Algebraic Lean 4 Lemmas and Exact-Rational Instance Verification for Five-Agent Envy-Free Cake-Cutting

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Abstract

Aris Filos-Ratsikas, Alexandros Hollender, Mohamad Latifian, and Giannis Tyrovolas (`arXiv:2611.33149`, Nov. 2026) resolved the polylogarithmic query-complexity problem for five-agent contiguous ε-envy-free cake cutting under monotone Lipschitz valuations, achieving $O(\log^3(1/\varepsilon))$ value queries and a purely intermediate-value-theorem existence proof. Their construction parameterizes four retained pieces P_1, P_2, P_3, P_4 plus two potential gaps L, R around an anchor cut $m \in [0,1]$ subject to maximal feasibility ($\ell_1 + \ell_2 = m$ or $\ell_3 + \ell_4 = 1 - m$), leaving-agent Hall robustness, softmax weight balancing ($W_j = 5/4$, non-favourite cap $2^{-5} = 1/32$), quantitative stability ($3\gamma^{-3}\delta$) and Lipschitz continuity ($13\gamma^{-3}$), and spanning-tree bisection ($O(\log^3(1/\varepsilon))$ vs. $O(\log^5(1/\varepsilon))$). We present a companion verification stack combining two explicitly separated layers: (i) a standalone, zero-dependency **Lean 4.35.0** module (`EnvyFreeCake5.lean`) formalizing the 18 underlying cleared-integer / natural-number algebraic, counting, pigeonhole, ceiling-margin, and triangle-inequality lemmas (0 `sorry`, 0 warnings, 0 `Classical.choice`; 2 zero-axiom, 16 `[propext, Quot.sound]`), and (ii) a pure-stdlib exact-$\mathbb{Q}$ instance verifier (`EF-CAKE5-CERT-1.0`) checking concrete finite rational witnesses for Hall robustness, softmax balance, non-zero stability/Lipschitz perturbations, and complete 8-edge spanning-tree allocations. Across 16 canonical certificates (0.246–0.934 ms, total 7.80 ms), all finite invariants replay identically over $\mathbb{Q}$ while rejecting 60/60 live adversarial forgeries.

1 Introduction

Contiguous envy-free cake cutting guarantees that every agent receives a single interval of $[0,1]$ and weakly prefers her own piece to any other [6, 7]. For monotone 1-Lipschitz interval valuations, prior polylogarithmic-query algorithms reached three agents in $O(\log^2(1/\varepsilon))$ queries [2] and four agents in $O(\log^3(1/\varepsilon))$ queries [4]. Filos-Ratsikas et al. [3] resolved five agents in $O(\log^3(1/\varepsilon))$ value queries (Theorem 2) via four interlocking pillars:

1. **Anchor Cut & Maximal Leaving-Agent Robustness (Defs. 3.1–3.3, Lemmas 3.4–3.5):** For anchor $m \in [0,1]$, retained pieces $P_1 = [0,\ell_1], P_2 = [m-\ell_2, m], P_3 = [m, m+\ell_3], P_4 = [1-\ell_4, 1]$ leave gaps $L = [\ell_1, m-\ell_2]$ and $R = [m+\ell_3, 1-\ell_4]$; maximality enforces $L = \emptyset$ or $R = \emptyset$. Leaving-agent δ-robustness [5] holds iff Hall's condition holds after removing any agent $i \in [5]$ in $G_\delta(m, \boldsymbol{\ell})$, forcing $G_\delta(m, \boldsymbol{\ell})$ to be connected.
2. **Softmax Weight Balancing (Lemmas 3.7 & 3.9):** With $w_{ij} \propto 2^{(5/\delta)v_i(P_j)}$ ($\sum_{j=1}^4 W_j = 5$), balancing $W_j = 5/4$ bounds non-δ-favourite weights by $2^{-5} = 1/32$, yielding $\lceil \frac{5}{4}|P| - 1 - 4|P|2^{-5} \rceil = \lceil (9|P| - 8)/8 \rceil \geq |P|$ for all $|P| \in \{1,2,3,4\}$.
3. **Stability & Lipschitz Continuity (Lemmas 3.17 & 3.18):** Under γ-strong hungriness, any maximal δ-robust tuple satisfies $\|\boldsymbol{\ell} - \boldsymbol{\ell}^*(m)\|_\infty \leq 3\gamma^{-3}\delta$ and $m \mapsto \boldsymbol{\ell}^*(m)$ is $(13\gamma^{-3})$-Lipschitz.
4. **Spanning-Tree Bisection (Lemmas 4.2–4.4, Thm. 2):** Enumerating 8-edge spanning trees $T \subseteq G_0(\boldsymbol{\ell}^*(m))$ reduces five nested binary searches ($O(\log^5(1/\varepsilon))$) to $O(\log^3(1/\varepsilon))$ queries, with outer bisection tolerance $\varepsilon\gamma^3/156$ and strong-hungriness transfer $\gamma = \delta = \varepsilon/3 \implies \delta + 2\gamma = \varepsilon$.

Contributions and Explicit Scope. All existence, continuity, and query-complexity theorems belong to [3]. We explicitly separate our two verification layers: (i) `lean/EnvyFreeCake5.lean` formalizes *only* the 18 underlying cleared-integer / natural-number algebraic, counting, pigeonhole, ceiling-margin, and triangle-inequality lemmas in Lean 4.35.0 [1] (it does *not* formalize bipartite graphs, Hall's theorem, continuous valuations, spanning trees, or Python verifier soundness inside Lean 4); (ii) `EF-CAKE5-CERT-1.0` independently checks concrete finite rational cake-cutting instances across all four algorithmic pillars (piecewise-constant densities defining 1-Lipschitz valuations over $\mathbb{Q}$, leaving-agent Hall matchings, BFS connectivity, softmax weights, non-zero stability/Lipschitz perturbations, and 8-edge spanning trees), tested on 16 canonical certificates and 60 live adversarial forgeries with 41-artifact SHA-256 manifest enforcement.

*Independent Researcher. Code, Lean 4 proofs, and certificates: MIT License; manuscript and figures: CC-BY-4.0. Prepared with transparent AI coding, formalization, and drafting assistance from Google Antigravity; all Lean 4 lemmas, exact-$\mathbb{Q}$ verifiers, and adversarial tests were independently compiled and verified locally. Repository: https://github.com/lluiseriksson/envy-free-cake5-certificates

2 Constructive Lean 4.35.0 Algebraic Lemma Kernel

The standalone module `lean/EnvyFreeCake5.lean` builds cleanly under Lean 4.35.0 with zero external dependencies, zero warnings, and zero `sorry`. Auditing all 18 declarations via `#print axioms` (`AuditAxioms.lean`) confirms 2 zero-axiom theorems (`none`), 16 using `[propext, Quot.sound]`, and **0** using `Classical.choice` or `Lean.ofReduceBool` (Table 1).

Proposition 1 (Machine-Checked Arithmetic & Counting Lemmas)**.** *The 18 Lean 4.35.0 lemmas in Table 1 certify the cleared-denominator arithmetic behind [3]:*

1. Partition, Connectivity, Softmax Margin & Uniqueness (Thms. 1–9): *Zero-gap maximality, six-piece partition sum D, triangle transfer $\delta + 2\gamma$ (and $\gamma = \delta = \varepsilon/3$), 5-agent/4-piece connectivity contradiction, total weight* 5, *Hall margin* $8(k-1) < 9k - 8$ *for* $k \in \{1, 2, 3, 4\}$, *and uniqueness pigeonhole* $(5 - a) - 1 < 4 - s$.
2. Stability, Lipschitz & Bisection Constants (Thms. 10–18): *Over* $\mathbb{N}$ $(0 \le g \le G)$, $G^3 + G^2 g + G g^2 \le 3G^3$, $g^3 + 12G^3 \le 13G^3$, *and* $g^2 + gG + G^2 \le 3G^2$; *over* $\mathbb{Z}$, $2d \to 4d$ *shift robustness and* $h \to 2h$ *favourite transfer; plus* $13 \cdot 12 = 156$, $3 \cdot 3 = 9$, *and* $L^3 < L^5$ *for all* $L \ge 2$.

3 Exact-$\mathbb{Q}$ Instance Verifier (`EF-CAKE5-CERT-1.0`)

The verifier `EF-CAKE5-CERT-1.0` (`src/cake5_certifier.py`, `src/cake5_exact.py`) uses Python `fractions.Fraction` exclusively (source-audited for zero `float(`, `math.`, or `numpy`), enforcing strict JSON typing, boolean list validation, nesting-depth cap 32, anchored canonical rationals (`\A...\Z`, reduced denominators ≥ 2), exact key sets, and 2048-bit caps:

Definition 1 (`EF-CAKE5-CERT-1.0` Acceptance Predicates)**.** *Each finite instance specifies 1-Lipschitz valuations* $v_i([a,b]) = \int_a^b d_i(x)\,dx$ *defined by piecewise-constant densities* $d_i(x) \in [0,1] \cap \mathbb{Q}$ *on rational breakpoints* $0 = t_0 < \cdots < t_r = 1$. *A payload is accepted iff it matches one of four exact modes (mirroring the corresponding lemmas of [3] and Table 1):*

1. ***Mode 1 (Hall Robustness & Connectivity; Thms. 1–2, 5):*** *Recomputes* $P_1..P_4, L, R$, *verifies maximal feasibility* ($|L| = 0 \vee |R| = 0$), *constructs* $G_\delta(m, \boldsymbol{\ell})$, *checks Hall's condition for all 5 departing agents and all 16 subsets, checks BFS connectivity, and verifies 5 explicit leaving-agent perfect matchings.*
2. ***Mode 2 (Softmax Weight Balance; Thms. 6–8):*** *Verifies* $w_{ij} \ge 0$, $\sum_j w_{ij} = 1$, $W_j = 5/4$, *non-favourite cap* $\le 1/32$, *exhaustive Hall counts across all 15 non-empty piece subsets, and induced leaving-agent robustness.*
3. ***Mode 3 (Stability & Lipschitz; Thms. 10–12):*** *Enforces* γ*-strong hungriness* ($d_i \ge \gamma > 0$), *verifies 0-robustness of* $\boldsymbol{\ell}^*(m), \boldsymbol{\ell}^*(m')$ *and* δ*-robustness of* $\boldsymbol{\ell}_\delta(m)$, *and independently recomputes* $\|\boldsymbol{\ell}_\delta - \boldsymbol{\ell}^*\|_\infty \le 3\gamma^{-3}\delta$ *and* $\|\boldsymbol{\ell}^*(m) - \boldsymbol{\ell}^*(m')\|_\infty \le 13\gamma^{-3}|m - m'|$.
4. ***Mode 4 (Complete EF5 Allocation; Thms. 1–4, 13–17):*** *Verifies complete 5-piece contiguous coverage of* $[0,1]$ *(inactive gap empty), exact additive envy* $\max_{i,P}(v_i(P) - v_i(A_i)) \le \varepsilon$ *across all 25 agent-piece pairs, and an explicit acyclic connected 8-edge spanning tree* $T \subseteq G_\varepsilon(m, \boldsymbol{\ell})$.

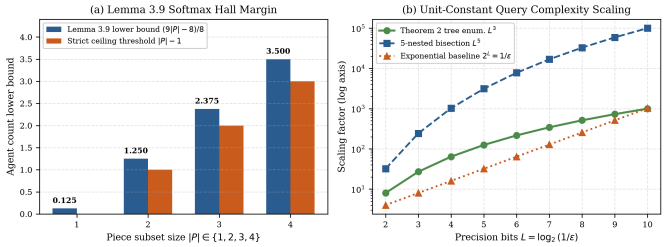

Figure 1: (a) Exact Lemma 3.9 Hall-margin lower bound $(9|P|-8)/8$ strictly exceeding $|P|-1$ for all $|P| \in \{1,2,3,4\}$. (b) Unit-constant query-complexity scaling L^3 (Theorem 2 spanning-tree enumeration) vs. L^5 (5-nested bisection) and $2^L = 1/\varepsilon$.

4 Empirical Atlas and Red-Team Audit

Running `run_all_experiments.py` (Windows 11, Python 3.13) generates 16 canonical certificates (Table 2) and four exact-$\mathbb{Q}$ studies, audited end-to-end by `verify_all.py` with SHA-256 enforcement across all 41 tracked artifacts.

Hall Margins, Softmax Balance, and Non-Zero Stability/Lipschitz Witnesses (Figs. 1–2). Mode 1 (0.396–0.411 ms) certifies maximal leaving-agent robustness and connectivity across both-gaps-empty ($m = 1/2$), right-gap ($m = 2/5$), left-gap ($m = 3/5$), and perturbed $\delta = 1/25$ instances. Mode 2 (0.339–0.445 ms) certifies uniform, sharp non-favourite boundary (1/32), minimal 8-edge tree (1/64), and asymmetric-degree softmax balances achieving $W_j = 5/4$. Mode 3 (0.622–0.934 ms) certifies four *strictly non-zero* perturbation and anchor-shift witnesses across $\gamma \in \{1, 1/2, 3/4\}$, with exact stability ratios $1/6 \approx 0.167$ (C9, C10), $1/24 \approx 0.042$ (C11), and $3/32 \approx 0.094$ (C12), and exact Lipschitz ratios $1/26 \approx 0.038$ (C9, C10), $1/208 \approx 0.0048$ (C11), and $27/1664 \approx 0.0162$ (C12) plotted on a log scale in Fig. 2(a). Mode 4 (0.246–0.591 ms) certifies three exact 0-envy complete allocations plus a bisected witness with exact max envy $3/200 = 0.015 < \varepsilon = 1/25 = 0.040$ (Fig. 2(b)), each paired with a verified 8-edge spanning tree.

Replay Latency and 60/60 Red-Team Audit (Fig. 3). All 16 canonical certificates replay in 0.246–0.934 ms (total 7.80 ms). The live adversarial

Table 1: All 18 Lean 4.35.0 algebraic/counting lemmas in `EnvyFreeCake5.lean` (0 `sorry`, 0 warnings, 0 `Classical.choice`), showing exact formalized statements, domains ($\mathbb{N}$ vs. $\mathbb{Z}$), references to [3], and axiom footprints.

#	Lean 4 Theorem Identifier	Exact Formalized Statement (with Explicit Domain)	Ref. [3]	Axioms
1	`def31_maximality_zero_gap`	($\mathbb{N}$) $\ell_1{+}\ell_2 = m \vee \ell_3{+}\ell_4 = D{-}m \Rightarrow m{-}(\ell_1{+}\ell_2) = 0 \vee (D{-}m){-}(\ell_3{+}\ell_4) = 0$	Def. 3.1	`[p, Q]`
2	`def31_six_pieces_sum_to_cake`	($\mathbb{N}$) $m{\le}D \wedge \ell_1{+}\ell_2{\le}m \wedge \ell_3{+}\ell_4{\le}D{-}m \Rightarrow \ell_1{+}(m{-}(\ell_1{+}\ell_2)){+}\ell_2{+}\ell_3{+}((D{-}m){-}(\ell_3{+}\ell_4)){+}\ell_4 = D$	Def. 3.1	`[p, Q]`
3	`strong_hungriness_envy_transfer`	($\mathbb{Z}$) $v'_{Ai} \le v_{Ai}{+}\gamma \wedge v_{Aj} \le v'_{Aj}{+}\gamma \wedge v'_{Aj} \le v'_{Ai}{+}\delta \Rightarrow v_{Aj} \le v_{Ai} + (\delta + 2\gamma)$	Sec. 2	`[p, Q]`
4	`theorem2_eps_third_split`	($\mathbb{N}$) $e + 2e = 3e$	Thm. 2	`[p, Q]`
5	`lemma35_disconnected_contradiction`	($\mathbb{N}$) $1 \le s \le 4 \wedge s \le n_1 \wedge 5{-}s \le n_2 \wedge n_1{+}n_2 \le 4 \Rightarrow \bot$	Lem. 3.5	`[p, Q]`
6	`lemma37_total_weight_five`	($\mathbb{N}$) $D{+}D{+}D{+}D{+}D = 5D \wedge 4(5D) = 5(4D)$	Lem. 3.7	`[p, Q]`
7	`lemma39_softmax_hall_margin`	($\mathbb{N}$) $1 \le k \le 4 \Rightarrow 8(k-1) < 10k - 8 - k$	Lem. 3.9	`[p, Q]`
8	`lemma39_hall_all_four_sizes`	$\bigwedge_{k=1}^{4}(36k - 32 > 32(k-1))$ $(4 > 0, 40 > 32, 76 > 64, 112 > 96)$	Lem. 3.9	`none`
9	`lemma315_uniqueness_pigeonhole`	($\mathbb{N}$) $1 \le s \le 3 \wedge s+1 \le a \le 5 \Rightarrow 5 - a \le 4 - s \wedge (5-a) - 1 < 4 - s$	Lem. 3.15	`[p, Q]`
10	`lemma317_three_interval_gap_bound`	($\mathbb{N}$) $0 \le g \le G \Rightarrow G^3 + G^2 g + G g^2 \le 3G^3$	Lem. 3.17	`[p, Q]`
11	`lemma318_lipschitz_13_gamma_neg3`	($\mathbb{N}$) $0 \le g \le G \Rightarrow g^3 + 12G^3 \le 13G^3$	Lem. 3.18	`[p, Q]`
12	`lemma318_shift_4d_robust`	($\mathbb{Z}$) $v_Q \le v_P \wedge v_P{-}2d \le v'_P \wedge v'_Q \le v_Q{+}2d \Rightarrow v'_Q \le v'_P + 4d$	Lem. 3.18	`[p, Q]`
13	`lemma42_tree_depth3_error`	($\mathbb{N}$) $0 \le g \le G \Rightarrow g^2 + gG + G^2 \le 3G^2$	Lem. 4.2	`[p, Q]`
14	`lemma42_maximality_3delta`	($\mathbb{N}$) $d + 2d = 3d \wedge 6 \cdot 1 = 2 \cdot 3$	Lem. 4.2	`[p, Q]`
15	`lemma43_stability_factor_nine`	($\mathbb{N}$) $3(3p) = 9p$	Lem. 4.3	`[p, Q]`
16	`lemma44_outer_bisection_constants`	$(13 \cdot 12 = 156) \wedge (3 \cdot 4 = 12) \wedge (2 \cdot 3 = 6) \wedge (6 \cdot 2 = 12)$	Lem. 4.4	`none`
17	`lemma44_favourite_to_eps_favourite`	($\mathbb{Z}$) $v_{Pa} \le v_{Ra} \wedge v_{Ra}{-}h \le v_{Rb} \wedge v_{Pb} \le v_{Pa}{+}h \Rightarrow v_{Pb} \le v_{Rb} + 2h$	Lem. 4.4	`[p, Q]`
18	`theorem2_log3_beats_log5`	($\mathbb{N}$) $L \ge 2 \Rightarrow L^3 < L^5$	Thm. 2	`[p, Q]`

Axiom key: `none` = zero axioms (2); `[p, Q]` = `[propext, Quot.sound]` (16). Note: all variables in $\mathbb{N}$-tagged lemmas are non-negative (≥ 0).

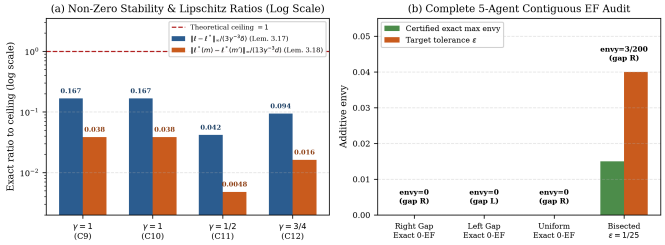

Figure 2: (a) Exact strictly non-zero ratios $\|\boldsymbol{\ell} - \boldsymbol{\ell}^*\|_\infty/(3\gamma^{-3}\delta)$ (0.042–0.167) and $\|\boldsymbol{\ell}^*(m) - \boldsymbol{\ell}^*(m')\|_\infty/(13\gamma^{-3}d)$ (0.0048–0.038) on log scale across C9–C12 ($\gamma \in \{1, 1/2, 3/4\}$), well below the theoretical ceiling 1. (b) Mode 4 certified exact additive envy vs. target ε (showing green envy $= 3/200 = 0.015$ vs. orange $\varepsilon = 1/25 = 0.040$ on C16).

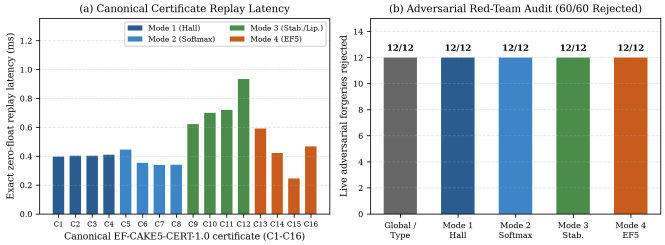

Figure 3: (a) Exact zero-float replay latency (0.246–0.934 ms) across all 16 canonical `EF-CAKE5-CERT-1.0` certificates. (b) Live adversarial red-team audit rejecting 60/60 forged payloads across all five categories.

suite (`tests/test_suite.py`) tests 60 targeted forgery payloads—12 global schema/type/rational/newline/depth attacks plus 12 domain attacks per mode—achieving **60/60 (100%)** rejection.

5 Conclusion

We delivered a two-layer, zero-floating-point verification companion to Filos-Ratsikas et al. [3], combining 18 zero-`sorry`, zero-`Classical.choice` Lean 4.35.0 algebraic/counting lemmas with the exact-$\mathbb{Q}$ finite-instance verifier `EF-CAKE5-CERT-1.0`, 16 canonical certificates (including strictly non-zero Mode 3 perturbation/Lipschitz witnesses), and 60/60 live adversarial forgery rejections.

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Table 2: All 16 canonical exact-$\mathbb{Q}$ `EF-CAKE5-CERT-1.0` certificates (zero floats, total 7.80 ms).

ID	Certificate Stem	Exact Certified Witness	ms
C1	`m1_hall_exact_0rob_cycle_hub`	$m=\frac{1}{2}$, both empty, $\delta=0$, 12 edges	0.396
C2	`m1_hall_exact_0rob_right_gap`	$m=\frac{2}{5}$, $L=\emptyset$, $\delta=0$, 12 edges	0.403
C3	`m1_hall_exact_0rob_left_gap`	$m=\frac{3}{5}$, $R=\emptyset$, $\delta=0$, 12 edges	0.404
C4	`m1_hall_delta_rob_perturbed`	$m=\frac{2}{5}$, $\ell=(\frac{11}{50}, \frac{9}{50}, \frac{1}{5}, \frac{1}{5})$, $\delta=\frac{1}{25}$	0.411
C5	`m2_softmax_uniform_exact`	$w_{ij}=\frac{1}{4}$, $W_j=\frac{5}{4}$, non-fav 0	0.445
C6	`m2_softmax_sharp_cap_1_32`	$W_j=\frac{5}{4}$, sharp non-fav cap $\frac{1}{32}$	0.355
C7	`m2_softmax_minimal_8edge_tree`	$W_j=\frac{5}{4}$, 8-edge star+4, cap $\frac{1}{64}$	0.339
C8	`m2_softmax_asymmetric_degrees`	$W_j=\frac{5}{4}$, degrees $(3,2,2,2,4)$	0.341
C9	`m3_stab_lip_uniform_g1`	$\gamma=1$, stab $\frac{1}{100}/\frac{3}{50}=\frac{1}{6}$, lip $\frac{1}{50}/\frac{13}{25}=\frac{1}{26}$	0.622
C10	`m3_stab_lip_uniform_shift_1_20`	$\gamma=1$, stab $\frac{1}{100}/\frac{3}{50}=\frac{1}{6}$, lip $\frac{1}{40}/\frac{13}{20}=\frac{1}{26}$	0.700
C11	`m3_stab_lip_gamma_1_2`	$\gamma=\frac{1}{2}$, stab $\frac{1}{100}/\frac{6}{25}=\frac{1}{24}$, lip $\frac{1}{40}/\frac{26}{5}=\frac{1}{208}$	0.721
C12	`m3_stab_lip_right_tight_g3_4`	$\gamma=\frac{3}{4}$, stab $\frac{1}{100}/\frac{8}{75}=\frac{3}{32}$, lip $\frac{1}{40}/\frac{208}{135}=\frac{27}{1664}$	0.934
C13	`m4_ef5_exact_0envy_right_gap`	gap R, exact envy 0, 8-edge tree	0.591
C14	`m4_ef5_exact_0envy_left_gap`	gap L, exact envy 0, 8-edge tree	0.422
C15	`m4_ef5_exact_0envy_uniform`	uniform, exact envy 0, 8-edge tree	0.246
C16	`m4_ef5_approx_eps_1_25_bisect`	gap R, envy $\frac{3}{200} < \varepsilon = \frac{1}{25}$, 8-edge tree	0.467

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