(3)$ subgroup is implemented by unitary conjugation.

6 An exact finite-step depolarizing semigroup

Choose the unnormalized tetrahedral kicks

$$h_1 = X + Y + Z, \quad h_2 = X - Y - Z, \quad h_3 = -X + Y - Z, \quad h_4 = -X - Y + Z. \quad (20)$$

Here $r = \sqrt{3}$, $S = 4\sqrt{3}$, and Theorem 5.2 gives gap $64/3$.

Proposition 6.1 (Exact geometric-generator orbit). *As σ ranges over S_4 , the geometric Hamiltonians (2) are*

$$H_\sigma \in \{\pm 4X, \pm 4Y, \pm 4Z\}, \quad (21)$$

each with multiplicity four.

Proof. Substitution of (20) into (2) gives the following exhaustive partition of S_4 (we suppress commas inside an order):

$$\begin{array}{ll} 4X & 1324, 2413, 3241, 4132 \\ -4X & 1423, 2314, 3142, 4231 \\ 4Y & 1432, 2143, 3214, 4321 \\ -4Y & 1234, 2341, 3412, 4123 \\ 4Z & 1243, 2431, 3124, 4312 \\ -4Z & 1342, 2134, 3421, 4213. \end{array}$$

Hence each displayed value occurs four times; reversal pairs opposite rows. The ancillary exact enumeration independently checks the table. $\square$

The proposition turns the asymptotic frame statement into a finite identity. Define

$$\mathcal{T}_u = \frac{1}{24} \sum_{\sigma \in S_4} \text{Ad}\left(e^{-i\sqrt{u}H_\sigma}\right). \quad (22)$$

Theorem 6.2 (Exactly calibrated isotropic qubit depolarization). *The channel (22) fixes the identity and multiplies every Pauli observable by*

$$\eta(u) = \frac{1 + 2\cos(8\sqrt{u})}{3}. \quad (23)$$

Equivalently,

$$\mathcal{T}_u(\rho) = \eta(u)\rho + [1 - \eta(u)]\text{Tr}(\rho)I/2. \quad (24)$$

For any Bloch decay rate $\gamma \geq 0$, time $t \geq 0$, and integer $n \geq 1$, set

$$u_n(\gamma, t) = \frac{1}{64} \arccos^2\left(\frac{3e^{-\gamma t/n} - 1}{2}\right). \quad (25)$$

where arccos denotes its principal branch, selecting the smallest nonnegative pulse, with $0 \leq u_n < \pi^2/144$. Then

$$\mathcal{T}_{u_n}^n(I) = I, \quad \mathcal{T}_{u_n}^n(P) = e^{-\gamma t}P \quad (P = X, Y, Z). \quad (26)$$

Thus the target isotropic unital qubit depolarizing semigroup is realized exactly at every finite n . The calibrated step u_n depends on n ; exactness concerns the ensemble-averaged effective-holonomy channel, not an individual random trajectory or the complete physical word.

Proof. Conjugation by $e^{\mp i4\sqrt{u}X}$ fixes X and rotates the YZ plane through angle $\pm 8\sqrt{u}$. Averaging both signs removes the sine terms. Averaging the three axes gives (23). Equation (25) is the exact inverse relation $\eta(u_n) = e^{-\gamma t/n}$, so multiplication over n steps proves (26). $\square$

The word “exact” here concerns the effective holonomy twirl. The next section asks the harder question: what happens when each holonomy must itself arise from the underlying short kicks and pinchings?

7 Diffusion from complete echoed physical words

For a retained kick define

$$T_{j,y} = \mathcal{E} \circ \text{Ad}(e^{-iyh_j}) \circ \mathcal{E}, \quad W_{\sigma,y} = T_{\sigma_m,y} \circ \cdots \circ T_{\sigma_1,y}. \quad (27)$$

Let $\sigma^R = (\sigma_m, \dots, \sigma_1)$ and form the relative echo

$$R_{\sigma,y} = W_{\sigma,y} \circ W_{\sigma^R,y}^*. \quad (28)$$

The adjoint is implemented physically by reversing the order and the signs of the kicks; no inverse of a dissipative pinching is assumed. Each macro-step contains $2m$ kicks and intervening pinchings.

Let μ be a probability law on orders satisfying

$$\mu(\sigma) = \mu(\sigma^R). \quad (29)$$

Define $\Phi_y^\mu = \sum_\sigma \mu(\sigma) R_{\sigma,y}$ and

$$\mathcal{Q}_\mu = -\frac{1}{2} \sum_\sigma \mu(\sigma) \text{ad}_{H_\sigma}^2. \quad (30)$$

For the uniform law, permutation covariance gives $\mathcal{Q}_\mu = \mathcal{Q}$ from (3).

Lemma 7.1 (Reversal removes the cubic logarithm). *After subtracting the central balance $B = \sum_j h_j$, the unitary word has Magnus expansion*

$$\log U_\sigma(y) = -iyB + y^2\Omega_{2,\sigma} + y^3\Omega_{3,\sigma} + O(y^4),$$

where reversal changes the sign of Ω_2 but not Ω_3 . Consequently

$$\log(U_\sigma(y)U_{\sigma^R(y)}^*) = -2iy^2H_\sigma + O(y^4). \quad (31)$$

Proof. Time reversal changes the sign of the even Magnus terms and preserves the odd ones [11]. The first term $-iyB$ is central and therefore contributes neither to commutators nor to the Baker–Campbell–Hausdorff corrections. Subtracting the reversed logarithm cancels both B and the cubic term, while the second term doubles. Equation (2) fixes the normalization. $\square$

Theorem 7.2 (Complete-word forgotten-order limit). *Let*

$$S = \sum_j \text{dist}(h_j, Z(\mathcal{A})) \quad (32)$$

in operator norm, and choose norm-minimizing central representatives. Their sum remains central by balance, which is all the reversal argument requires. Under (29),

$$\Phi_y^\mu = \mathcal{E} + 4y^4\mathcal{Q}_\mu\mathcal{E} + \mathcal{R}_y, \quad \|\mathcal{R}_y\|_\diamond \leq \frac{(4S)^6}{720}|y|^6. \quad (33)$$

With $y_n(t) = (t/4n)^{1/4}$ and $g_\mu = \|\mathcal{Q}_\mu\|_\diamond$,

$$\left\| (\Phi_{y_n(t)}^\mu)^n - e^{t\mathcal{Q}_\mu}\mathcal{E} \right\|_\diamond \leq \frac{32}{45} \frac{S^6 t^{3/2}}{\sqrt{n}} + \frac{g_\mu^2 t^2}{2n} e^{g_\mu t/n}. \quad (34)$$

The convergence is uniform for t in compact intervals.

Proof. Within $\mathcal{A}$, $R_{\sigma R, y} = R_{\sigma, y}^* = R_{\sigma, y}^{-1}$. Equation (29) pairs every branch with its inverse. By Lemma 7.1, write its logarithm as

$$K_\sigma(y) = -2iy^2 \text{ad}_{H_\sigma} + O(y^4).$$

Thus

$$\frac{e^{K_\sigma(y)} + e^{-K_\sigma(y)}}{2} = I - 2y^4 \text{ad}_{H_\sigma}^2 + O_\diamond(y^6).$$

There is no fifth-order term: the echo logarithm has no cubic term and the inverse pair is a hyperbolic cosine. Averaging gives $4y^4 \mathcal{Q}_\mu$.

The sum of the diamond norms of the $2m$ channel generators is at most $4S$. The sixth derivative of their product is therefore bounded by $(4S)^6$, and Taylor's theorem yields the remainder in (33). Finally compare each step with $e^{t\mathcal{Q}_\mu/n}$. Contractivity of both channels and the exponential Taylor remainder give

$$n \|\mathcal{R}_{y_n}\|_\diamond = \frac{32}{45} \frac{S^6 t^{3/2}}{\sqrt{n}},$$

and

$$n \left\| e^{t\mathcal{Q}_\mu/n} - I - t\mathcal{Q}_\mu/n \right\|_\diamond \leq \frac{g_\mu^2 t^2}{2n} e^{g_\mu t/n}.$$

Telescoping proves (34). $\square$

No polarity bit is needed in Theorem 7.2. Balance and reversal symmetry already remove the fifth order. If the order law is not reversal symmetric, a second-order coherent mean generally survives and accumulates as $\sqrt{n}$ on the fourth-order clock. A fresh random polarity can restore centering, but reversal symmetry is the cleaner hypothesis for retained words.

For (20), Theorem 7.2 gives the primitive limit

$$X, Y, Z \longmapsto e^{-64t/3}(X, Y, Z). \quad (35)$$

Unlike Theorem 6.2, this is an approximation by complete physical words; its certified rate is $O(n^{-1/2})$.

8 Off-block perturbations and the critical clock

Now perturb each retained Hamiltonian by an off-block term $v_j = v_j^*$ with $\mathcal{E}(v_j) = 0$. In each physical kick use

$$\eta(yh_j + y^{1+\beta}v_j), \quad \eta \in \{-1, +1\}, \quad \beta > 0, \quad (36)$$

with a fresh uniform polarity η per macro-step. Here polarity is retained because pinching makes the reduced word nonunitary and reversal alone need not cancel every odd mixed term. Define the positive leakage form on $\mathcal{A}$ by

$$\mathcal{L} = \sum_j \mathcal{E} \text{ad}_{v_j}^2 \mathcal{E}|_{\mathcal{A}}, \quad \langle X, \mathcal{L}X \rangle_{\text{HS}} = \sum_j \|[v_j, X]\|_2^2. \quad (37)$$

Theorem 8.1 (Echo-critical expansion). *Let $\widehat{\Phi}_y^{(\beta)}$ be the uniform average of the pinched echoes over orders and polarities. Then*

$$\widehat{\Phi}_y^{(\beta)} = \mathcal{E} + 4y^4 \mathcal{Q}\mathcal{E} - y^{2+2\beta} \mathcal{L}\mathcal{E} + \mathcal{R}_y^{(\beta)}. \quad (38)$$

Writing $V = \sum_j \|v_j\|_\infty$ and $z = 4(Sy + Vy^{1+\beta})$, a conservative explicit remainder is

$$\begin{aligned} \left\| \mathcal{R}_y^{(\beta)} \right\|_\diamond &\leq 64S^2 V^2 y^{4+2\beta} + \frac{128}{3} S V^3 y^{4+3\beta} \\ &\quad + \frac{32}{3} V^4 y^{4+4\beta} + \frac{e^z z^6}{720}. \end{aligned} \quad (39)$$

Proof. Expand the $2m$ -kick channel product. A single off-block insertion maps a retained operator into an off-block sector and is killed by the terminal pinching. Two insertions in the same second-order leakage excursion give $-y^{2+2\beta}\mathcal{L}$. Four retained insertions give $4y^4\mathcal{Q}$ by Theorem 7.2. Polarity cancels every odd total pulse degree. The remaining terms below total degree six require at least two off-block insertions. Their fourth-degree multinomial bounds are, respectively,

$$\binom{4}{2} \frac{(4S)^2(4V)^2}{4!} = 64S^2V^2, \quad \binom{4}{1} \frac{(4S)(4V)^3}{4!} = \frac{128}{3}SV^3, \quad \frac{(4V)^4}{4!} = \frac{32}{3}V^4.$$

Restoring the corresponding powers of y gives the first three terms of (39). The derivative estimate with total generator norm $z/|y|$ bounds the tail. In three or more blocks, the SV^3 term cannot be discarded because three off-block excursions can close a block cycle. $\square$

Set $y = y_n(t) = (t/4n)^{1/4}$ and

$$a_n(t) = ny_n(t)^{2+2\beta} = 2^{-(1+\beta)}t^{(1+\beta)/2}n^{(1-\beta)/2}. \quad (40)$$

Equation (39) and telescoping imply

$$\left(\widehat{\Phi}_{y_n(t)}^{(\beta)}\right)^n = e^{t\mathcal{Q}-a_n(t)\mathcal{L}}\mathcal{E} + O_\diamond\left(n^{-\min\{\beta/2, 1/2\}}\right). \quad (41)$$

The singular branch below uses the following finite-dimensional estimate. It records both the suppressed excursions and the nonuniform initial layer rather than appealing only to a formal Zeno limit.

Lemma 8.2 (Quantitative compression at strong coupling). *Let $B = B^* \leq 0$ and $L = L^* \geq 0$ act on a finite-dimensional Hilbert space. Let P project onto $\ker L$, and suppose*

$$L|_{(\ker L)^\perp} \geq \lambda I, \quad \lambda > 0.$$

Then for $b > 0$ and $s \geq 0$,

$$\left\|e^{s(B-bL)} - e^{sPBP}P\right\| \leq e^{-b\lambda s} + \frac{2\|B\| + s\|B\|^2}{b\lambda}. \quad (42)$$

Consequently the convergence to $e^{sPBP}P$ is uniform for $s \in [\delta, T]$, while at $s = 0$ it is nonuniform whenever $P \neq I$.

Proof. Write a solution of $u' = (B - bL)u$ as $u = (x, y)$ in $P\mathcal{H} \oplus (I - P)\mathcal{H}$, and denote the corresponding blocks of B by B_{ij} . Since $B \leq 0$, the full evolution and the diagonal semigroups generated by B_{00} and $B_{11} - bL_1$ are contractions, with

$$\left\|e^{s(B_{11}-bL_1)}\right\| \leq e^{-b\lambda s}.$$

Duhamel's formula therefore gives

$$\|y(s)\| \leq e^{-b\lambda s} \|y(0)\| + \frac{\|B_{10}\|}{b\lambda} \|u(0)\|,$$

and, after inserting this estimate into the x equation,

$$\|x(s) - e^{sB_{00}}x(0)\| \leq \frac{\|B_{01}\|}{b\lambda} \|y(0)\| + \frac{s\|B_{01}\|\|B_{10}\|}{b\lambda} \|u(0)\|.$$

Adding the two components and using $\|B_{ij}\| \leq \|B\|$ proves (42). The last assertion follows because the left side at $s = 0$ is $\|I - P\|$. $\square$

Theorem 8.3 (Clock trichotomy). *Let P be the Hilbert–Schmidt projection onto*

$$\ker \mathcal{L} = \mathcal{A} \cap \bigcap_j \{v_j\}'. \quad (43)$$

Then the complete physical words have the limits

$$\boxed{\begin{array}{ll} \beta > 1 & : e^{t\mathcal{Q}}\mathcal{E}, \\ \beta = 1 & : e^{t(\mathcal{Q}-\mathcal{L}/4)}\mathcal{E}, \\ 0 < \beta < 1 & : e^{tP\mathcal{Q}P}P\mathcal{E}, \quad t > 0. \end{array}} \quad (44)$$

The last convergence is uniform on $t \in [\delta, T]$ for $\delta > 0$, but not at $t = 0$ unless $P = I$. For $0 < \beta < 1$ the initial layer has the explicit profile

$$t_n(\tau) = 4\tau n^{-(1-\beta)/(1+\beta)}, \quad \left(\widehat{\Phi}_{y_n(t_n(\tau))}^{(\beta)}\right)^n \longrightarrow e^{-\tau^{(1+\beta)/2}\mathcal{L}}\mathcal{E} \quad (45)$$

for every fixed $\tau \geq 0$.

Proof. For $\beta > 1$, $a_n(t) \rightarrow 0$; for $\beta = 1$, $a_n(t) = t/4$. This proves the first two branches using (41). Suppose $0 < \beta < 1$ and put

$$b_n(t) = \frac{a_n(t)}{t} = 2^{-(1+\beta)} t^{(\beta-1)/2} n^{(1-\beta)/2}, \quad t > 0.$$

Apply Lemma 8.2 with $B = \mathcal{Q}$, $L = \mathcal{L}$, $b = b_n(t)$, and $s = t$. On every $[\delta, T]$, $b_n(t)$ diverges uniformly, so (42) gives

$$\sup_{\delta \leq t \leq T} \left\| e^{t\mathcal{Q}-a_n(t)\mathcal{L}} - e^{tP\mathcal{Q}P}P \right\| \longrightarrow 0.$$

All norms on this finite-dimensional channel space are equivalent, and the remainder in (41) is uniform on compact t -intervals. This proves the stated diamond-norm convergence.

Finally, direct substitution of $t_n(\tau)$ into (40) gives $a_n(t_n(\tau)) = \tau^{(1+\beta)/2}$, while $t_n(\tau)\mathcal{Q} \rightarrow 0$. The remainders in (39) also vanish after multiplication by n , proving (45). Thus the layer interpolates from I on $\mathcal{A}$ at $\tau = 0$ to P as $\tau \rightarrow \infty$; its width is $\Theta(n^{-(1-\beta)/(1+\beta)})$. $\square$

At the critical clock, self-adjoint negativity yields the exact fixed algebra

$$\boxed{\text{Fix}(\mathcal{Q} - \mathcal{L}/4) = \mathcal{A} \cap \bigcap_{j < k} \{C_{jk}\}' \cap \bigcap_j \{v_j\}'.} \quad (46)$$

Indeed, the Dirichlet form is

$$-\langle X, (\mathcal{Q} - \mathcal{L}/4)X \rangle_{\text{HS}} = \frac{1}{24} \sum_{j < k} \| [C_{jk}, X] \|_2^2 + \frac{1}{4} \sum_j \| [v_j, X] \|_2^2. \quad (47)$$

9 The $n^{3/4}$ complete-word clock barrier

The effective holonomy in Theorem 6.2 is conditioned at scale $\sqrt{u} \sim n^{-1/2}$. Its synthesis from a balanced physical loop uses the shorter pulse scale $y \sim n^{-1/4}$ because holonomy is second order, and diffusion is then fourth order. This distinction changes the resource law.

Proposition 9.1 (Serial action of the physical echo). *Each macro-step contains two copies of every retained kick. At $y_n(t) = (t/4n)^{1/4}$, the total retained serial action is*

$$\boxed{\mathfrak{A}_n = 2ny_n(t)S = \sqrt{2}St^{1/4}n^{3/4}.} \quad (48)$$

If the total laboratory time is T and the kick amplitude is bounded by Λ , then

$$\Lambda T \geq \sqrt{2}St^{1/4}n^{3/4}. \quad (49)$$

The $n^{3/4}$ exponent is not a generic Zeno speed limit. It is the bookkeeping cost of this specific complete-word architecture: second-order balanced holonomy, order/reverse echo, and fourth-order accumulation. It is stronger than the $\sqrt{n}$ amplitude barrier for maintaining a conditioned second-order holonomy, but it does not exclude alternative hardware that implements the effective Hamiltonians directly.

10 Certificates and reproducibility

The symbolic certificate uses exact arithmetic for four independent checks:

- (i) both qutrit weighted-shift branches in (10)–(11);
- (ii) the representative equilateral loop for $F = \text{diag}(2, -1, -1)$ and action $36\sqrt{3}$;
- (iii) all 24 tetrahedral holonomies, the $64/3$ frame gap, and physical-echo coefficients through degree five, including exact vanishing of degrees 1, 2, 3, 5 without polarity;
- (iv) a two-block critical architecture with retained kicks $\text{diag}(h_j, h_j)$ and one off-block swap $v_1 = 4 \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$, whose restricted critical spectrum is

$$\{0, -16, -64/3, -112/3\} \quad \text{with multiplicities } (1, 1, 3, 3). \quad (50)$$

The simple zero in (50) certifies a primitive critical semigroup in the retained algebra.

The deterministic numerical suite adds 256 random qutrit targets, 4096 balanced equal-norm qubit quadrilaterals, exact finite-twirl and schedule comparisons, diamond-norm convergence of complete physical echoes, and all three off-block clock regimes. The qutrit target residual is below 1.3×10^{-15} , the largest sampled qubit gap remains below $64/27$, the finite schedule agrees to 2.3×10^{-14} , and the fitted complete-word diamond slope is -0.462 , consistent with the predicted $-1/2$ asymptotic law.

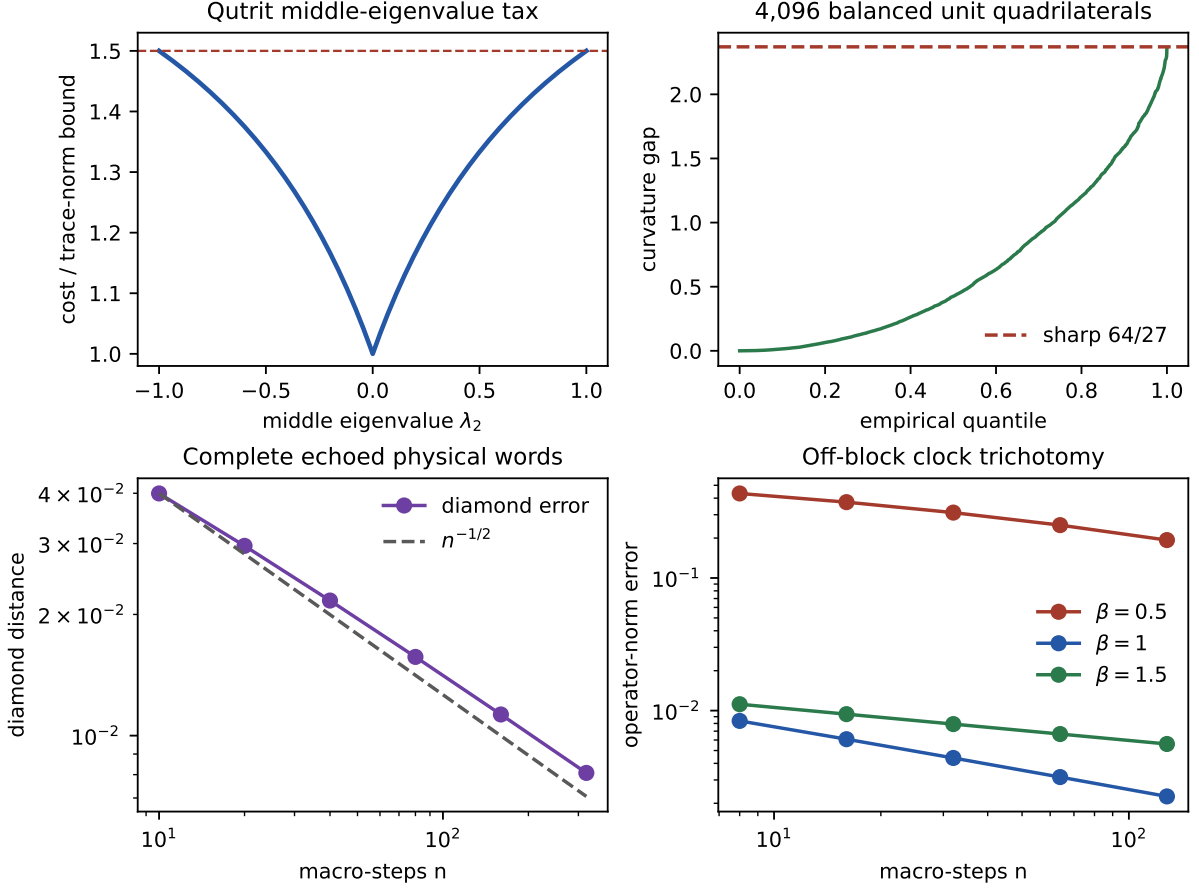


Figure 1: Certified regimes. Top left: the exact qutrit middle-eigenvalue tax (15). Top right: 4096 random balanced unit quadrilaterals below the sharp tetrahedral gap 64/27. Bottom left: complete physical echoes converge in diamond distance with the predicted $n^{-1/2}$ scale. Bottom right: operator-norm convergence in the three off-block clock regimes.

All reported numbers are regenerated from source by one command. The archive contains the manuscript source, exact and numerical scripts, machine-readable JSON outputs, dependency pins, a claim ledger, and SHA-256 manifests. Numerical evidence is not used in place of the analytic proofs.

11 Scope, limitations, and conclusion

The theorem chain is finite-dimensional and uses ideal fixed-block pinchings. Fresh order randomization is required at each macro-step. Equation (34) needs both central balance and reversal symmetry; without them a lower-order coherent term can survive. The polarity in the off-block theorem is not cosmetic, because reduced pinched words need not be invertible. For $0 < \beta < 1$, the correct limit is $e^{tPQP}P$, not $e^{tQ}P$ unless P commutes with Q . The exact finite schedule applies to the effective tetrahedral twirl, whereas the physical-word construction retains its $O(n^{-1/2})$ approximation and $n^{3/4}$ action cost. We make no claim about infinite-dimensional unbounded generators, detector energetics, or optimality over all possible control architectures.

Within that scope, the paper provides three closures. The qutrit target cost is no longer controlled only by trace norm: the middle eigenvalue has an exact operational price. The tetrahedron is no longer merely a convenient primitive example: it is the unique equal-norm maximizer, and its finite twirl admits exact semigroup calibration. Finally, forgotten-order diffusion is no longer only an effective-holonomy statement: it emerges from complete echoed words with a diamond-norm rate, a critical leakage exponent, and a transparent physical clock bill.

AI-assistance disclosure

The author used OpenAI Codex for literature organization, symbolic and numerical code generation, algebraic cross-checks, manuscript drafting, and layout verification. The author directed the research programme, selected the claims, and remains responsible for every theorem, proof, citation, computation, and submission decision. Exact certificates and deterministic replay files are supplied so that the AI-assisted parts can be independently audited.

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