The Reconstructed Theory Has One Mass: a Machine-Checked Volume-Uniform Spectral Gap with Exact Identification Against the Gibbs Sums

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Abstract

For the spatial $\mathbb{Z}_2$ (Ising-slice) system in the Dobrushin window $2\tanh|\beta| + 2\tanh|\gamma| \leq \alpha < 1$, we machine-check in Lean 4 an end-to-end chain: (i) the Osterwalder–Schrader (site-form) reconstruction of the transfer operator is unitarily conjugate — by the explicit $\sqrt{w}$ boundary dressing — to the symmetrised Dobrushin kernel; (ii) the *unnormalised Gibbs sums themselves* are exact matrix elements of the reconstructed operator's powers, $\mathrm{gibbsPathSum}\, w\, \beta\, N\, A\, B = \lambda^N \langle \hat{T}^N QA,\, QB\rangle$ and the partition function is the same shape at the dressed constant — identities, not bounds; and (iii) there is *one* mass $m > 0$ such that for *every* spatial extent L the projected operator norm is at most e^{-m} and every *mixed* connected correlator of the reconstructed operator obeys $\left|\langle u,\, \hat{T}^n v\rangle - \langle \Omega,\, u\rangle\langle \Omega,\, v\rangle\right| \leq \|u\|\,\|v\|\,(e^{-m})^n$, the zero-time case included; (iv) the connected two-point function of the *normalised* Gibbs measure obeys $|\langle A(X_0)B(X_N)\rangle - \langle A(X_0)\rangle\langle B(X_N)\rangle| \leq C(A,B)(e^{-m})^N$ with C independent of the time depth, the division licensed by a denominator floor uniform in N that the positive cone supplies and the spectrum does not (it controls only the even powers); (v) the $N \to \infty$ limit state exists, *is* the vacuum state of the reconstructed operator, and does not depend on the strictly positive observable terminating the chain; and (vi) the reconstructed operator *is* a reversible Markov chain — stochastic and in detailed balance for $\pi = \Omega^2$, both proved — and in that stationary state the connected correlator of bounded observables obeys $|\mathbb{E}_\pi[f\, P^N g] - \mathbb{E}_\pi[f]\mathbb{E}_\pi[g]| \leq K_f K_g\,(e^{-m})^N$ with the quantifier order $\exists m\, \forall L$: *no factor depending on the spatial extent.* So the rate is uniform in the volume throughout, and in the stationary state so is the constant; the end-of-chain state of (v) keeps a boundary-dependent one, and we keep the two apart. The window is non-empty at an interacting point ($\beta = \gamma = 1/10$, $\alpha = 1/2$), machine-checked, so none of these conditionals is vacuous. All quantifiers live in the Lean statements; the prose here copies the kernel's cut. To situate the claim in the 2026 landscape of machine-checked constructive field theory: the abstract construction (reflection-positive data $\to$ physical Hilbert space $\to$ self-adjoint contractive transfer operator) has been machine-checked, sorry-free, in the Harvard formalization programme, *with the spectral gap carried as an undischarged hypothesis* (`GappedTransfer`); the Osterwalder–Schrader axioms have been machine-checked for the free field; and measure-side Dobrushin clustering has been machine-checked for lattice Yang–Mills. What we add, and what to our knowledge no machine-checked artefact yet contains, is the combination {concrete interacting model, gap *proved* rather than hypothesised, uniform in the spatial

*With machine-checked artefacts produced on the programme's Colab plane; toolchain `leanprover/lean4:v4.29.0-rc6`, Mathlib pinned to `0764272048...`. Repository: THE-ERIKSSON-PROGRAMME, branch `d3-closure`, module anchor commit `7230b97b286775781728952d71720902b6e78073`; verification anchor `813658de4a1ac5f4365611c444ebf1785ed679cc` (fresh clone, Addendum 637).

volume} on the reconstructed operator, *identified exactly against the raw Gibbs sums* — i.e. a concrete discharge of exactly the hypothesis shape the abstract frameworks carry. We do not claim Wightman reconstruction, a continuum limit, $SU(N)$, or any progress on the Clay problem.

1 Introduction and landscape

The Osterwalder–Schrader route builds a physical theory out of Euclidean data: a reflection-positive form gives a Hilbert space, and composition with the time shift gives a self-adjoint transfer operator. For a lattice slice system the construction is elementary — the content is never the construction but the *spectrum*: a spectral gap at every fixed spatial extent L is classical (Perron–Frobenius; in this programme it is the earlier `coupled_gap_all_eigenvalues`), and it degenerates as $L \to \infty$ unless something uniform survives. What a thermodynamic limit can actually consume is *one* $m > 0$ valid for every L at once.

That uniform input exists in this repository: the Dobrushin corollary `dobrushin_ising_uniform_gap` produces, inside the window $2\tanh|\beta| + 2\tanh|\gamma| \leq \alpha < 1$, a single $m > 0$ bounding the projected transfer operator of the tilted kernel at every extent. This paper's contribution is the *identification*, and then the *normalisation*. First (Section 3): the operator that bound governs *is* — unitarily, by the explicit $\sqrt{w}$ boundary dressing, with no spectral-transport theory — the transfer operator forced on the reconstructed (site-form) theory. Second (Section 5): the *unnormalised Gibbs sums themselves* are exact matrix elements of that operator's powers — λ^N times an inner product, an identity valid at every real β and every positive weight, with the window entering only when the gap is invoked. Third (Sections 8–13): dividing by the partition function needs a denominator that does not degrade, and the positive cone supplies one where the spectrum cannot, which turns the operator gap into decay of the *measure's* connected correlator and into an infinite-time limit state that forgets its boundary. Fourth (Section 10): read in the normalisation the operator itself provides — the reversible Markov chain it is — the constant loses its dependence on the extent as well.

The analytic content is deliberately cheap: the same witnesses as D-6, no new estimate, no wider window. Cheapness is the honesty of the claim — the uniformity is real because it is D-6's, and what the reconstruction contributes is commuting squares, exact identities, and the right normalisation, all machine-checked.

1.1 Machine-checked prior work, cited before a referee does

The following table is load-bearing and its rows were collected by a four-modality search on 2026-08-04 (evidence: `docs/OS-R-PRIORITY-COLLATION.md` in the repository; the four raw agent reports are appended there verbatim).

Artefact	What is machine-checked	What is not
`mrdouglasny/reflection-positivity` (sorry-free ∼2026-06)	RP data → Hilbert space (completion/quotient) → self-adjoint contraction T; the dictionary $\langle [f], T^n[g]\rangle = \int f\,(g \circ \tau^n \circ \theta)\,d\mu$; trace formulas; susceptibility bounds *from* a gap	the gap itself: carried as the `GappedTransfer` *hypothesis* (`GroundGap.lean`); no spatial-volume-uniform statement; concrete instance is finite-lattice RP only
`OSforGFF` (arXiv:2603.15770)	all five GJ/OS axioms for the d=4 Gaussian free field, 0 sorries	reconstruction (listed as next milestone); any interacting model
`xiyin137/OSreconstruction`	Wightman-side GNS lane of the full OS↔Wightman theorem	the theorem: 53 sorries + 12 axioms at last README snapshot
`mrdouglasny/lgt` (complete 2026-05-04)	Chatterjee-style Dobrushin uniqueness for lattice YM, $U(n)$, with exponential clustering whose constants depend only on (n, d, β)	an operator gap; any reconstruction; the statement is per-volume on the measure side
`pphi2`	transfer-matrix spectral machinery for φ^4_2	rests on 27–29 project axioms
This paper	the $\sqrt{w}$-dressing unitary; the transported volume-uniform gap; the *exact* identities gibbsPathSum $= \lambda^N \langle \hat{T}^N QA, QB\rangle$ and $Z_N = \lambda^N \langle \hat{T}^N Q\mathbf{1}, Q\mathbf{1}\rangle$; volume-uniform *mixed* clustering with the zero-time case; the positive-cone denominator floor, diagonal and off-diagonal, uniform in N; the normalised Gibbs connected correlator; the infinite-time limit state, identified as the vacuum and independent of the positive terminating observable; and the reconstructed operator read as a reversible Markov chain, whose stationary-state clustering has *both* rate and constant free of the extent (final anchor: core 8481, oracle 3055, 0 sorry, fresh-clone verified)	Wightman axioms; continuum limit; $SU(N)$; Clay; the $L \to \infty$ state *itself* — the constants are extent-free, the limit is not constructed (no inclusion maps, no compatibility between π_L and π_{L+1}); boundary independence beyond strictly positive time-terminations

There is a coordinated, active programme (Douglas et al., Harvard CMSA) whose declared live fronts are the OS reconstruction theorem, $P(\varphi)_2$, and YM mass gap at strong coupling, and whose framework carries the spectral gap as an undischarged hypothesis. The endpoint below is a concrete discharge of that hypothesis *shape* (different framework, same mathematical hole), which is why the comparison table above is printed in full before any claim wording.

2 The two geometries and the forced operator

Everything in this section is prior work of the repository, cited by module (the public trace of this lane is the ai.viXra `2607.009x` cluster); no new mathematics appears here. The slice system carries two pairings on complex boundary vectors: the *site* form, which weighs by $1/w$ (`siteForm`, module `SpatialReconstruction`), and the *bond* form carrying the decoupled kernel spatialKernel β (`bondForm`). The transfer operator is *forced*: $T = D_w K$ is the unique operator with $\mathrm{site}(u, Tv) = \mathrm{bond}(u, v)$ (`transferOp_unique`), self-adjoint for the site form at every β (`transferOp_selfAdjoint`) and positive for $\beta \geq 0$; reflection positivity of the underlying measure is the lane's earlier bond/site pair (modules `SpatialReflection`, `SpatialOS`). On the measure side, the same repository carries the Gibbs bridge (module `SpatialGibbs`): the unnormalised two-point path sum `gibbsPathSum` and partition function `gibbsPartition` of the coupled slice system, with the in-tree operator bridge `gibbsPathSum_eq_iterate` (the path sum as an N-fold kernel iterate against dressed observables, $\mathrm{dress}\, w\, A = \sqrt{w}\, A$). On the Dobrushin side it carries `symWeighted` $= \sqrt{w} K \sqrt{w}$ (module `PerronGap`), its normalisation `tiltKernel` $= \mathrm{symWeighted}/\lambda$ (module `DobrushinTilt`), the Euclidean packaging `opOf`, `vacOf`, `vacuumTransfer_opOf` (module `DobrushinTransport`), and the corollary `dobrushin_ising_uniform_gap` (module `DobrushinCorollary`). The seam fact this paper depends on — one `spatialKernel` declaration serving both geometries — is enforced by the elaborator in the module of Section 3.

2.1 The OS provenance: reflection positivity, collapse, null space, physical quotient

The operator of Section 2 is not an ansatz; its Osterwalder–Schrader provenance is machine-checked in the same tree, and we present it here because the reconstruction claim of this paper rests on it. All items below are green, in-tree prior work of the lane (modules `SpatialReflection`, `SpatialOS`, `SpatialReconstruction`); nothing in this subsection is new.

- *Reflection positivity over the measure.* The bond OS pairing of two half-chain observables is literally a sum over whole Gibbs paths — `osPairingBond_eq_gibbsSum`, through the splitting equivalence `bondEquiv` and the weight identity `gibbsWeight_joinBond` — and it is nonnegative on the diagonal for $\beta \geq 0$ (`osPairingBond_nonneg`; the restriction is sharp: `gibbsPathSum_neg_of_neg_beta` exhibits negativity at $\beta < 0$).

- *The collapse and its null space.* Half-chain observables collapse linearly onto boundary vectors (`collapseL`, surjective by `collapseL_surjective`), and the null space of the reflected site form is *exactly* the kernel of the collapse: $F \in \ker(\mathrm{collapseL}) \iff \mathrm{osPairingSite}\, F = 0$ (`mem_ker_collapseL_iff`). The quotient is by something identified, not by an abstract null space that happens to be there.

- *The physical quotient.* The quotient of half-chain observables by that kernel *is* the boundary space, as a bundled linear equivalence (`physicalEquiv`, via `quotKerEquivOfSurjective`).

- *The operator's provenance.* The transfer operator's matrix element between two collapsed half-chain observables is the reflected two-point sum of the Gibbs measure over whole paths, one slice further apart — `osPairing_transfer_gibbsSum`, stated for two observables, not one. Together with forcing (`transferOp_unique`), self-adjointness at every β (`transferOp_selfAdjoint`) and positivity for $\beta \geq 0$ (`transferOp_nonneg`), this is the OS construction of the operator whose spectrum Sections 4–7 control.

At finite extent the site form with $w > 0$ is definite, so on boundary vectors no further quotient intervenes; the quotient content lives one level up, on half-chain observables, where it is the `physicalEquiv` above. This is the honest division of labour between construction and spectrum in this model.

3 The dressing unitary

All statements of this and the following sections live in one module (`YangMills/OS/OSReconstructionUniform.le` sha256 `d9919e256ae59ac9...27bac149`, 20,428 bytes, ten-pass ladder on the plane, final `ELAB_EXIT 0`). Write $Qu := \sqrt{w}\, u$ for the boundary dressing of a real Euclidean vector (`qEmbed`; it coincides with the measure side's `dress`, complexified — `qEmbed_eq_dress` is definitional); the site form weighs by $1/w$.

Theorem 3.1 (A1, `siteForm_qEmbed`). *For $w > 0$ and real u, v:* $\mathrm{siteForm}\, w\, (Qu)\, (Qv) = \sum_\sigma u_\sigma v_\sigma$. *Per site the cancellation is elementary:* $\sqrt{w}u \cdot \sqrt{w}v \cdot (1/w) = uv$.

Theorem 3.2 (A2, `transferOp_qEmbed`). $T(Qu) = Q\big(\mathrm{act}\, (\mathrm{symWeighted}\, w\, \beta)\, u\big)$ *pointwise: the forced transfer operator on a dressed vector is the dressing of the symmetrised kernel's action.*

Theorem 3.3 (A3, `transferOp_iterate_qEmbed`). $T^{[n]}(Qu) = Q\big((\mathrm{act}\, S)^{[n]} u\big)$ *— the shape the in-tree measure bridge `gibbsPathSum_eq_iterate` consumes.*

Together with A1, the dressing is a unitary between the Euclidean ℓ^2 pairing and the site form at every finite extent (the site form with $w > 0$ is definite, so no quotient intervenes — this is the elementary, finite-volume case of the OS construction, and we say so plainly).

4 The transported volume-uniform gap

Theorem 4.1 (`os_reconstruction_uniform_gap`). *Let $\beta, \gamma \in \mathbb{R}$ and $0 < \alpha < 1$ with $2 \tanh|\beta| + 2 \tanh|\gamma| \leq \alpha$. There is $m > 0$ such that for every L there are $\lambda > 0$ and $\Omega > 0$ with the fixed-point equation $\sum_\tau \mathrm{tiltKernel}\, (\mathrm{sliceW}\, \gamma\, L)\, \beta\, \lambda\, \sigma\, \tau\, \Omega_\tau = \Omega_\sigma$, the projected bound $\|\mathrm{projectedTransfer}\, (\mathrm{opOf}\, (\mathrm{tiltKerne}$ e^{-m}, and the A2 identity at $w := \mathrm{sliceW}\, \gamma\, L$.*

The proof is an `obtain` on `dobrushin_ising_uniform_gap` followed by A2: *the same witnesses*, no new analytic content — which is precisely the point. The uniformity is real because it is D-6's; the reconstruction contributes the identification, not a new estimate.

5 Exact identification against the Gibbs sums

The bridge from the raw measure to the operator is a chain of *identities* — no bound appears until the gap is invoked, and none of the statements below needs the window: they hold at every real β, every $\lambda \neq 0$, every positive weight.

Lemma 5.1 (B0b, `act_symWeighted_eq_smul_act_tilt`). $\mathrm{act}\, S\, u = \lambda \cdot \mathrm{act}\,(\mathrm{tilt})\, u$ *for* $\lambda \neq 0$*, and iterating (*`act_symWeighted_iterate_eq_smul_tilt`*, via the scalar transport lemmas* `act_smul_fun`*,* `act_iterate_smul_fun`*):* $(\mathrm{act}\, S)^{[N]} u = \lambda^N \cdot (\mathrm{act}\,\mathrm{tilt})^{[N]} u$*.*

Lemma 5.2 (B0a, `act_iterate_eq_opOf_pow`). $(\mathrm{act}\, M)^{[n]} u = (\mathrm{opOf}\, M)^n (\mathrm{toLp}\, u)$ *pointwise, for any kernel matrix* M*.*

Theorem 5.3 (B1, `gibbsPathSum_eq_inner_pow`). *For* $w > 0$*,* $\beta \in \mathbb{R}$*,* $\lambda \neq 0$*,* $N \in \mathbb{N}$ *and real observables* A, B*:*

$$\mathrm{gibbsPathSum}\, w\, \beta\, N\, A\, B \;=\; \lambda^N \left\langle (\mathrm{opOf}\,(\mathrm{tiltKernel}\, w\, \beta\, \lambda))^N\, (QA),\; QB \right\rangle,$$

an exact identity: the unnormalised Gibbs two-point sum is λ^N *times a matrix element of the reconstructed operator's powers, taken between dressed observables.*

Theorem 5.4 (B2, `gibbsPartition_eq_inner_pow`). *Under the same hypotheses, the partition function is the same shape at the dressed constant:* $\mathrm{gibbsPartition}\, w\, \beta\, N = \lambda^N \langle \hat{T}^N (Q\mathbf{1}),\, Q\mathbf{1}\rangle$*.*

Remark 5.5. *Dividing B1 by B2 cancels* λ^N*, so the* normalised *two-point function equals* $\langle \hat{T}^N QA,\, QB\rangle / \langle \hat{T}^N Q\mathbf{1},\, Q\mathbf{1}\rangle$ *— plain arithmetic from the two identities, recorded here as prose. The quantitative six-term* connected *bound is a theorem of the same module, in division-free form: Section 8. We print the boundary between identity and estimate exactly where the kernel puts it.*

6 Mixed clustering on a generic Hilbert space

The clustering machinery is stated for an arbitrary real Hilbert space H (a `section` with `[NormedAddCommGroup H] [InnerProductSpace ℝ H] [CompleteSpace H]`), so the three lemmas below are reusable against any transfer operator packaged as a `VacuumTransfer` — in particular they are exactly the consumer shape of the `GappedTransfer` hypothesis of the abstract frameworks in Section 1.1.

Lemma 6.1 (`norm_sub_inner_smul_le`). *For* $\|\Omega\| = 1$*:* $\|v - \langle \Omega,\, v\rangle\, \Omega\| \leq \|v\|$ *(Pythagoras; removing the vacuum component does not grow the norm).*

Lemma 6.2 (`pow_apply_norm_le`). $\|S^n v\| \leq r^n \|v\|$ *whenever* $\|S\| \leq r$*,* $r \geq 0$*.*

Theorem 6.3 (`mixed_connCorr_bound`). *Let* T *be a* `VacuumTransfer` *with vacuum* Ω*, and suppose* $\|\mathrm{projectedTransfer}\, T\, \Omega\| \leq r$ *with* $r \geq 0$*. Then for all* $u, v \in H$ *and all* $n \in \mathbb{N}$ including $n = 0$*:*

$$\left|\langle u,\, T^n v\rangle - \langle \Omega,\, u\rangle\, \langle \Omega,\, v\rangle\right| \;\leq\; \|u\|\, \|v\|\, r^n.$$

The $n = 0$ *case is the Pythagoras lemma;* $n \geq 1$ *goes through the in-tree* `projected_pow_succ`*.*

This strictly extends the lane's earlier diagonal clustering (`os_reconstruction_uniform_clustering`, $u = v$, constant $\|v\|^2$), which remains in the module: two *different* observables, and the time-zero correlator is bounded too — the zero-time case is where a naive geometric-series argument is silent, and it is carried by the projection geometry instead.

7 The endpoint

Theorem 7.1 (`os_reconstruction_measure_uniform`). *Let $\beta, \gamma \in \mathbb{R}$ and $0 < \alpha < 1$ with $2\tanh|\beta| + 2\tanh|\gamma| \le \alpha$. There is* one *$m > 0$ such that for* every *spatial extent L there are $\lambda > 0$ and $\Omega > 0$ with:*

(i) the fixed-point equation for tiltKernel (sliceW γ L) β λ *at* Ω*;*

(ii) exact identification*: for every N and all real observables A, B,* gibbsPathSum (sliceW γ L) β N A B $=$ $\lambda^N \langle \hat{T}^N(QA),\, QB\rangle$ *with* $\hat{T} = \mathrm{opOf}\,(\mathrm{tiltKernel}\ldots)$*;*

(iii) volume-uniform mixed decay*: for all u, v in the slice Hilbert space and all $n \ge 0$,* $\big|\langle u,\, \hat{T}^n v\rangle - \langle \mathrm{vacOf}\,\Omega,\, u\rangle\, \langle \mathrm{vacOf}\,\Omega,\, v\rangle\big| \le \|u\|\,\|v\|\,(e^{-m})^n$.

The witnesses of (i)–(iii) are `dobrushin_ising_uniform_gap`'s, consumed verbatim; the packaging is `vacuumTransfer_opOf` + `tiltKernel_symm`; (ii) is B1 at $w = \mathrm{sliceW}\,\gamma\,L$ (whose positivity is the in-tree `sliceW_pos`); (iii) is Theorem `mixed_connCorr_bound` at $r = e^{-m}$. One theorem, three clauses, no new constant: the mass that clause (iii) exhibits is the mass D-6 proved, now measured against the raw Gibbs sums through the exact identity of clause (ii).

8 The division-safe six-term connected bound

The one step the previous version named open is now a theorem, in two layers. The generic layer lives in the same arbitrary-Hilbert-space section as the mixed bound:

Theorem 8.1 (`six_term_connected_bound`). *Let T be a `VacuumTransfer` with vacuum Ω, let $0 \le r \le 1$ with* $\|\mathrm{projectedTransfer}\,T\,\Omega\| \le r$*, and write $F(u,v) := \langle u,\, T^N v\rangle$. Then for all q_A, q_B, q_1 and all N:*

$$\big|F(q_1,q_1)\,F(q_A,q_B) - F(q_A,q_1)\,F(q_1,q_B)\big| \;\le\; 6\,\|q_A\|\,\|q_B\|\,\|q_1\|^2\, r^N.$$

The proof is a ring identity in E-terms ($E(u,v) := F(u,v) - \langle\Omega,\, u\rangle\langle\Omega,\, v\rangle$): the pure product terms cancel *exactly*, six terms survive, each carrying at least one E factor bounded by Theorem `mixed_connCorr_bound`, and the two $E\cdot E$ terms spend one factor through $r^N \le 1$. The constant 6 is the honest count of surviving terms, not an optimised one.

The model layer consumes it at the dressed observables, *in raw Gibbs terms*:

Theorem 8.2 (`os_reconstruction_connected_uniform`). *Let $\beta, \gamma \in \mathbb{R}$ and $0 < \alpha < 1$ with $2\tanh|\beta| + 2\tanh|\gamma| \le \alpha$. There is* one *$m > 0$ such that for* every *spatial extent L there are $\lambda > 0$, $\Omega > 0$ with the fixed-point equation, and for every time depth N and all real observables A, B:*

$$\big|Z_N\, S_N(A,B) - S_N(A,\mathbf{1})\, S_N(\mathbf{1},B)\big| \;\le\; \lambda^{2N}\,\big(6\,\|QA\|\,\|QB\|\,\|Q\mathbf{1}\|^2\big)\,(e^{-m})^N,$$

where $S_N = \mathrm{gibbsPathSum}$*,* $Z_N = \mathrm{gibbsPartition}$*, and* $Q\cdot = \mathrm{toLp}(\mathrm{dress}\,\cdot)$.

This is the normalised bound multiplied through by the positive scale of Z_N^2. To divide it one needs a lower bound for $\langle \hat{T}^N Q\mathbf{1},\, Q\mathbf{1}\rangle$ that does not degrade with N, and that is the content of the next section.

9 The positive cone: a denominator floor uniform in N

A spectral argument gives the floor only along the even powers: writing $Q\mathbf{1} = \langle \Omega,\, Q\mathbf{1}\rangle\Omega + q^{\perp}$ one has $\langle \hat{T}^N Q\mathbf{1},\, Q\mathbf{1}\rangle = \langle \Omega,\, Q\mathbf{1}\rangle^2 + \langle S^N q^{\perp},\, q^{\perp}\rangle$ with S the projected operator, and the correction is sign-controlled only for N even (S^{2k} is positive semidefinite; S^{2k+1} is not). The obstruction is an artefact of the method, not of the object: the kernel here is entrywise *positive*, so the order-theoretic route closes at every N.

Lemma 9.1 (`tiltKernel_pos`). *For $w > 0$ and $\lambda > 0$,* tiltKernel $w\,\beta\,\lambda$ *is entrywise positive — at* every *real β, since* spatialKernel *is a product of positive bond factors (`spatialKernel_pos`). The vacuum is positive too (`vacOf_pos`).*

Lemma 9.2 (cone propagation, `opOf_pow_ge_smul_fix`). *Let M be entrywise nonnegative with* opOf $M\,\Omega = \Omega$*. If $c\,\Omega \le v$ entrywise then $c\,\Omega \le T^n v$ entrywise, for every n. (Induction on n; the step is $\sum_\tau M_{\sigma\tau}(c\,\Omega_\tau) \le \sum_\tau M_{\sigma\tau}(T^n v)_\tau$ and the fixed-point equation. No parity, no spectral input.)*

Theorem 9.3 (the floor, `inner_pow_floor`). *Let M be entrywise nonnegative, Ω the normalised positive fixed vector, v entrywise positive. Then there is $f > 0$ with $f \le \langle T^n v,\, v\rangle$ for* every *n. Explicitly $f = c^2\|\Omega\|^2$ where $c = \min_\sigma(v_\sigma/\Omega_\sigma) > 0$ is the in-tree finite minimum `minWeight`.*

Theorem 9.4 (`os_reconstruction_normalised_clustering`). *In the window there is* one *$m > 0$ such that for* every *spatial extent L: the partition function is positive at every depth, and there is a constant $C(A,B)$ — depending on the observables and on L, but* not *on the time depth N — with*

$$\left|\frac{S_N(A,B)}{Z_N} - \frac{S_N(A,\mathbf{1})}{Z_N}\,\frac{S_N(\mathbf{1},B)}{Z_N}\right| \le C(A,B)\,(e^{-m})^N, \qquad C(A,B) = \frac{6\,\|QA\|\,\|QB\|\,\|Q\mathbf{1}\|^2}{f^2}.$$

The three quotients are the expectations of the normalised Gibbs measure: $S_N(A,B)/Z_N = \langle A(X_0)B(X_N)\rangle$, $S_N(A,\mathbf{1})/Z_N = \langle A(X_0)\rangle$ and $S_N(\mathbf{1},B)/Z_N = \langle B(X_N)\rangle$. So the statement is that the connected two-point function *of the measure* decays at the volume-uniform rate, with a time-independent prefactor. This is the step the earlier version of this manuscript named as open; it is now a theorem, and the honest residue is stated in Section 14: C still depends on L.

10 Constants uniform in the extent: the ground-state chain

The L-dependence of the constant in the *stationary-state* bound is an artefact of *normalisation*, not a property of the object, and we remove it here rather than list it as a limitation. The scope of that claim should be stated before it is made, because it is not the whole of Sections 9–13: what loses its extent is the constant in the stationary state $\pi = \Omega^2$ constructed below. The end-of-chain bound of Section 13 keeps a constant depending on the extent *and* on the terminating observable, and it is not repaired here — that state is a different object, and Remark 10.5 says exactly how.

The dressed observables carry an unnormalised partition function in their Euclidean norm: $\|Q\mathbf{1}\|^2 = \sum_\sigma w_\sigma$, which grows with the extent. Any bound written in those norms therefore has an extent-dependent prefactor *however sharp the operator estimate is.* The right normalisation is the one the operator itself provides.

Theorem 10.1 (the reconstructed dynamics is a reversible Markov chain)**.** *Write* $P(\sigma,\tau) :=$ $\mathrm{tiltKernel}(\sigma,\tau)\,\Omega_\tau/\Omega_\sigma$ *(*`groundKernel`*; the Doob h-transform by the positive Perron vector). Then P is entrywise nonnegative (*`groundKernel_nonneg`*), stochastic — $\sum_\tau P(\sigma,\tau) = 1$, by the fixed-point equation (*`groundKernel_row_sum`*) — and reversible for the ground-state measure $\pi_\sigma := \Omega_\sigma^2$: $\pi_\sigma P(\sigma,\tau) = \pi_\tau P(\tau,\sigma)$ (*`groundKernel_reversible`*). Moreover π is a probability measure, $\sum_\sigma \Omega_\sigma^2 = 1$ (*`sum_vacOf_sq`*), because the vacuum is a unit vector.*

Lemma 10.2 (the intertwining, `opOf_dressVac` and its iterate)**.** *Let $(Uf)_\sigma := \Omega_\sigma f_\sigma$. Then $\hat{T}(Uf) = U(Pf)$, hence $\hat{T}^N(Uf) = U(P^N f)$: the reconstructed operator* is *the Markov step, conjugated by the vacuum. The normalisation of Ω cancels between the two ends.*

Lemma 10.3 (`norm_dressVac_le`)**.** *If $|f_\sigma| \le K$ for all σ then $\|Uf\| \le K$.* This is where the extent disappears*: $\|Uf\|^2 = \sum_\sigma \Omega_\sigma^2 f_\sigma^2 \le K^2 \sum_\sigma \Omega_\sigma^2 = K^2$, and the last step is an equality precisely because Ω is a unit vector — no partition function survives.*

Theorem 10.4 (`os_reconstruction_ground_state_clustering`)**.** *In the window there is* one *$m > 0$ such that for* every *spatial extent L, and for all observables f, g with $|f| \le K_f$, $|g| \le K_g$ pointwise, and every N:*

$$\Big|\mathbb{E}_\pi\big[f \cdot P^N g\big] - \mathbb{E}_\pi[f]\,\mathbb{E}_\pi[g]\Big| \;\le\; K_f\,K_g\,(e^{-m})^N.$$

The quantifier order is $\exists m\ \forall L\ \forall f, g, N$: the constant is $K_f K_g$, supplied by the observables alone, and *nothing in the bound depends on the spatial extent.* This is the same gap as before — m is still D-6's — read in the normalisation that does not smuggle a partition function into the constant. In particular, for observables bounded by 1 the connected correlator is at most e^{-mN}, uniformly in the volume.

Remark 10.5. *The two states in this paper are different objects and we keep them apart. The state of Section 13 is the $N \to \infty$ limit at the* end *of a chain terminated by B; its density against Ω is $Q\mathbf{1}$, and its approach constant is boundary-dependent. The state $\pi = \Omega^2$ of this section is the stationary state of the chain — the marginal in the* middle *of a doubly infinite chain — and it is the one in which the constant is absolute. Both are legitimate; only the second gives an L-uniform bound, and saying which is which is the point.*

11 Non-vacuity, and the summed form

Every theorem above is conditional on the Dobrushin window. A conditional theorem whose hypothesis cannot be satisfied says nothing, so the window's non-emptiness is machine-checked here rather than assumed — and at an *interacting* point, not at the free one.

Lemma 11.1 (`tanh_tenth_le`)**.** $\tanh(1/10) \le 1/9$. *No decimal arithmetic is used: the identity* $\tanh x = 1 - 2/(e^{2x} + 1)$ *reduces the claim to $e^{1/5} \le 5/4$, and that follows from $1 + x \le e^x$ at $x = -1/5$ together with $e^{1/5}e^{-1/5} = 1$.*

Theorem 11.2 (`window_nonempty_interacting`)**.** *There are $\beta, \gamma \ne 0$ and α with $0 < \alpha < 1$ and $2\tanh|\beta| + 2\tanh|\gamma| \le \alpha$ — namely $\beta = \gamma = 1/10$, $\alpha = 1/2$, since $4/9 \le 1/2$.*

So the hypothesis of every theorem in this paper is satisfiable with both couplings switched on, and none of the conditionals is vacuous. We emphasise what this does and does not say:

it makes the statements non-empty, and it says nothing about how large the window is. The window is sufficient, not optimal.

Finally, the extent-free bound of Section 10 is put in the form a thermodynamic-limit argument actually consumes.

Theorem 11.3 (`summed_clustering_of_uniform_bound`). *If $|c_N| \le K_f K_g\, r^N$ for all N, with $r = e^{-m} < 1$, then for every cut-off M*

$$\sum_{N<M} |c_N| \;\le\; \frac{K_f K_g}{1-e^{-m}}.$$

Applied to the ground-state correlator this bounds the summed connected correlation — a susceptibility — by a quantity independent of the cut-off *and* of the spatial extent. The summed form exists only because the constant of Section 10 carries no extent; with an L-dependent prefactor the sum would still converge for each L but would carry that L into every consumer.

12 Reproduction

All numbers below are read from committed run artefacts; predictions were committed *before* their measurements (the programme's standing discipline — the ledger records nine consecutive exact count predictions for this lane before this paper's two).

Stage OS-R-1 (measured, ledger Addenda 616–619)

Quantity	Predicted	Measured
core jobs, unwired	8480 (ledger baseline)	8480
core jobs, wired	$8480 + 1$	8481
oracle reports, wired	$3010 + 7$	3017
`sorryAx` occurrences	0	0
endpoint axiom sets	standard triple	exactly `[propext, Classical.choice, Quot.sound]`

Fresh-clone witness at anchor `8127e5b52d75116b19c172ea022e12c1cab0f8c2`: judges 110/110 in normal and `-O`; core 8481; oracle 3017; module sha256 bit-identical (Addendum 619; run cell `scripts/osr1_pass7_freshclone.py`).

Stage OS-R-2 (predictions Addendum 621, committed before the run; measurements Addendum 622)

Quantity	**Predicted (committed first)**	**Measured**
core jobs	8481 exact (file grew, no module added)	8481
oracle reports	$3017 + 9 = 3026$ exact	3026
`sorryAx` occurrences	0	0
new endpoint axiom sets	standard triple	exactly `[propext, Classical.choice,`
six-term stage (predictions Addendum 623, committed before the run; measured Addendum 624):		
core jobs	8481 exact (file grew again)	8481
oracle reports	$3026 + 2 = 3028$ exact	3028
`sorryAx` occurrences	0	0
new endpoint axiom sets	standard triple	exactly the triple $\times 2$
fresh-clone reproduction	bit-identical module, same counts	judges 110/110 both modes; 8481; 3028;
positive-cone stage (predictions Addendum 626, committed before the run; measured Addendum 627):		
core jobs	8481 exact (file grew a third time)	8481
oracle reports	$3028 + 7 = 3035$ exact	3035
`sorryAx` occurrences	0	0
new endpoint axiom sets	standard triple	exactly the triple $\times 7$
fresh-clone reproduction	bit-identical module, same counts	judges 110/110 both modes; 8481; 3035;
limit-state stage (predictions Addendum 629, committed before the run; measured Addendum 630):		
core jobs	8481 exact (file grew a fourth time)	8481
oracle reports	$3035 + 5 = 3040$ exact	3040
`sorryAx` occurrences	0	0
new endpoint axiom sets	standard triple	exactly the triple $\times 5$
fresh-clone reproduction	bit-identical module, same counts	judges 110/110 both modes; 8481; 3040;
non-vacuity stage (predictions Addendum 635, committed before the run):		
core jobs	8481 exact (import set unchanged)	8481
oracle reports	$3050 + 5 = 3055$ exact	3055
`sorryAx` occurrences	0	0
fresh-clone reproduction	bit-identical module, same counts	judges 110/110 both modes; 8481; 3055;
ground-state stage (predictions Addendum 632, committed before the run; measured Addendum 633):		
core jobs	8481 exact (file grew a fifth time)	8481
oracle reports	$3040 + 10 = 3050$ exact	3050
`sorryAx` occurrences	0	0
new endpoint axiom sets	standard triple	exactly the triple $\times 10$
fresh-clone reproduction	bit-identical module, same counts	judges 110/110 both modes; 8481; 3050;

The core count is the tenth consecutive exact count prediction of this lane; the six-term, cone, limit-state and ground-state stages made it fourteen: the module file grew from 12,347 to 20,428 to 32,533 to 43,213 to 54,745 to 65,975 bytes across the five stages and the job total never moved. Module: `YangMills/OS/OSReconstructionUniform.lean`, final sha256 `b60120d38d0632d7088b34ddeec8ba004f8` (70,976 bytes), banked at commit `7230b97b2` (ground-state stage: `dac8e3f5...`, 65,975 B at `5812678bb`; limit-state stage: `ec19ffa7...`, 54,745 B at `6ff9074b3`; cone stage: `2f2f4cf9...`, 43,213 B at `6ef9aee03`; six-term stage: `3b3598eb...`, 32,533 B at `b47dda50e`; measure stage: `d9919e25...`, 20,428 B at `3b0dfa9e8`) after the plane printed `ELAB_EXIT 0` on byte-identical source (the cell prints size and sha256 before elaborating; both matched the local file before banking). Pass history: passes 1–5 (OS-R-1, Addenda 614–616), passes 8–10 (OS-R-2, Addendum 621: the inner-product notation trap at this Mathlib pin, then a single `nlinarith` orientation failure in the Pythagoras lemma, then green). The wiring unit with the registered predictions is committed as `scripts/osr2_pass11_wiring.py` *before* its run, at commit `520e177a7`.

13 The infinite-time limit state

The off-diagonal floor costs one line more than the diagonal one and buys a genuine state-theoretic statement. Fix a strictly positive boundary observable B and terminate the chain with it: the one-point expectation of A at the near end is $\langle A\rangle_{N,B} := S_N(A, B)/S_N(\mathbf{1}, B)$, a quantity that depends on the chain length N and on the boundary condition B.

Lemma 13.1 (off-diagonal floor, `inner_pow_floor_offdiag`). *For M entrywise nonnegative with positive fixed vector Ω and u, v entrywise positive there is $f > 0$ with $f \le \langle T^n u,\, v\rangle$ for every n — namely $f = c\,\langle \Omega,\, v\rangle$ with c the cone constant of u.*

Theorem 13.2 (`inner_ratio_approx`, generic Hilbert space). *With T a `VacuumTransfer`, $\|\text{projectedTransfer}\| \le r$, and $d \le \langle q_1,\, T^N q_B\rangle$, $d > 0$, $\langle \Omega,\, q_1\rangle > 0$:*

$$\left|\frac{\langle q_A,\, T^N q_B\rangle}{\langle q_1,\, T^N q_B\rangle} - \frac{\langle \Omega,\, q_A\rangle}{\langle \Omega,\, q_1\rangle}\right| \;\le\; \frac{2\|q_A\|\|q_B\|\|q_1\|}{d\,\langle \Omega,\, q_1\rangle}\, r^N.$$

Theorem 13.3 (`os_reconstruction_vacuum_state_limit`). *In the window there is* one $m > 0$ *such that for* every *spatial extent, every observable A and every strictly positive boundary observable B, there is a constant C (independent of N) with*

$$\left|\langle A\rangle_{N,B} - \frac{\langle \Omega,\, QA\rangle}{\langle \Omega,\, Q\mathbf{1}\rangle}\right| \;\le\; C\,(e^{-m})^N.$$

Three readings sit in that one inequality. *(i) The limit exists*, and is approached exponentially at the volume-uniform rate. *(ii) The limit is the vacuum state*: the right-hand expression is the Perron vector's expectation, so the $N \to \infty$ state of the finite-extent chain is the ground state of the reconstructed transfer operator. *(iii) The limit does not depend on the boundary condition*: B appears on the left and *not* on the right, so any two strictly positive terminations give the same limit — boundary condition independence, in the time direction, at fixed spatial extent. None of the three is available without the floor: every one of them is a statement about a quotient, and the quotient's denominator is exactly what a spectral argument fails to control at odd N.

14 What is not claimed

The claims above are finite-extent and time-directional, and the gap between them and the objects a constructive field theorist would want is worth naming item by item rather than leaving to inference.

- *An infinite-volume transfer operator.* Not here. Every operator in this paper acts on the finite-dimensional slice space at a fixed extent L; no inductive limit of the spaces is constructed.
- *A thermodynamic state.* Only in the *time* direction: Section 13 produces the $N \to \infty$ limit state at fixed L and identifies it. The $L \to \infty$ limit is *not* constructed: there is no inductive limit of the slice spaces, no compatibility between the ground states of different extents,

and no state on an infinite spatial lattice. What Section 10 does supply is the missing *uniformity in the constant*: in the stationary state $\pi = \Omega^2$ the connected correlator is bounded by $K_f K_g e^{-mN}$ with no extent-dependent factor, so the finite-L bounds no longer degrade as L grows. That is a necessary condition for a spatial thermodynamic limit; it is not the limit itself, and we do not present it as one. (The end-of-chain state of Section 13 keeps its boundary-dependent constant; the two states are different objects.)

- *General boundary-condition independence.* Only for strictly positive terminations of the chain in the time direction (Section 13). Nothing is proved about spatial boundary conditions, about non-positive or signed terminations, or about independence in the infinite-volume limit.
- *A continuum limit.* Nothing. The lattice spacing never appears.
- *A Wightman theory.* Nothing: no analytic continuation, no OS→Wightman reconstruction, no relativistic spectrum condition. The reconstruction here is the elementary finite-volume site-form construction (the quotient content lives on half-chain observables, Section 2.1), and the theorems are about its *spectrum and its exact relation to the Gibbs sums*, not about the construction being difficult.
- *A consequence for Yang–Mills.* None. The model is the $\mathbb{Z}_2$ Ising slice, not a gauge theory; importance is not inherited by thematic proximity, and only a proved reduction would transfer it. This programme's own audit of its Yang–Mills chain records that the chain dies at the bridge, and nothing in this paper repairs it.

Finally, the window is the Dobrushin window, and it is sufficient, not optimal; outside it the programme's own kill-test (repository, Addendum 610) shows the standard-cone Birkhoff route is projectively blind to the site weight, which is why no claim beyond the window appears here.

References

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