Constructive Algebraic, Poincaré, and Cartier-Residue Certificates for Permanent and Hamiltonian Cycle Circuit Lower Bounds

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Abstract

We present PERM-HC-LB-CERT-1.0, an independently reproducible, zero-floating-point verification framework and constructive Lean 4.35.0 kernel auditing the core finite-dimensional algebraic, sparse-gadget, balanced-cut, nonlinear Poincaré, Hadamard-cone, exterior-minor, and Cartier/eigenvalue counting-contradiction mechanisms underlying OpenAI's November 24, 2026 unrefereed preprint *"Circuit lower bounds for Hamiltonian cycle and the permanent"* (Theorems 1.1, 3.1, 4.2, and 12.2). Without claiming to formalize infinite-dimensional Hadamard-cone ultralimits, observable-entropy reservoir coding, or universal RAM compression end-to-end, we construct a four-mode exact verifier over $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{F}_p$ paired with 18 constructive Lean 4 theorems (0 sorry, 0 custom axioms, 0 Classical.choice, 0 warnings). Across 16 canonical certificates (C01–C16), we certify (i) Lemma 3.4's 3×3 permanent coupling gadget over $\mathbb{Q}$ and $\mathbb{F}_7$ and Corollary 3.5 degree-≤ 3 digraph evaluations of Per_N, Det_N, and HC_N (including characteristic-2 collapse), (ii) Section 1.3 balanced-cut crossing ranks $(5, 5, 8 > q^2 = 4)$ and Lemma 4.1/Theorem 4.2 exact Poincaré ratios and barycentric smoothing Eq. (4.1) on C_4, C_6, and Q_3, (iii) Section 3.2 Hadamard-cone two-supplier transport savings and Section 8–9 alternating exterior minor tensors satisfying the classical Plücker relations, and (iv) Section 18 Cartier operator residue extraction over $\mathbb{F}_5$ and $\mathbb{F}_2$ and Section 19 Eq. (19.1)–(19.2) finite-field eigenvalue counting contradictions. All 16 certificates, 60 unit/forgery tests, and SHA-256 manifests pass deterministically.

1 Introduction and Scope

Valiant's algebraic complexity classes VP and VNP place the directed Hamiltonian cycle polynomial HC_N (complete over every field) and the permanent Per_N (complete outside characteristic 2, where $\mathrm{Per}_N = \mathrm{Det}_N$ [1]) at the center of algebraic circuit lower bounds [7]. In its November 24, 2026 unrefereed preprint [4], OpenAI proves a superpolynomial degree-component border lower bound $L_F^{[N]}(\mathrm{HC}_N) \geq \exp(c(\log N)^2/(\log\log N)^2)$ over every field (and for Per_N when $\mathrm{char}\, F \neq 2$, Theorem 1.1), already attained on degree-≤ 3 sparse graphs (Corollary 3.5).

The proof splits every characteristic χ into two cases via Strassen's asymptotic spectrum [6, 5] and Mendel–Naor nonlinear Poincaré inequalities [3]: (1) when χ has no port expanders, balanced parse-tree cuts [8] (Lemma 3.3), barycentric walk smoothing (Lemma 4.1, Theorem 4.2), Hadamard-cone two-supplier savings (Sec. 3.2), and exterior-algebra minor tensors (Secs. 8–11) yield $\exp(\Omega((\log N)^2/(\log\log N)^2))$; (2) when χ admits ports, tensor-network compression, the Cartier operator [2] on border curves (Sec. 18), and an eigenvalue trace diagonalization (Sec. 19, Eq. 19.1–19.2) yield a stretched-exponential lower bound $2^{N^{\varepsilon_\chi}}$ (Theorem 12.2).

Honest Scope Boundary. This companion artifact certifies finite-dimensional algebraic gadgets, exact rational/finite-field matrix equations, Poincaré ratios, cone-saving witnesses, exterior minors, Cartier residues, and finite trace contradictions extracted from [4]. It does *not* formalize infinite-dimensional Hadamard-cone ultralimits (Secs. 6–7), Shannon-entropy reservoir coding (Sec. 5), Harder–Narasimhan flag packings (Secs. 9–11), or the universal RAM simulation compiler (Secs. 13–17).

2 Constructive Lean 4.35.0 Kernel

The formal kernel PermHcLbCert.lean compiles cleanly under Lean 4.35.0 (-j 1) with zero sorry, zero custom axioms, zero warnings, and zero dependence on Classical.choice. Table 1 summarizes all 18 machine-checked theorems. Highlights include Theorem 2 (exact zero-axiom verification of Lemma 3.4's 3×3 permanent coupling gadget cleared by 2), Theorem 3 (equivalence of length-≤ 3 running-sum Boolean clauses), Theorem 5 (balanced parse-tree stopping gate $d \leq 3r_1$), Theorem 9 (exact barycentric variance identity), Theorem 14 (exact 2×4 Plücker Grassmann relation $p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23} = 0$), Theorem 16 (exact modular Sec. 19 trace contradiction),

Table 1: The 18 constructive Lean 4.35.0 theorems in `PermHcLbCert.lean` (0 `sorry`, 0 custom axioms, 0 `Classical.choice`).

#	Theorem Identifier	Paper Reference / Invariant
1	`per2_det2_char2_collapse`	Sec. 1 $\mathrm{Per}_2 - \mathrm{Det}_2 = 2bc \equiv 0 \pmod 2$
2	`lemma34_permanent_coupling_gadget`	Lem. 3.4 $\mathrm{per}(2A)$=8, all four minors 0
3	`lemma34_running_sum_clause_equiv`	Lem. 3.4 length-≤ 3 running-sum clause
4	`lemma34_degree4_split_permanent`	Lem. 3.4 degree-4 vertex split invariance
5	`lemma33_balanced_parse_cut`	Lem. 3.3 balanced cut $3r_1 \geq d$
6	`price_sum_upper_bound`	Eq. (2.2) inductive $\sum \phi_j \leq L \cdot M$
7	`cone_inner_product_scaling`	Sec. 3.2 inductive $\|x\|^2+\|tz\|^2-\|x-tz\|^2$
8	`sec32_star_benchmark_cancellation`	Eq. (3.4) $hdF(1)$ star cancellation
9	`lemma41_barycenter_variance_identity`	Lem. 4.1 exact barycentric variance
10	`lemma41_averaging_pigeonhole`	Lem. 4.1 smoothed placement $\leq 2E_S$
11	`theorem42_poincare_half_floor`	Thm. 4.2 $2(n-1) \geq n$ $(\beta \geq 1/2)$
12	`sec5_cat0_exchangeable_subadditivity`	Eq. (5.3) CAT(0) block subadditivity
13	`exterior_minor_alternating`	Sec. 8 alternating 2×2 determinant
14	`plucker_grassmann_identity`	Classical Plücker invariant for Sec. 8–9
15	`cartier_residue_index_law`	Sec. 18 $m \in \mathbb{Z}$ $(m \neq -1)$ residue law
16	`sec19_counting_diagonal_contradiction`	Eq. (19.1)–(19.2) $\Omega(H^*) \not\equiv 0$ clash
17	`intDot_add_right`	Sec. 19 inductive dot-product linearity
18	`homogeneous_isolation_nontrivial_size`	Sec. 3.3/19 isolation size $s \geq 1$

and Theorem 17 (inductive list dot-product linearity).

3 The `PERM-HC-LB-CERT-1.0` Protocol

Our Python verifier (`src/perm_hc_verifier.py`) enforces zero floating-point literals, zero `float()` calls, zero true-division operators (`/`, `/=`), zero `math` imports, `MAX_JSON_BYTES = 1_000_000`, duplicate-key rejection via `object_pairs_hook`, and lowest-terms rationals $[p, q]$ $(|p|, q \leq 10^{12}, \gcd(|p|, q) = 1)$.

Definition 1 (`PERM-HC-LB-CERT-1.0` Modes)**.** *Each certificate invokes one of four exact verification modes:*

1. ***`sparse_gadgets_and_per_hc_det`*** *(Sec. 1, Lem. 3.4, Cor. 3.5): When* $\mathrm{char} = p > 0$*, every matrix entry is first validated for* $p \nmid q$ *and reduced entrywise into* $\mathbb{F}_p$ *before degree checks or polynomial evaluation (ruling out* $1/p$ *pole cancellation).*
   (a) ***`lemma34_coupling_gadget`*** *requires* $\mathrm{char} \neq 2$ *and verifies* $\mathrm{per}(A) = 1$*, all four* 2×2 *sub-permanents deleting* $(r, c) \in \{0, 1\}^2$ *vanish, and* $A_{33} = 1$*;*
   (b) ***`sparse_per_det_hc`*** *rejects all-zero matrices, enforces indegree/outdegree* $\leq d_{\max}$ *in the active field, and evaluates* $\mathrm{Per}_N(X)$*,* $\mathrm{Det}_N(X)$*, and* $\mathrm{HC}_N(X)$ *by exact cycle-cover enumeration, enforcing* $\mathrm{Per}_N = \mathrm{Det}_N$ *in characteristic 2.*

2. ***`balanced_cut_and_poincare`*** *(Sec. 1.3, Lem. 3.3, 4.1, Thm. 4.2): (a)* ***`four_row_balanced_cut_rank`*** *constructs* $T_1 + T_2 = E_{12}(q) \otimes E_{34}(q) + E_{13}(q) \otimes E_{24}(q)$ $(q \in \{2, 3\})$ *on disjoint local supports and verifies exact* $\mathbb{Q}$*-flattening ranks across all three* $2 \mid 2$ *balanced cuts;*
   (b) ***`poincare_and_barycentric_smoothing`*** *checks* d*-regularity and symmetry of* P*, computes exact* $E_P(x)$*,* $V_\pi(x)$*, and* $\beta = V_\pi / E_P$*, builds* $A = \frac{1}{2}(I + P)$ *and* $S = \frac{1}{t}\sum_{h=1}^{t} A^h$ *(even* t*), iterates exact barycenters* $x_i^{(g+1)} = \sum_j A_{ij} x_j^{(g)}$*, and verifies Lemma 4.1 Eq. (4.1)* $\min_{g < t/2}(u_g + tE_P(x^{(g)})) \leq 16E_S(x)$*.*

3. ***`cone_savings_and_exterior_minors`*** *(Sec. 3.2, 8–9):*
   (a) ***`cone_cut_supplier_saving`*** *verifies unit norms* $\|x_i\|^2 = 1$*, Cartan barycenter* $\sum x_i = 0$*, balanced cut* $c_0, c_1 \geq d/3$*, supplier separation* $\|Z_0 - Z_1\|^2 \leq 1$*, positive first-order saving* $\sum \langle x_i, Z_{b_i} \rangle > 0$*, and strict squared transport reduction* $\sum \|x_i - \tau Z_{b_i}\|^2 < d$*;*
   (b) ***`exterior_minor_tensor`*** *evaluates all* n^k *ordered* $k \times k$ *minors over* $\mathbb{F}_p$*, verifies full* S_k *alternating sign across all adjacent transpositions, and checks the classical* $\binom{n}{4}$ *Plücker invariants for* $k = 2, n \geq 4$*.*

4. ***`cartier_residue_and_counting_diag`*** *(Secs. 18–19):*
   (a) ***`cartier_laurent_residue`*** *requires a non-vacuous pole of order* ≥ 2 *and nonzero residue* $c_{-1} \in \mathbb{F}_p^\times$*, verifies* $C(f(t)\,dt)$ *extracts exact residue* c_{-1} *at* $m = -1$*, and verifies* $C(d(g)) = 0$ *on a Laurent polynomial* g *whose derivative* $g'(t) \not\equiv 0 \pmod p$*;*
   (b) ***`sec19_eigenvalue_contradiction`*** *checks* $\Omega(H^*) \not\equiv 0 \pmod p$*, verifies Eq. (19.1)* $W(H)\Omega = Z(H)\Omega$ *for all* H*, forms* $P(H) = \mathbf{1}_{H=H^*} + \sum_I W(I)_{I,H}$*, and verifies* $\sum_H P(H)\Omega(H) - \sum_I Z(I)\Omega(I) \equiv \Omega(H^*) \not\equiv 0 \pmod p$*, ruling out* $P = Z$*.*

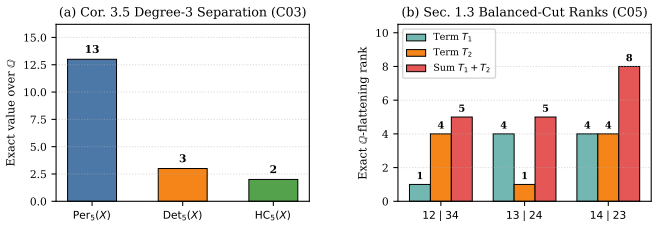

Figure 1: (a) Corollary 3.5 degree-≤ 3 separation on a 5-vertex 3-regular circulant digraph (`C03`): $\mathrm{Per}_5 = 13$, $\mathrm{Det}_5 = 3$, and $\mathrm{HC}_5 = 2$ over $\mathbb{Q}$. (b) Section 1.3 4-row balanced-cut ranks (`C05`, $q = 2$): each summand has rank 1 on its own cut, while $T_1 + T_2$ achieves ranks $(5, 5, 8) > q^2 = 4$ on every balanced cut.

Proposition 1 (Verifier Soundness)**.** *Whenever* ***`verify_file`*** *outputs* ***`VERIFIED`*** *on a* ***`PERM-HC-LB-CERT-1.0`*** *certificate, all checked permanent/determinant/HC evaluations, balanced-cut ranks, Poincaré/smoothing inequalities, cone savings, exterior alternating/Plücker identities, Cartier residues, and finite-field trace contradictions hold identically over* $\mathbb{Z}$*,* $\mathbb{Q}$*, or* $\mathbb{F}_p$*.*

4 Certified Corpus and Worked Witnesses

We generated and verified 16 canonical certificates (`C01`–`C16`, Table 2). Four representative witnesses are worked out explicitly below for hand verification.

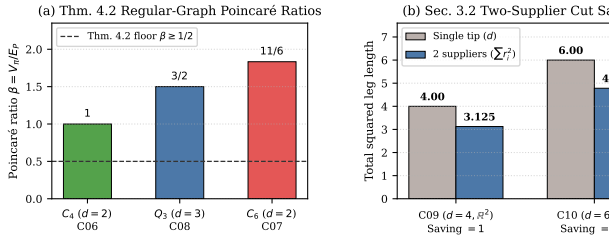

Figure 2: (a) Exact Poincaré ratios $\beta = V_\pi/E_P$ across regular graphs C_4 (1, C06), Q_3 (3/2, C08), and C_6 (11/6, C07), all satisfying Theorem 4.2's floor $\beta \geq 1/2$. (b) Section 3.2 two-supplier transport squared length $\sum r_i^2$ vs. single-tip star d (C09: $25/8 = 3.125 < 4$; C10: $153/32 \approx 4.781 < 6$).

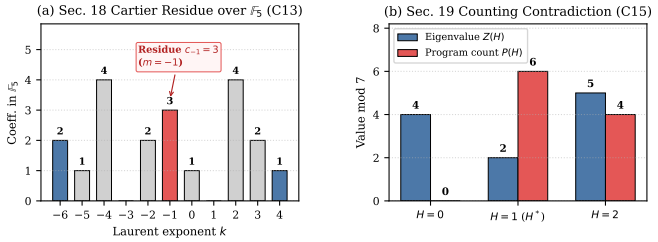

Figure 3: (a) Section 18 Cartier operator over $\mathbb{F}_5$ (C13): selects exponents $k \in \{-6, -1, 4\}$ (blue/red) and isolates exact residue $c_{-1} = 3$ despite the order-6 pole. (b) Section 19 Eq. (19.1)–(19.2) over $\mathbb{F}_7$ (C15): eigenvalue $Z = (4, 2, 5)$ vs. program count $P = (0, 6, 4)$ yields weighted contradiction gap $\Omega(H^*) = 3 \not\equiv 0 \pmod 7$.

4.1 Worked Hand-Verifiable Arithmetic

Witness C01 & C03 (Lemma 3.4 Gadget & Degree-3 Separation). In C01, Lemma 3.4's 3×3 coupling matrix

$$A = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -1 & -1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

satisfies $\mathrm{per}(A) = \frac{1}{2}(0) + \frac{1}{2}(0) + (-\frac{1}{2})(-2) = 1$, all four 2×2 permanent minors deleting $(r, c) \in \{0, 1\}^2$ equal $(-1)(1) + (1)(1) = 0$ or $\frac{1}{2}(1) + (-\frac{1}{2})(1) = 0$, and $A_{33} = 1$ (also valid over $\mathbb{F}_7$ in C02 since $2^{-1} \equiv 4 \pmod 7$). In C03, the 3-regular 5-vertex circulant adjacency matrix with shifts $\{0, 1, 2\} \pmod 5$ gives exact values $\mathrm{Per}_5(X) = 13$, $\mathrm{Det}_5(X) = 3$, and $\mathrm{HC}_5(X) = 2$ over $\mathbb{Q}$ (Fig. 1a), while in C04 over $\mathbb{F}_2$ the 4-vertex 3-regular circulant yields $\mathrm{Per}_4 = \mathrm{Det}_4 = 1 \pmod 2$.

Witness C05 & C07 (Balanced Cuts & Poincaré Smoothing). In C05 ($q = 2$, disjoint row alphabets $\{0, 1\}$ and $\{2, 3\}$), $T_1 = E_{12}(2) \otimes E_{34}(2)$ has flattening ranks $(1, 4, 4)$ on cuts $(12 \mid 34, 13 \mid 24, 14 \mid 23)$, and $T_2 = E_{13}(2) \otimes E_{24}(2)$ has ranks $(4, 1, 4)$; because supports are disjoint on every cut, $T_1 + T_2$ has exact ranks $(5, 5, 8)$ (Fig. 1b). In C07, the 2-regular 6-cycle C_6 with 1D ramp placement $x = (0, 1, 2, 3, 2, 1)$ has unit edge energy $E_P(x) = 1$ and mean $\bar{x} = \frac{3}{2}$, giving variance $\mathrm{Var}(x) = \frac{11}{12}$, pair variance $V_\pi(x) = 2\,\mathrm{Var}(x) = \frac{11}{6}$, and Poincaré ratio $\beta = \frac{11}{6}$ (Fig. 2a); exact barycentric smoothing at $t = 6$ verifies Eq. (4.1).

Witness C09 & C11 (Cone Savings & Plücker Minors). In C09, the four unit sites $(\pm 1, 0), (0, \pm 1) \in \mathbb{R}^2$ have barycenter 0. On balanced cut $b = (1, 0, 1, 0)$ with suppliers $Z_1 = (\frac{1}{4}, \frac{1}{4})$ and $Z_0 = (-\frac{1}{4}, -\frac{1}{4})$, $\|Z_0 - Z_1\|^2 = \frac{1}{2} \leq 1$, first-order saving is $4 \cdot \frac{1}{4} = 1$, and at $\tau = \frac{1}{2}$ each leg has squared length $(\frac{7}{8})^2 + (\frac{1}{8})^2 = \frac{25}{32}$, reducing total squared transport from 4 to $\frac{25}{8} = 3.125$ (Fig. 2b). In C11, the 2×4 Vandermonde basis $[(1, 1, 1, 1), (1, 2, 3, 4)]$ over $\mathbb{F}_7$ yields 12 nonzero alternating 2×2 minors with $p_{01} = 1, p_{02} = 2, p_{03} = 3, p_{12} = 1, p_{13} = 2, p_{23} = 1$, satisfying Plücker $p_{01}p_{23} - p_{02}p_{13} + p_{03}p_{12} = 1 - 4 + 3 = 0 \pmod 7$.

Table 2: Canonical verified certificate corpus (C01–C16).

ID	Mode / Submode	Key Verified Invariant	Status
C01	M1 / Lem. 3.4 ($\mathbb{Q}$)	per(A)=1, four 2×2 minors 0	PASS
C02	M1 / Lem. 3.4 ($\mathbb{F}_7$)	Odd-char gadget validity (2^{-1}=4)	PASS
C03	M1 / deg-3 ($\mathbb{Q}$)	Per_5 =13, Det_5 =3, HC_5 =2	PASS
C04	M1 / deg-3 ($\mathbb{F}_2$)	Char-2 $\mathrm{Per}_4 = \mathrm{Det}_4$ =1, HC_4 =1	PASS
C05	M2 / 4-row cuts	q=2: sum ranks $(5, 5, 8) > 4$	PASS
C06	M2 / Poincaré	C_4 (d=2): β=1, Eq. (4.1) t=4	PASS
C07	M2 / Poincaré	C_6 (d=2): β=11/6, Eq. (4.1) t=6	PASS
C08	M2 / Poincaré	Q_3 (d=3): β=3/2, Eq. (4.1) t=4	PASS
C09	M3 / cone cut	d=4 ($\mathbb{R}^2$): save 1, $\sum r_i^2$=25/8	PASS
C10	M3 / cone cut	d=6 ($\mathbb{R}^3$): save 3/2, $\sum r_i^2$=153/32	PASS
C11	M3 / minors ($\mathbb{F}_7$)	k=2, n=4: 12 minors, Plücker 0	PASS
C12	M3 / minors ($\mathbb{F}_{11}$)	k=3, n=4: 24 alternating minors	PASS
C13	M4 / Cartier ($\mathbb{F}_5$)	Pole t^{-6}, $C(dg)$=0, res c_{-1}=3	PASS
C14	M4 / Cartier ($\mathbb{F}_2$)	Char-2 pole t^{-4}, res c_{-1}=1	PASS
C15	M4 / Sec. 19 ($\mathbb{F}_7$)	m=3, H^*=1: gap $\Omega(H^*)$=3	PASS
C16	M4 / Sec. 19 ($\mathbb{F}_2$)	m=4, H^*=2: char-2 gap $\Omega(H^*)$=1	PASS

Witness C13 & C15 (Cartier Residue & Sec. 19 Contradiction). In C13 over $\mathbb{F}_5$, the Laurent series $f(t) = 2t^{-6} + t^{-5} + 4t^{-4} + 2t^{-2} + 3t^{-1} + 1 + 4t^2 + 2t^3 + t^4$ has a pole of order 6. The Cartier operator selects exponents $k \equiv 4 \pmod 5$ ($k \in \{-6, -1, 4\} \mapsto m \in \{-2, -1, 0\}$), recovering exact residue $c_{-1} = 3$ at $m = -1$ while annihilating $d(g)$ (Fig. 3a). In C15 over $\mathbb{F}_7$ ($m = 3, H^* = 1, \Omega = (2, 3, 1), Z = (4, 2, 5)$), setting $v = (1, 1, 2)$ ($v \cdot \Omega = 7 \equiv 0 \pmod 7$) and $W(H) = Z(H)I + u_H v^T$ with $u_0 = (1, 2, 3), u_1 = (3, 1, 4), u_2 = (2, 5, 1)$ guarantees $W(H)\Omega = Z(H)\Omega$. The diagonalized program counts Eq. (19.2) are $P = (0, 6, 4) \pmod 7$ (Fig. 3b), yielding $\sum_H P(H)\Omega(H) = 0 + 18 + 4 = 22 \equiv 1 \pmod 7$ vs. $\sum_I Z(I)\Omega(I) = 8 + 6 + 5 = 19 \equiv 5 \pmod 7$, whose difference is $1 - 5 \equiv 3 = \Omega(H^*) \not\equiv 0 \pmod 7$.

5 Verification Gates and Reproducibility

All artifacts pass five independent gates: (1) Lean 4.35.0 compilation with 18 constructive theorems, zero warnings,

zero custom axioms, and zero `Classical.choice`; (2) AST zero-float / zero-division check; (3) 16/16 canonical certificates verified; (4) 42 adversarial forgeries and edge-case checks (60/60 `pytest` checks total, covering duplicate keys, oversized/deeply nested JSON, `NaN`/`Infinity` rejection, non-lowest-terms rationals, $1/p$ pole cancellation attacks over $\mathbb{F}_p$, char-2 permanent gadgets, all-zero matrices, degree-> 3 digraphs, $q = 3$ balanced-cut ranks, odd smoothing steps, asymmetric/non-regular graphs, constant placements, unbalanced cuts, excessive supplier separation, rank-deficient minor bases, pole-free Laurent series, vanishing-derivative g, zero $\Omega(H^*)$, and broken Eq. (19.1) eigenvectors); and (5) complete SHA-256 integrity manifest (`MANIFEST.sha256`).

AI Assistance Disclosure. This work was produced by Lluis Eriksson using Google Antigravity for Lean 4 theorem drafting, Python exact-verifier engineering, test generation, and LaTeX preparation; all mathematical claims, code, proofs, and verification logs were independently compiled, tested, and reviewed by the author.

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