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Graph Slepian Area Laws: Exact Variational Commutator Duality, Non-Asymptotic Landau $2WT$ Counting, Dirichlet–Ky Fan Localization, and Binomial-Tail Hypercube Spectra

Lluis Eriksson
Independent Researcher, Östersund, Sweden
editor@airr.science

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Abstract

Let $G = (V, E, w)$ be a finite undirected weighted graph ($n = |V| \geq 2$) with combinatorial Laplacian L, spatial projection $P_S = \mathrm{diag}(\mathbf{1}_S)$ onto a vertex subset $S \subseteq V$, and spectral projection $Q_K = \Phi_K \Phi_K^\top$ onto a frequency band $K \subseteq [n]$ separated from K^c by spectral band gap $\Delta_K := \min_{j\in K, \ell \in K^c} |\mu_\ell - \mu_j| > 0$. We establish a unified spectral, variational, and harmonic-analytic theory for the graph Slepian concentration operator $T_{S,K} := P_S Q_K P_S$ ($1 \geq \lambda_1 \geq \cdots \geq \lambda_n \geq 0$), its effective Shannon dimension $N_{\mathrm{eff}}(S, K) := \mathrm{Tr}(T_{S,K})$, and its plunge-region variance $\Pi(S, K) := \mathrm{Tr}(T_{S,K}(I - T_{S,K})) = \sum_{i=1}^n \lambda_i(1 - \lambda_i)$.

First, we prove the exact variational commutator duality $\Pi(S, K) = \inf_f \frac{1}{2}\|[f(L), P_S]\|_F^2$ over all spectral filters $f : \mathrm{spec}(L) \to \mathbb{R}$ satisfying $|f(\mu_\ell) - f(\mu_j)| \geq 1$ across $K \times K^c$, together with its exact orthogonal defect identity. Taking $f(\mu) = \mu/\Delta_K$ yields the universal *Graph Slepian Area Law* $\Pi(S, K) \leq |\partial_2 S|/\Delta_K^2$, where $|\partial_2 S| := \sum_{u\in S, v\in S^c} w_{uv}^2$ is the squared edge boundary (attained with exact equality on disjoint matchings mK_2), implying the non-asymptotic graph Landau $2WT$ counting bounds $-\Pi/(1-\tau) \leq M_\tau - N_{\mathrm{eff}}(S, K) \leq \Pi/\tau$ for every $\tau \in (0, 1)$ and sharp symmetric threshold $|M_{1/2} - N_{\mathrm{eff}}(S, K)| \leq 2\Pi(S, K) \leq 2|\partial_2 S|/\Delta_K^2$.

Second, via Jordan's principal-angle decomposition of (P_S, Q_K), we prove four-block plunge invariance $\Pi(S, K) = \Pi(S, K^c) = \Pi(S^c, K) = \Pi(S^c, K^c)$, an elementary three-line multi-mode Ky Fan Dirichlet–Cheeger localization bound $\sum_{i=1}^r (1 - \lambda_i(T_{S,K})) \leq \omega_+^{-1} \sum_{i=1}^r \nu_i(S)$ linking Slepian concentration directly to the Dirichlet eigenvalues $\nu_i(S)$ of L_{SS} (and showing that Weyl's Dirichlet eigenvalue count lower-bounds the Slepian well-concentrated count), and a graph uncertainty ceiling ruling out joint localization whenever $N_{\mathrm{eff}}(S, K) < 1$.

Third, on the d-dimensional hypercube Q_d with any codimension-c subcube S ($|S| = 2^{d-c}$) and Hamming-degree band $K_r = \{A \subseteq [d] : |A| \leq r\}$, we derive the exact closed-form binomial-tail Slepian spectrum $\lambda^{(j)} = 2^{-c} \sum_{t=0}^{\min(c, r-j)} \binom{c}{t}$ with multiplicity $\binom{d-c}{j}$ ($0 \leq j \leq \min(r, d-c)$), and for arbitrary Cartesian product graphs $G_1 \square G_2$ we prove the exact multiplicative plunge identity $\Pi(S_1 \times S_2, K_1 \times K_2) = N_{\mathrm{eff}}^{(1)} \Pi_2 + N_{\mathrm{eff}}^{(2)} \Pi_1 - \Pi_1 \Pi_2$.

Fourth, we resolve the minimax least-squares recovery problem for K-bandlimited graph signals observed on S, proving exact amplification $1/\sqrt{\lambda_{|K|}(T_{S,K})}$ and a sharp recovery phase transition on punctured hypercubes $Q_d \setminus S_{\mathrm{miss}}$ at critical codimension $c_c = r + 1$: recovery is ill-posed for $c \leq r$ and well-posed for every $c \geq r + 1$ with exact minimum eigenvalue $1 - 2^{-c} \sum_{t=0}^r \binom{c}{t}$ (e.g., $2^{-3} = 1/8$ at $r = 2, c = 3$). All theorems are verified to machine precision in an open-source reproducibility suite.

Keywords: graph signal processing, Slepian concentration operators, Landau $2WT$ theorem, commutator area laws, Dirichlet Laplacian, hypercube harmonic analysis, bandlimited signal recovery.

1 Introduction

In classical harmonic analysis, Slepian, Landau, and Pollak [6, 3, 2, 5] discovered that the time–frequency concentration operator $P_T Q_W P_T$ exhibits a dramatic spectral dichotomy: its eigenvalues cluster near 1 for the first $\approx 2WT$ modes, plunge sharply across a narrow transition region of width $O(\log(2WT))$, and collapse near 0 thereafter. Extending this phenomenon to irregular finite graphs $G = (V, E, w)$—where spatial domains $S \subseteq V$ have combinatorial boundaries and frequencies are eigenmodes of the graph Laplacian L—is a central problem in discrete harmonic analysis, graph signal processing, and network sampling theory [7, 8, 4].

Existing graph Slepian studies [8] define the concentration operator $T_{S,K} := P_S Q_K P_S$ and empirically observe sharp eigenvalue plunges on community-structured graphs and regular lattices, yet four foundational theoretical questions have remained open or fragmented:

1. **Non-asymptotic graph area law for the plunge width.** On an arbitrary weighted graph G—without assuming Euclidean geometry, regularity, or a commuting tridiagonal differential operator—what exact graph-theoretic invariant bounds the plunge-region variance $\Pi(S, K) := \sum_i \lambda_i(1 - \lambda_i)$ and the deviation $|M_\tau - N_{\text{eff}}(S, K)|$ of the Slepian mode count from the Shannon number $N_{\text{eff}}(S, K) = \text{Tr}(T_{S,K})$?
2. **Deterministic localization lower bounds from Dirichlet spectra.** Given a spatial subset S with small Cheeger edge expansion or low Dirichlet eigenvalues $\nu_1(S) \le \nu_2(S) \le \dots$, how many Slepian modes are guaranteed to concentrate with energy $\ge 1 - \varepsilon$ inside S?
3. **Exact non-trivial closed-form Slepian spectra beyond the cycle.** Beyond the 1D cycle (discrete prolate spheroidal sequences [5]), do high-dimensional canonical graph families such as the d-dimensional hypercube Q_d admit completely exact closed-form Slepian spectra for arbitrary subcubes and bandwidths?
4. **Exact recovery thresholds for bandlimited extrapolation.** When reconstructing a K-bandlimited graph signal from noisy observations on S, what is the sharp geometric threshold separating ill-posed from well-posed sampling sets on high-dimensional structured graphs?

Summary of Main Results

We resolve all four questions through four unified theorems:

- **Theorem 2.2 (Variational Commutator Duality, Graph Slepian Area Law, and Non-Asymptotic Landau $2WT$ Law):** We prove $\Pi(S, K) = \inf_f \frac{1}{2}\|[f(L), P_S]\|_F^2$ over gap-normalized spectral filters, deduce the universal Graph Slepian Area Law $\Pi(S, K) \le |\partial_2 S|/\Delta_K^2$ with an exact equality characterization (saturated by disjoint matchings mK_2), and derive sharp non-asymptotic Landau counting bounds $-\Pi/(1-\tau) \le M_\tau - N_{\text{eff}}(S, K) \le \Pi/\tau$ for all $\tau \in (0, 1)$.
- **Theorem 3.1 (Four-Block Symmetry, Ky Fan Dirichlet–Cheeger Localization, and Uncertainty):** We establish four-block plunge invariance and index relations via Jordan's principal-angle theorem, prove the multi-mode Ky Fan inequality $\sum_{i=1}^r (1 - \lambda_i(T_{S,K})) \le \omega_+^{-1} \sum_{i=1}^r \nu_i(S)$ (implying that the Dirichlet Weyl count lower-bounds the Slepian count), and obtain a spectral-spatial uncertainty ceiling ruling out exact localization whenever $N_{\text{eff}}(S, K) < 1$.
- **Theorem 4.1 (Exact Binomial-Tail Hypercube Spectrum and Cartesian-Product Plunge Identity):** On Q_d with codimension-c subcube S and Hamming-degree band K_r, we prove that T_{S,K_r} has exact eigenvalues $\lambda^{(j)} = 2^{-c} \sum_{t=0}^{\min(c, r-j)} \binom{c}{t}$ of multiplicity $\binom{d-c}{j}$ $(0 \le j \le \min(r, d-c))$, and on arbitrary Cartesian products $G_1 \square G_2$ we prove $\Pi(S_1 \times S_2, K_1 \times K_2) = N_{\text{eff}}^{(1)} \Pi_2 + N_{\text{eff}}^{(2)} \Pi_1 - \Pi_1 \Pi_2$.

- **Theorem 5.1 (Minimax Bandlimited Recovery and Sharp Subcube Phase Transition):** We characterize the exact least-squares noise amplification $1/\sqrt{\lambda_{|K|}(T_{S,K})}$ and prove a sharp recovery phase transition on punctured hypercubes $Q_d \setminus S_{\text{miss}}$ at critical codimension $c_c = r + 1$, with exact minimum recovery eigenvalue $1 - 2^{-c}\sum_{t=0}^{r}\binom{c}{t}$ for all $c \geq r + 1$.

All analytical results, benchmark tables, and automated tests (4/4 pytest modules) are openly archived at https://github.com/lluiseriksson/graph-slepian-area-laws.

2 Variational Commutator Duality and the Graph Slepian Area Law

Let $G = (V, E, w)$ be a finite undirected graph on $n = |V| \geq 2$ vertices with symmetric nonnegative edge weights $w_{uv} = w_{vu} \geq 0$ ($w_{uu} = 0$) and combinatorial Laplacian $L := \operatorname{diag}(W\mathbf{1}) - W \succeq 0$. Fix an orthonormal eigenbasis $\Phi = [\phi_1, \ldots, \phi_n] \in \mathbb{R}^{n\times n}$ of L with sorted eigenvalues $0 = \mu_1 \leq \mu_2 \leq \cdots \leq \mu_n$.

Given a spatial subset $S \subseteq V$ ($s := |S|$) and a spectral band $K \subseteq [n]$ ($k := |K|$, $K^c := [n] \setminus K$), let $P_S := \operatorname{diag}(\mathbf{1}_S)$ and $Q_K := \Phi_K \Phi_K^\top = \sum_{j\in K} \phi_j \phi_j^\top$ be the corresponding orthogonal projections on $\mathbb{R}^n$. Note that whenever K is eigenspace-closed (i.e., $\Delta_K := \min_{j\in K, \ell\in K^c} |\mu_\ell - \mu_j| > 0$), Q_K is uniquely determined by L independent of the choice of eigenbasis.

Definition 2.1 (Graph Slepian Operator, Shannon Dimension, and Plunge Variance)**.** The *graph Slepian concentration operator* associated with (S, K) is the positive semidefinite contraction

$$T_{S,K} := P_S Q_K P_S \in \mathbb{R}^{n\times n},$$

with sorted eigenvalues $1 \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$ (at most $\min(s, k)$ nonzero). Define its *effective Shannon dimension* $N_{\text{eff}}(S, K)$, *plunge-region variance* $\Pi(S, K)$, *threshold mode count* $M_\tau(S, K)$, *transition-zone count* $N_{\text{trans}}(\varepsilon)$, and *squared edge boundary* $|\partial_2 S|$ by

$$\begin{aligned}
N_{\text{eff}}(S, K) &:= \operatorname{Tr}(T_{S,K}) = \sum_{i=1}^{n} \lambda_i = \|P_S \Phi_K\|_F^2, \\
\Pi(S, K) &:= \operatorname{Tr}(T_{S,K}(I - T_{S,K})) = \sum_{i=1}^{n} \lambda_i(1 - \lambda_i) = \|P_S Q_K P_{S^c}\|_F^2, \\
M_\tau(S, K) &:= \#\{i \in [n] : \lambda_i \geq \tau\}, \qquad N_{\text{trans}}(\varepsilon) := \#\{i \in [n] : \varepsilon < \lambda_i < 1 - \varepsilon\}, \\
|\partial_2 S| &:= \sum_{u\in S,\, v\in S^c} w_{uv}^2.
\end{aligned}$$

For unweighted graphs ($w_{uv} \in \{0, 1\}$), $|\partial_2 S| = |\partial S|$ is precisely the edge boundary (cut size) of S. For any function $f : \operatorname{spec}(L) \to \mathbb{R}$, we write $f(L) := \sum_{j=1}^{n} f(\mu_j)\phi_j\phi_j^\top$ and $[A, B] := AB - BA$.

Theorem 2.2 (Variational Commutator Duality, Graph Slepian Area Law, and Non-Asymptotic Landau $2WT$ Law)**.** *Let $G = (V, E, w)$ be a finite weighted graph with Laplacian L, $S \subseteq V$, and $K \subseteq [n]$ a non-trivial band ($1 \leq |K| \leq n-1$) with spectral band gap $\Delta_K := \min_{j\in K, \ell\in K^c} |\mu_\ell - \mu_j| > 0$.*

(a) ***Exact Variational Commutator Duality and Defect Identity.*** *For every function $f : \operatorname{spec}(L) \to \mathbb{R}$,*

$$\begin{aligned}
\frac{1}{2}\,\|[f(L), P_S]\|_F^2 \;-\; \Pi(S, K) = \frac{1}{2}\big(\|Q_K[f(L), P_S]Q_K\|_F^2 + \|Q_{K^c}[f(L), P_S]Q_{K^c}\|_F^2\big) \\
+ \sum_{j\in K}\sum_{\ell\in K^c} \big((f(\mu_\ell) - f(\mu_j))^2 - 1\big)\,|\langle \phi_j, P_S\phi_\ell\rangle|^2.
\end{aligned} \tag{1}$$

Consequently, $\Pi(S,K)$ *obeys the exact variational characterization*

$$\Pi(S,K) \;=\; \inf_{\substack{f:\mathrm{spec}(L)\to\mathbb{R} \\ |f(\mu_\ell)-f(\mu_j)|\geq 1 \;\forall j\in K,\,\ell\in K^c}} \frac{1}{2}\,\|[f(L),P_S]\|_F^2, \tag{2}$$

with the infimum attained by the spectral step filter $f(\mu)=\mathbf{1}_{\{\mu\in\mu(K^c)\}}$*; more generally, a feasible filter* f *attains* (2) *if and only if* $Q_K[f(L),P_S]Q_K=0$*,* $Q_{K^c}[f(L),P_S]Q_{K^c}=0$*, and* $|f(\mu_\ell)-f(\mu_j)|=1$ *whenever* $\langle\phi_j,P_S\phi_\ell\rangle\neq 0$*. In particular, defining the* bipartite cross-band eigenspace coupling graph *on vertex set* $\mu(K)\sqcup\mu(K^c)$ *with edges* $\{(\mu,\nu)\in\mu(K)\times\mu(K^c):Q_\mu P_S Q_\nu\neq 0\}$ *(where* Q_μ *denotes the orthogonal projection onto the* μ*-eigenspace of* L*), among one-sided filters satisfying* $f(\mu_\ell)-f(\mu_j)\geq 1$ *on* $K\times K^c$ *the minimizer is unique up to an additive constant whenever this bipartite coupling graph is connected.*

(b) ***Universal Graph Slepian Area Law and Exact Equality Characterization.*** *Taking* $f(\mu)=\mu/\Delta_K$ *yields*

$$\Pi(S,K) \;\leq\; \frac{|\partial_2 S|}{\Delta_K^2}, \tag{3}$$

with equality if and only if $Q_K[L,P_S]Q_K=0$*,* $Q_{K^c}[L,P_S]Q_{K^c}=0$*, and every cross-band pair* $(j\in K,\ell\in K^c)$ *with* $\langle\phi_j,P_S\phi_\ell\rangle\neq 0$ *satisfies* $|\mu_\ell-\mu_j|=\Delta_K$*. In particular, for any disjoint union of* $m\geq 1$ *unit-weight edges* $G=mK_2$ *with* S *picking one endpoint per edge and* $K=\{1,\dots,m\}$ *(*$\mu=0$*), (3) holds with exact equality* $\Pi(S,K)=m/4=|\partial_2 S|/\Delta_K^2$*.*

(c) ***Non-Asymptotic Graph Landau*** $2WT$ ***Counting Law.*** *For every* $\varepsilon\in(0,1/2)$ *and every* $\tau\in(0,1)$*,*

$$N_{\mathrm{trans}}(\varepsilon) \;\leq\; \frac{\Pi(S,K)}{\varepsilon(1-\varepsilon)} \;\leq\; \frac{|\partial_2 S|}{\varepsilon(1-\varepsilon)\,\Delta_K^2}, \tag{4}$$

$$-\frac{\Pi(S,K)}{1-\tau} \;\leq\; M_\tau(S,K)-N_{\mathrm{eff}}(S,K) \;\leq\; \frac{\Pi(S,K)}{\tau}, \tag{5}$$

and in particular at the half-energy threshold $\tau=1/2$*,*

$$\big|\,M_{1/2}(S,K)-N_{\mathrm{eff}}(S,K)\,\big| \;\leq\; 2\,\Pi(S,K) \;\leq\; \frac{2\,|\partial_2 S|}{\Delta_K^2}, \tag{6}$$

where the constant 2 *is universally sharp (attained whenever every nonzero eigenvalue equals* $1/2$*, e.g.,* $G=mK_2$ *above).*

Proof. **(a)** Because $Q_K+Q_{K^c}=I_n$ is an orthogonal decomposition of $\mathbb{R}^n$, decomposing the skew-symmetric matrix $[f(L),P_S]$ into its four orthogonal blocks $Q_K[\cdot]Q_K$, $Q_{K^c}[\cdot]Q_{K^c}$, $Q_K[\cdot]Q_{K^c}$, and $Q_{K^c}[\cdot]Q_K$ yields

$$\|[f(L),P_S]\|_F^2=\|Q_K[f(L),P_S]Q_K\|_F^2+\|Q_{K^c}[f(L),P_S]Q_{K^c}\|_F^2+2\,\|Q_K[f(L),P_S]Q_{K^c}\|_F^2,$$

since $(Q_K[f(L),P_S]Q_{K^c})^\top=-Q_{K^c}[f(L),P_S]Q_K$. For any $j\in K$ and $\ell\in K^c$, since $f(L)\phi_r=f(\mu_r)\phi_r$,

$$\langle\phi_j,\,[f(L),P_S]\phi_\ell\rangle=\big(f(\mu_j)-f(\mu_\ell)\big)\,\langle\phi_j,P_S\phi_\ell\rangle,$$

so $\|Q_K[f(L),P_S]Q_{K^c}\|_F^2=\sum_{j\in K,\ell\in K^c}(f(\mu_\ell)-f(\mu_j))^2|\langle\phi_j,P_S\phi_\ell\rangle|^2$. Meanwhile, because $P_S^2=P_S$ and $Q_K^2=Q_K$,

$$\begin{aligned}\Pi(S,K)&=\mathrm{Tr}\big(P_SQ_KP_S(I-P_SQ_KP_S)\big)\\&=\mathrm{Tr}(P_SQ_KP_SQ_{K^c})=\|Q_KP_SQ_{K^c}\|_F^2=\sum_{j\in K}\sum_{\ell\in K^c}|\langle\phi_j,P_S\phi_\ell\rangle|^2,\end{aligned}$$

and identically $\|P_S Q_K P_{S^c}\|_F^2 = \mathrm{Tr}(P_S Q_K (I - P_S) Q_K) = \mathrm{Tr}(P_S Q_K P_S - (P_S Q_K P_S)^2) = \Pi(S, K)$. Subtracting $\Pi(S, K)$ from $\frac{1}{2}\|[f(L), P_S]\|_F^2$ establishes (1). Whenever $|f(\mu_\ell) - f(\mu_j)| \geq 1$ for all $j \in K, \ell \in K^c$, every term on the right-hand side of (1) is nonnegative, giving $\frac{1}{2}\|[f(L), P_S]\|_F^2 \geq \Pi(S, K)$; setting $f(\mu) = \mathbf{1}_{\{\mu \in \mu(K^c)\}}$ gives $f(L) = Q_{K^c}$, for which both diagonal blocks vanish and $|f(\mu_\ell) - f(\mu_j)| = 1$ identically, proving (2). By (1), any feasible f attains the infimum iff $Q_K[f(L), P_S]Q_K = 0$, $Q_{K^c}[f(L), P_S]Q_{K^c} = 0$, and $|f(\mu_\ell) - f(\mu_j)| = 1$ whenever $\langle \phi_j, P_S \phi_\ell \rangle \neq 0$. Under the one-sided constraint $f(\mu_\ell) - f(\mu_j) \geq 1$ for all $j \in K, \ell \in K^c$, every coupled cross-band edge forces $f(\nu) - f(\mu) = 1$, so connectedness of the bipartite cross-band eigenspace coupling graph forces f to take a single constant c on $\mu(K)$ and $c + 1$ on $\mu(K^c)$.

(b) For $f(\mu) = \mu/\Delta_K$, we have $|f(\mu_\ell) - f(\mu_j)| = |\mu_\ell - \mu_j|/\Delta_K \geq 1$ across $K \times K^c$. Since $P_S = \mathrm{diag}(\mathbf{1}_S)$ is diagonal and L is symmetric with off-diagonal entries $L_{uv} = -w_{uv}$, the (u, v) entry of $[L, P_S]$ is 0 whenever $u, v \in S$ or $u, v \in S^c$, $+L_{uv} = -w_{uv}$ when $u \in S^c, v \in S$, and $-L_{uv} = +w_{uv}$ when $u \in S, v \in S^c$. Hence

$$\frac{1}{2}\|[L, P_S]\|_F^2 = \sum_{u \in S,\, v \in S^c} w_{uv}^2 = |\partial_2 S|,$$

and (1) yields (3) together with its exact equality conditions. For $G = mK_2$, the eigenvalues of L are 0 (multiplicity m) and 2 (multiplicity m), so $\Delta_K = 2$, $|\partial_2 S| = m$, each edge mode has Slepian eigenvalue $1/2$, and $\Pi(S, K) = m \cdot \frac{1}{4} = |\partial_2 S|/\Delta_K^2$.

(c) For (4), every eigenvalue $\lambda_i \in (\varepsilon, 1-\varepsilon)$ satisfies $\lambda_i(1-\lambda_i) > \varepsilon(1-\varepsilon)$, so Markov's inequality gives $\varepsilon(1-\varepsilon) N_{\mathrm{trans}}(\varepsilon) \leq \sum_{i=1}^n \lambda_i(1-\lambda_i) = \Pi(S, K)$. For (5), write

$$M_\tau(S, K) - N_{\mathrm{eff}}(S, K) = \sum_{\lambda_i \geq \tau} (1 - \lambda_i) - \sum_{\lambda_i < \tau} \lambda_i.$$

For $\lambda_i \geq \tau$, $1-\lambda_i \leq \frac{1}{\tau}\lambda_i(1-\lambda_i)$, so $M_\tau - N_{\mathrm{eff}} \leq \frac{1}{\tau}\sum_{\lambda_i \geq \tau} \lambda_i(1-\lambda_i) \leq \Pi(S, K)/\tau$. Symmetrically, for $\lambda_i < \tau$, $\lambda_i \leq \frac{1}{1-\tau}\lambda_i(1-\lambda_i)$, so $N_{\mathrm{eff}} - M_\tau \leq \frac{1}{1-\tau}\sum_{\lambda_i < \tau} \lambda_i(1-\lambda_i) \leq \Pi(S, K)/(1-\tau)$. Setting $\tau = 1/2$ gives (6). On $G = mK_2$, $M_{1/2} = m$, $N_{\mathrm{eff}} = m/2$, and $\Pi = m/4$, so $|M_{1/2} - N_{\mathrm{eff}}| = m/2 = 2\Pi$ achieves exact sharpness. □

Remark 2.3 (Sharp Sigmoid Approximation of the Variational Infimum). Theorem 2.2(a) explains why smooth band-separating filters substantially tighten the linear area law $|\partial_2 S|/\Delta_K^2$ when the full spectrum spans a wide range relative to Δ_K. Suppose K is a low-pass band with $\mu_j \leq \omega_- := \max_{i \in K} \mu_i < \omega_+ := \min_{\ell \in K^c} \mu_\ell \leq \mu_\ell$ for all $j \in K, \ell \in K^c$, and gap midpoint $\omega_0 := \frac{1}{2}(\omega_- + \omega_+)$. Taking $f_\beta(\mu) := \tanh(\beta(\mu - \omega_0))/\Delta_\beta$ with normalization $\Delta_\beta := \tanh(\beta(\omega_+ - \omega_0)) - \tanh(\beta(\omega_- - \omega_0))$ makes f_β converge pointwise to $-\frac{1}{2}$ on K and $+\frac{1}{2}$ on K^c as $\beta \to \infty$, suppressing both in-band and high-frequency commutator energy and driving $\frac{1}{2}\|[f_\beta(L), P_S]\|_F^2 \to \Pi(S, K)$ (Figure 1(a)).

3 Four-Block Symmetry, Dirichlet–Ky Fan Localization, and Uncertainty

We next connect the graph Slepian spectrum to the complementary projections (P_{S^c}, Q_{K^c}) and to the Dirichlet Laplacian $L_{SS} \in \mathbb{R}^{|S| \times |S|}$ (the principal submatrix of L indexed by S), whose sorted eigenvalues we denote by $0 \leq \nu_1(S) \leq \nu_2(S) \leq \cdots \leq \nu_{|S|}(S)$. Define the four exact-localization multiplicities

$$d_{11} := \dim(\mathrm{ran}\, P_S \cap \mathrm{ran}\, Q_K), \qquad d_{10} := \dim(\mathrm{ran}\, P_S \cap \mathrm{ran}\, Q_{K^c}),$$
$$d_{01} := \dim(\mathrm{ran}\, P_{S^c} \cap \mathrm{ran}\, Q_K), \qquad d_{00} := \dim(\mathrm{ran}\, P_{S^c} \cap \mathrm{ran}\, Q_{K^c}).$$

Define also the *spectral incoherence* of a non-empty band K by

$$M_{\mathrm{loc}}(K) := \max_{v \in V} \sqrt{\frac{n}{|K|} \sum_{j \in K} |\phi_j(v)|^2} \;\geq\; 1,$$

noting that $M_{\mathrm{loc}}(K) = 1$ identically whenever G is vertex-transitive and K is eigenspace-closed (since $\sum_{j\in K} |\phi_j(v)|^2 = |K|/n$ is constant over $v \in V$).

Theorem 3.1 (Four-Block Symmetry, Ky Fan Dirichlet–Cheeger Bound, and Uncertainty)**.** *Let $G = (V, E, w)$ be a finite weighted graph with Laplacian L, $S \subseteq V$ $(1 \le |S| \le n-1)$, and $K \subseteq [n]$ $(1 \le |K| \le n-1)$.*

(a) ***Four-Block Complementary Symmetry and Exact Index Laws.*** *The four concentration operators $T_{S,K}$, T_{S,K^c}, $T_{S^c,K}$, and T_{S^c,K^c} share the exact same multiset of eigenvalues in the open interval $(0,1)$ up to the involution $\lambda \leftrightarrow 1-\lambda$: specifically, $\lambda \in (0,1)$ occurs with identical multiplicity in $T_{S,K}$ and T_{S^c,K^c}, while $1-\lambda$ occurs with that same multiplicity in T_{S,K^c} and $T_{S^c,K}$. Consequently,*

$$\Pi(S,K) = \Pi(S,K^c) = \Pi(S^c,K) = \Pi(S^c,K^c), \tag{7}$$

and the exact-localization dimensions obey the index identities

$$d_{11} - d_{00} = |S| + |K| - n, \qquad d_{10} - d_{01} = |S| - |K|. \tag{8}$$

(b) ***Multi-Mode Ky Fan Dirichlet–Cheeger Localization Lower Bound.*** *Suppose K is a low-pass band with $\mu_\ell \ge \omega_+ > 0$ for all $\ell \in K^c$. Then for every $r \in \{1, \dots, |S|\}$,*

$$\sum_{i=1}^{r} (1 - \lambda_i(T_{S,K})) \;\le\; \frac{1}{\omega_+} \sum_{i=1}^{r} \nu_i(S), \qquad \text{and} \qquad \lambda_r(T_{S,K}) \;\ge\; 1 - \frac{\nu_r(S)}{\omega_+}. \tag{9}$$

In particular, for $r = 1$,

$$\lambda_1(T_{S,K}) \;\ge\; 1 - \frac{\nu_1(S)}{\omega_+} \;\ge\; 1 - \frac{|\partial_1 S|/|S|}{\omega_+}, \qquad \text{where } |\partial_1 S| := \sum_{u\in S,\, v\in S^c} w_{uv}, \tag{10}$$

and for every $\varepsilon \in (0,1)$, the Dirichlet Weyl counting function lower-bounds the well-concentrated Slepian count:

$$M_{1-\varepsilon}(S,K) = \#\{i \in [n] : \lambda_i(T_{S,K}) \ge 1-\varepsilon\} \;\ge\; \#\{i \in \{1,\dots,|S|\} : \nu_i(S) \le \varepsilon\, \omega_+\}. \tag{11}$$

(c) ***Graph Spectral–Spatial Uncertainty Principle.*** *Every pair of nonzero signals $x \in \operatorname{ran} P_S$ and $y \in \operatorname{ran} Q_K$ satisfies*

$$\frac{|\langle x, y\rangle|^2}{\|x\|_2^2\, \|y\|_2^2} \;\le\; \lambda_1(T_{S,K}) \;\le\; \min\left\{1,\; N_{\mathrm{eff}}(S,K),\; \frac{|S||K|}{n}\, M_{\mathrm{loc}}(K)^2\right\}. \tag{12}$$

In particular, whenever $N_{\mathrm{eff}}(S,K) < 1$ (and on any vertex-transitive graph with eigenspace-closed K whenever $|S||K| < n$), exact joint localization is impossible ($\operatorname{ran} P_S \cap \operatorname{ran} Q_K = \{0\}$) and $\lambda_1(T_{S,K}) \le N_{\mathrm{eff}}(S,K) < 1$.

Proof. **(a)** By Halmos's two-projections / Jordan principal-angle theorem [1], $\mathbb{R}^n$ decomposes orthogonally into the four eigenspaces $\operatorname{ran} P_S \cap \operatorname{ran} Q_K$ (dimension d_{11}), $\operatorname{ran} P_S \cap \operatorname{ran} Q_{K^c}$ (d_{10}), $\operatorname{ran} P_{S^c} \cap \operatorname{ran} Q_K$ (d_{01}), $\operatorname{ran} P_{S^c} \cap \operatorname{ran} Q_{K^c}$ (d_{00}), plus p two-dimensional invariant subspaces $W_1, \dots, W_p$ on each of which, in an orthonormal basis,

$$P_S|_{W_a} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad Q_K|_{W_a} = \begin{pmatrix} c_a^2 & c_a s_a \\ c_a s_a & s_a^2 \end{pmatrix}, \qquad c_a = \cos\theta_a,\; s_a = \sin\theta_a,\; \theta_a \in (0, \pi/2).$$

Direct 2×2 multiplication on W_a yields nonzero eigenvalues c_a^2 for $T_{S,K}$ and T_{S^c,K^c}, and $s_a^2 = 1 - c_a^2$ for T_{S,K^c} and $T_{S^c,K}$. Because $c_a^2(1-c_a^2) = s_a^2(1-s_a^2)$, summing over $a = 1, \dots, p$ proves (7).

Counting dimensions gives $|S| = d_{11}+d_{10}+p$, $|K| = d_{11}+d_{01}+p$, and $n = d_{11}+d_{10}+d_{01}+d_{00}+2p$, from which (8) follows immediately.

(b) Because $L \succeq 0$ on $\operatorname{ran} Q_K$ and $L \succeq \omega_+ I$ on $\operatorname{ran} Q_{K^c}$, we have the operator inequality $L \succeq \omega_+ Q_{K^c} = \omega_+(I_n - Q_K)$. Restricting to $\operatorname{ran} P_S$ via $P_S(\cdot)P_S$ gives

$$P_S L P_S \;\succeq\; \omega_+(P_S - P_S Q_K P_S) = \omega_+(P_S - T_{S,K}). \tag{13}$$

Viewing both sides as symmetric operators on the $|S|$-dimensional subspace $\operatorname{ran} P_S \cong \mathbb{R}^{|S|}$, the eigenvalues of $P_S L P_S\big|_S = L_{SS}$ are $\nu_1(S) \le \cdots \le \nu_{|S|}(S)$, while the eigenvalues of $\omega_+(I_{|S|} - T_{S,K}\big|_S)$ in non-decreasing order are $\omega_+(1-\lambda_1(T_{S,K})) \le \cdots \le \omega_+(1-\lambda_{|S|}(T_{S,K}))$. Applying Weyl's monotonicity theorem to (13) yields $\omega_+(1-\lambda_r(T_{S,K})) \le \nu_r(S)$ for every $r \in \{1,\dots,|S|\}$, and summing $i = 1,\dots,r$ yields the Ky Fan inequality (9). For $r = 1$, testing $\nu_1(S) = \min_{\|u\|_2=1} u^\top L_{SS} u$ against the constant unit vector $u = |S|^{-1/2}\mathbf{1}_S$ gives $\nu_1(S) \le |S|^{-1}\mathbf{1}_S^\top L \mathbf{1}_S = |\partial_1 S|/|S|$, establishing (10). Finally, if $\nu_r(S) \le \varepsilon\omega_+$, then (9) forces $\lambda_r(T_{S,K}) \ge 1-\varepsilon$, proving (11).

(c) By Cauchy–Schwarz, for $x = P_S x$ and $y = Q_K y$,

$$|\langle x, y\rangle|^2 = |\langle Q_K P_S x, y\rangle|^2 \le \|Q_K P_S x\|_2^2 \|y\|_2^2 = \langle x, T_{S,K} x\rangle \|y\|_2^2 \le \lambda_1(T_{S,K})\|x\|_2^2\|y\|_2^2.$$

Since $T_{S,K} \succeq 0$ satisfies $\lambda_1 \le \sum_{i=1}^n \lambda_i = N_{\mathrm{eff}}(S,K)$ and $\lambda_1 \le 1$, while $N_{\mathrm{eff}}(S,K) = \sum_{v\in S}\sum_{j\in K}|\phi_j(v)|^2 \le |S|\,(|K|/n)\,M_{\mathrm{loc}}(K)^2$, (12) follows immediately. □

4 Exact Hypercube Binomial-Tail Spectra and Product Plunge Laws

We now exhibit two rich families where the entire graph Slepian spectrum and plunge variance admit exact closed-form formulas: subcubes of the d-dimensional hypercube Q_d, and arbitrary Cartesian graph products $G_1 \square G_2$.

Recall that Q_d has vertex set $\{0,1\}^d$ ($n = 2^d$), unit-weight edges joining vertices at Hamming distance 1, and exact Laplacian eigenbasis given by the orthonormal Walsh–Fourier characters $\chi_A(x) := 2^{-d/2}(-1)^{\langle A,x\rangle}$ ($A \subseteq [d]$) with exact eigenvalues $\mu_A = 2|A| \in \{0,2,\dots,2d\}$ (multiplicity $\binom{d}{j}$ at $\mu = 2j$). For any bandwidth $r \in \{0,1,\dots,d\}$, let $K_r := \{A \subseteq [d] : |A| \le r\}$ be the eigenspace-closed low-pass Hamming-degree band (with $|K_r| = \sum_{j=0}^r \binom{d}{j}$ and exact spectral band gap $\Delta_{K_r} = 2$ whenever $r < d$).

Theorem 4.1 (Exact Binomial-Tail Hypercube Spectrum and Cartesian-Product Plunge Identity)**.** (a) ***Exact Hypercube Subcube Spectrum.*** *Let $d \ge 2$, let $S \subseteq \{0,1\}^d$ be any subcube of codimension $c \in \{1,\dots,d\}$ ($|S| = 2^{d-c}$, obtained by fixing c coordinates), and let $K_r = \{A \subseteq [d] : |A| \le r\}$ ($0 \le r \le d$). Then the nonzero spectrum of T_{S,K_r} consists exclusively of the binomial-tail eigenvalues*

$$\lambda^{(j)} \;=\; 2^{-c}\sum_{t=0}^{\min(c,\,r-j)}\binom{c}{t} \;=\; \mathbb{P}(\mathrm{Bin}(c,\tfrac12) \le r-j), \qquad 0 \le j \le \min(r, d-c), \tag{14}$$

each occurring with exact multiplicity $\binom{d-c}{j}$ (and eigenvalue 0 with remaining multiplicity $2^d - \sum_{j=0}^{\min(r,d-c)}\binom{d-c}{j}$). In particular:

- *For a half-cube ($c = 1$, $1 \le r < d$), the entire nonzero spectrum is purely two-valued: $\lambda = 1$ with multiplicity $\sum_{j=0}^{r-1}\binom{d-1}{j}$ and $\lambda = 1/2$ with multiplicity $\binom{d-1}{r}$, giving exact plunge variance $\Pi(S,K_r) = \frac14\binom{d-1}{r}$.*

- *For a quarter-cube ($c = 2$, $2 \le r \le d-2$), the nonzero eigenvalues are 1 (multiplicity $\sum_{j=0}^{r-2}\binom{d-2}{j}$), 3/4 (multiplicity $\binom{d-2}{r-1}$), and 1/4 (multiplicity $\binom{d-2}{r}$), giving exact plunge variance $\Pi(S,K_r) = \frac{3}{16}\binom{d-1}{r}$ (for $r = d-1$ with $d \ge 3$, the 1/4 eigenspace has multiplicity $\binom{d-2}{d-1} = 0$, while the plunge formula $\frac{3}{16}\binom{d-1}{r}$ continues to hold).*

(b) ***Cartesian-Product Exact Tensor and Plunge Law.*** *Let $G_1 = (V_1, E_1, w_1)$ and $G_2 = (V_2, E_2, w_2)$ be finite weighted graphs with Laplacians L_1, L_2 and Cartesian product $G_1 \square G_2$ (Laplacian $L_{1\square 2} = L_1 \otimes I_{n_2} + I_{n_1} \otimes L_2$). For any product spatial domain $S = S_1 \times S_2$ and product spectral band $K = K_1 \times K_2$ (spanned by $\{\phi_a^{(1)} \otimes \phi_b^{(2)} : a \in K_1, b \in K_2\}$), the Slepian operator factorizes as $T_{S,K} = T_{S_1,K_1} \otimes T_{S_2,K_2}$, with exact Shannon dimension and plunge variance*

$$N_{\text{eff}}(S,K) = N_{\text{eff}}^{(1)} N_{\text{eff}}^{(2)}, \qquad \Pi(S,K) = N_{\text{eff}}^{(1)}\,\Pi_2 + N_{\text{eff}}^{(2)}\,\Pi_1 - \Pi_1\Pi_2. \tag{15}$$

Proof. **(a)** By coordinate symmetry of Q_d (automorphism invariance of L and K_r), we may assume $S = \{x \in \{0,1\}^d : x_1 = \cdots = x_c = 0\}$. Every subset $A \subseteq [d]$ decomposes uniquely as $A = A_1 \sqcup A_2$ with $A_1 \subseteq \{1,\dots,c\}$ and $A_2 \subseteq \{c+1,\dots,d\}$. For any $x \in S$, $\langle A_1 \sqcup A_2, x\rangle = \langle A_2, x_{c+1:d}\rangle$ is completely independent of A_1. Define the orthonormal Walsh basis on $\operatorname{ran} P_S \cong \mathbb{R}^{2^{d-c}}$ by

$$\psi_{A_2}(x) := \begin{cases} 2^{-(d-c)/2}(-1)^{\langle A_2, x_{c+1:d}\rangle}, & x \in S, \\ 0, & x \notin S, \end{cases} \qquad A_2 \subseteq \{c+1,\dots,d\}.$$

Then for every $A = A_1 \sqcup A_2$, $P_S \chi_{A_1 \sqcup A_2} = 2^{-c/2}\psi_{A_2}$. Substituting into $T_{S,K_r} = \sum_{A \in K_r} (P_S\chi_A)(P_S\chi_A)^\top$ yields the exact spectral decomposition

$$\begin{aligned} T_{S,K_r} &= \sum_{\substack{A_2 \subseteq \{c+1,\dots,d\} \\ |A_2| \le r}} \left(2^{-c} \sum_{\substack{A_1 \subseteq \{1,\dots,c\} \\ |A_1| \le r - |A_2|}} 1 \right) \psi_{A_2}\psi_{A_2}^\top \\ &= \sum_{j=0}^{\min(r,d-c)} \lambda^{(j)} \sum_{|A_2| = j} \psi_{A_2}\psi_{A_2}^\top. \end{aligned}$$

Since $\{\psi_{A_2}\}$ is orthonormal and there are $\binom{d-c}{j}$ subsets $A_2 \subseteq \{c+1,\dots,d\}$ of cardinality j, (14) follows immediately. The $c = 1$ and $c = 2$ specializations follow directly by Pascal's identity $\binom{d-2}{r-1} + \binom{d-2}{r} = \binom{d-1}{r}$.

(b) Because $P_{S_1\times S_2} = P_{S_1} \otimes P_{S_2}$ and $Q_{K_1\times K_2} = Q_{K_1} \otimes Q_{K_2}$, we obtain $T_{S,K} = (P_{S_1} \otimes P_{S_2})(Q_{K_1} \otimes Q_{K_2})(P_{S_1} \otimes P_{S_2}) = T_{S_1,K_1} \otimes T_{S_2,K_2}$. Writing the eigenvalues as $\{\lambda_a^{(1)}\lambda_b^{(2)}\}$, we have $N_{\text{eff}} = (\sum_a \lambda_a^{(1)})(\sum_b \lambda_b^{(2)}) = N_{\text{eff}}^{(1)} N_{\text{eff}}^{(2)}$ and

$$\begin{aligned} \Pi(S,K) &= \sum_{a,b} \lambda_a^{(1)}\lambda_b^{(2)}(1 - \lambda_a^{(1)}\lambda_b^{(2)}) \\ &= \sum_{a,b} \lambda_a^{(1)}\lambda_b^{(2)}\Big((1-\lambda_a^{(1)}) + \lambda_a^{(1)}(1-\lambda_b^{(2)})\Big) = N_{\text{eff}}^{(2)}\Pi_1 + (N_{\text{eff}}^{(1)} - \Pi_1)\Pi_2, \end{aligned}$$

using $\sum_a (\lambda_a^{(1)})^2 = \sum_a \lambda_a^{(1)} - \sum_a \lambda_a^{(1)}(1-\lambda_a^{(1)}) = N_{\text{eff}}^{(1)} - \Pi_1$, which gives (15). □

5 Minimax Bandlimited Signal Recovery and Sharp Subcube Phase Transitions

Consider recovering an unknown K-bandlimited graph signal $x = Q_K x \in \operatorname{ran} Q_K$ from noisy spatial observations on $S \subseteq V$:

$$y = P_S x + \eta \in \operatorname{ran} P_S, \qquad \|\eta\|_2 \le \delta.$$

Writing $x = \Phi_K \alpha$ ($\alpha \in \mathbb{R}^k$), the observation operator $A := P_S \Phi_K : \mathbb{R}^k \to \operatorname{ran} P_S$ has Gram matrix $A^\top A = \Phi_K^\top P_S \Phi_K \in \mathbb{R}^{k \times k}$, whose nonzero eigenvalues coincide with those of $T_{S,K} = P_S \Phi_K \Phi_K^\top P_S$. When $A^\top A \succ 0$, the unique minimum-norm least-squares estimator is $\hat{x}_{\mathrm{LS}} := \Phi_K (A^\top A)^{-1} A^\top y$.

Theorem 5.1 (Minimax Bandlimited Recovery Stability and Sharp Hypercube Phase Transition)**.** *Let $G = (V, E, w)$ be a finite weighted graph, $S \subseteq V$, and $K \subseteq [n]$ ($k = |K| \geq 1$).*

(a) ***Exact Minimax Noise Amplification.*** *Recovery of arbitrary $x \in \operatorname{ran} Q_K$ from $P_S x$ is uniquely solvable (well-posed) if and only if $\lambda_k(T_{S,K}) > 0$ (equivalently $\operatorname{ran} P_{S^c} \cap \operatorname{ran} Q_K = \{0\}$, i.e., $\lambda_1(T_{S^c,K}) < 1$). Whenever well-posed, the least-squares estimator $\hat{x}_{\mathrm{LS}}$ achieves the exact estimator-independent minimax worst-case error over all maps $\hat{x} : \operatorname{ran} P_S \to \operatorname{ran} Q_K$:*

$$\inf_{\hat{x}} \sup_{\substack{x \in \operatorname{ran} Q_K \\ \|\eta\|_2 \leq \delta}} \frac{\|\hat{x}(P_S x + \eta) - x\|_2}{\delta} = \sup_{\substack{\eta \in \operatorname{ran} P_S \\ \eta \neq 0}} \frac{\|\hat{x}_{\mathrm{LS}} - x\|_2}{\|\eta\|_2} = \frac{1}{\sqrt{\lambda_k(T_{S,K})}} = \frac{1}{\sqrt{1 - \lambda_1(T_{S^c,K})}}. \tag{16}$$

In particular, whenever the missing-vertex set S^c satisfies $N_{\mathrm{eff}}(S^c, K) < 1$, recovery on S is unconditionally well-posed with explicit stability guarantee $\lambda_k(T_{S,K}) \geq 1 - N_{\mathrm{eff}}(S^c, K) > 0$ (holding trivially when $S = V$ since $N_{\mathrm{eff}}(\emptyset, K) = 0$ and $\lambda_k = 1$, and by Theorem 3.1(c) otherwise).

(b) ***Sharp Subcube Recovery Phase Transition on Punctured Hypercubes.*** *Let $G = Q_d$ ($d \geq 2$), $K_r = \{A \subseteq [d] : |A| \leq r\}$ ($1 \leq r < d$), and suppose observations are taken on $S_{\mathrm{keep}} := V(Q_d) \setminus S_{\mathrm{miss}}$ where S_{miss} is a missing subcube of codimension $c \in \{1, \ldots, d\}$ ($|S_{\mathrm{miss}}| = 2^{d-c}$). Then bandlimited recovery undergoes a sharp phase transition at critical codimension $c_c = r + 1$:*

- *For every $c \leq r$, recovery is* ill-posed*: $\lambda_{|K_r|}(T_{S_{\mathrm{keep}},K_r}) = 0$, and the observation operator $A = P_{S_{\mathrm{keep}}} \Phi_{K_r}$ has exact nullspace dimension $\dim(\operatorname{ran} P_{S_{\mathrm{miss}}} \cap \operatorname{ran} Q_{K_r}) = \sum_{j=0}^{r-c} \binom{d-c}{j} \geq 1$.*
- *For every $c \geq r + 1$, recovery is* well-posed *with exact minimum eigenvalue and minimax amplification*

$$\lambda_{|K_r|}(T_{S_{\mathrm{keep}},K_r}) = 1 - 2^{-c} \sum_{t=0}^{r} \binom{c}{t} = \mathbb{P}(\mathrm{Bin}(c, \tfrac{1}{2}) \geq r + 1) > 0. \tag{17}$$

Proof. **(a)** Because $A^\top A = \Phi_K^\top P_S \Phi_K = I_k - \Phi_K^\top P_{S^c} \Phi_K$ has sorted eigenvalues $\lambda_1(T_{S,K}) \geq \cdots \geq \lambda_k(T_{S,K})$, we have $\lambda_k(T_{S,K}) = \lambda_{\min}(A^\top A) = 1 - \lambda_{\max}(\Phi_K^\top P_{S^c} \Phi_K) = 1 - \lambda_1(T_{S^c,K})$. Thus A is injective on $\mathbb{R}^k$ iff $\lambda_k(T_{S,K}) > 0$. When well-posed, $\hat{x}_{\mathrm{LS}} - x = \Phi_K (A^\top A)^{-1} A^\top \eta$, and since Φ_K is isometric,

$$\|\hat{x}_{\mathrm{LS}} - x\|_2^2 = \eta^\top A (A^\top A)^{-2} A^\top \eta,$$

whose operator norm over $\eta \in \operatorname{ran} P_S$ equals the largest eigenvalue of $A(A^\top A)^{-2} A^\top$, namely $\lambda_{\min}(A^\top A)^{-1} = \lambda_k(T_{S,K})^{-1}$. For the estimator-independent minimax lower bound, let $u_k \in \mathbb{R}^k$ ($\|u_k\|_2 = 1$) be a unit eigenvector of $A^\top A$ with eigenvalue $\lambda_k(T_{S,K})$ and set $x_0 := \delta\, \Phi_K u_k / \sqrt{\lambda_k(T_{S,K})} \in \operatorname{ran} Q_K$; then $\|P_S x_0\|_2 = \delta$ and $\|x_0\|_2 = \delta / \sqrt{\lambda_k(T_{S,K})}$, so the observation $y = 0$ is compatible with both $(+x_0, -P_S x_0)$ and $(-x_0, +P_S x_0)$ under $\|\eta\|_2 \leq \delta$, forcing $\max\{\|\hat{x}(0) - x_0\|_2, \|\hat{x}(0) + x_0\|_2\} \geq \|x_0\|_2 = \delta / \sqrt{\lambda_k(T_{S,K})}$ for every estimator $\hat{x}$, proving (16). Substituting $\lambda_1(T_{S^c,K}) \leq N_{\mathrm{eff}}(S^c, K)$ from Theorem 3.1(c) yields $\lambda_k(T_{S,K}) \geq 1 - N_{\mathrm{eff}}(S^c, K)$.

(b) Applying Theorem 4.1(a) to S_{miss} (codimension c), the largest eigenvalue of $T_{S_{\mathrm{miss}},K_r}$ occurs at $j = 0$ and equals $\lambda^{(0)} = 2^{-c} \sum_{t=0}^{\min(c,r)} \binom{c}{t}$. If $c \leq r$, then $\min(c, r) = c$, so $\lambda^{(0)} = 2^{-c} \cdot 2^c = 1$ (with exact multiplicity $\sum_{j=0}^{r-c} \binom{d-c}{j} \geq 1$ of eigenvalues $\lambda^{(j)} = 1$, since $r - j \geq c$ for all $j \leq r - c$), hence $\lambda_{|K_r|}(T_{S_{\mathrm{keep}},K_r}) = 1 - 1 = 0$. If $c \geq r + 1$, then $\min(c, r) = r < c$, so $\lambda_1(T_{S_{\mathrm{miss}},K_r}) = 2^{-c} \sum_{t=0}^{r} \binom{c}{t} < 1$, and (16) yields (17). □

6 Computational Verification and Benchmark Tables

All four theorems were verified programmatically via `scripts/verify_all.py` and the automated test suite `tests/test_graph_slepian.py` (4/4 pytest modules passing). Key numerical benchmarks are reported in Tables 1 and 2 and visualized in Figures 1 and 2.

- **Theorem 2.2 (75 random connected weighted graphs, $n \in \{12, 20, 30\}$):** Graphs were generated with i.i.d. edge probability 0.35, edge weights $w_{uv} \sim \mathrm{Unif}[0.1, 2.0]$, a spanning path enforcing connectivity, and uniformly random subsets S ($3 \le |S| \le n-3$) and low-pass bands $K = \{1, \dots, k\}$ ($2 \le k \le n-3$). Exact commutator defect identity (1) matched to $\le 8.19 \times 10^{-12}$; zero violations of the area law (3) (0.0) and zero violations of the Landau $2WT$ bounds (5) across $\tau \in \{0.25, 0.50, 0.75\}$ (0.0). On disjoint matchings mK_2 ($m \in \{1, 5, 10\}$), exact area-law equality $\Pi = |\partial_2 S|/\Delta_K^2 = m/4$ held to 0.0. Table 1 demonstrates the monotone convergence of gap-normalized variational filters toward $\Pi(S, K) = 0.517333$ on C_{60} (contiguous arc $S = \{0, \dots, 29\}$ and eigenspace-closed low-pass band $K = \{1, \dots, 21\}$).
- **Theorem 3.1 (40 two-block stochastic block models, $n = 24$):** Balanced $12 + 12$ SBMs with $p_{\text{in}} = 0.65$, $p_{\text{out}} = 0.06$, unit weights, a spanning path, S equal to the first 12-vertex block, and low-pass bands $K = \{1, \dots, k\}$ ($2 \le k \le 9$). Four-block plunge invariance (7) held to 9.36×10^{-15}; zero violations of the multi-mode Ky Fan Dirichlet bound (9) (0.0), zero violations of the Weyl counting lower bound (11) across $\varepsilon \in \{0.1, 0.25, 0.5\}$ (0), and zero violations of the uncertainty ceiling (12) (0.0).
- **Theorem 4.1 (18 hypercube subcube configurations $d \in \{5, 6, 7\}$, $c \in \{1, 2, 3\}$, $r \in \{2, 3\}$ + Cartesian product $C_{10} \square C_8$):** The binomial-tail formula (14) matched numerical diagonalization across all 18 configurations to max eigenvalue error 3.22×10^{-15} (Table 2). On $C_{10} \square C_8$ with contiguous arcs $S_1 = \{0, \dots, 4\}$, $S_2 = \{0, \dots, 3\}$ and eigenspace-closed low-pass bands $K_1 = \{1, \dots, 5\}$, $K_2 = \{1, 2, 3\}$, (15) matched exact $N_{\text{eff}} = 3.750000$ to 8.88×10^{-16} and exact $\Pi = 1.221242$ to 6.66×10^{-16}.
- **Theorem 5.1 (C_{40} and punctured hypercube $Q_6 \setminus S_{\text{miss}}$ at $r = 2$, $|K_2| = 22$):** On C_{40} (contiguous arc $S = \{0, \dots, 23\}$, eigenspace-closed band $K = \{1, \dots, 13\}$), $\lambda_k = 2.1255 \times 10^{-7}$ matched $\lambda_{\min}(A^\top A)$ to 2.20×10^{-16} (amplification 2169.03). On $Q_6 \setminus S_{\text{miss}}$, recovery is ill-posed at $c = 1, 2$ ($\lambda_{22} \le 5.58 \times 10^{-16}$) and snaps to well-posed at $c_c = 3$ with exact $\lambda_{22} = 0.125000$ (amp $2\sqrt{2} = 2.8284$, err 1.11×10^{-16}) and $\lambda_{22} = 0.312500$ at $c = 4$ (amp $4/\sqrt{5} = 1.7889$, err 5.55×10^{-17}; Figure 2(b)).

Table 1: Variational commutator bounds $\frac{1}{2}\|[f(L), P_S]\|_F^2$ versus exact plunge variance $\Pi(S, K) = 0.517333$ on C_{60} ($s = 30, k = 21$, $\omega_- = 1.0000$, $\omega_+ = 1.1865$, Theorem 2.2). Each filter $h(L)$ is normalized as $f = h/\Delta_h$ with $\Delta_h := |h(\omega_+) - h(\omega_-)|$.

Gap-normalized filter $f(L) = h(L)/\Delta_h$	Bound $\frac{1}{2}\|[f(L), P_S]\|_F^2$	Exact Π	Ratio UB/Π	Defect err
Linear L/Δ_K (area law)	57.484124	0.517333	111.12×	4.26×10^{-14}
Resolvent $(L + 2I)^{-1}/\Delta_{\text{res}}$	36.480692	0.517333	70.52×	3.55×10^{-14}
Heat kernel $e^{-0.5L}/\Delta_{\text{heat}}$	33.197047	0.517333	64.17×	7.11×10^{-15}
Sigmoid $\tanh(5(L - \omega_0))/\Delta_5$	1.707043	0.517333	3.30×	6.66×10^{-16}
Sigmoid $\tanh(10(L - \omega_0))/\Delta_{10}$	0.760303	0.517333	1.47×	4.72×10^{-16}
Sigmoid $\tanh(20(L - \omega_0))/\Delta_{20}$	0.544457	0.517333	1.05×	5.48×10^{-16}

7 Discussion and Open Directions

We have established a non-asymptotic spectral and harmonic-analytic theory for graph Slepian concentration operators, unifying variational commutator area laws, Dirichlet–Ky Fan localization

Table 2: Exact binomial-tail Slepian spectra on hypercube subcubes Q_d (Theorem 4.1(a)) vs. linear area law $|\partial_2 S|/\Delta_K^2$.

d	c	r	$\lvert S\rvert$	$\lvert K_r\rvert$	$N_{\text{eff}}(S, K_r)$	Exact $\Pi(S, K_r)$	$\lvert\partial_2 S\rvert/\Delta_K^2$	Max spec. err
6	1	2	32	22	11.000	2.500000	8.000	3.22×10^{-15}
6	2	2	16	22	5.500	1.875000	8.000	1.83×10^{-15}
6	3	2	8	22	2.750	1.187500	6.000	1.67×10^{-15}
7	1	3	64	64	32.000	5.000000	16.000	2.78×10^{-15}
7	2	3	32	64	16.000	3.750000	16.000	1.44×10^{-15}
7	3	3	16	64	8.000	2.375000	12.000	1.05×10^{-15}

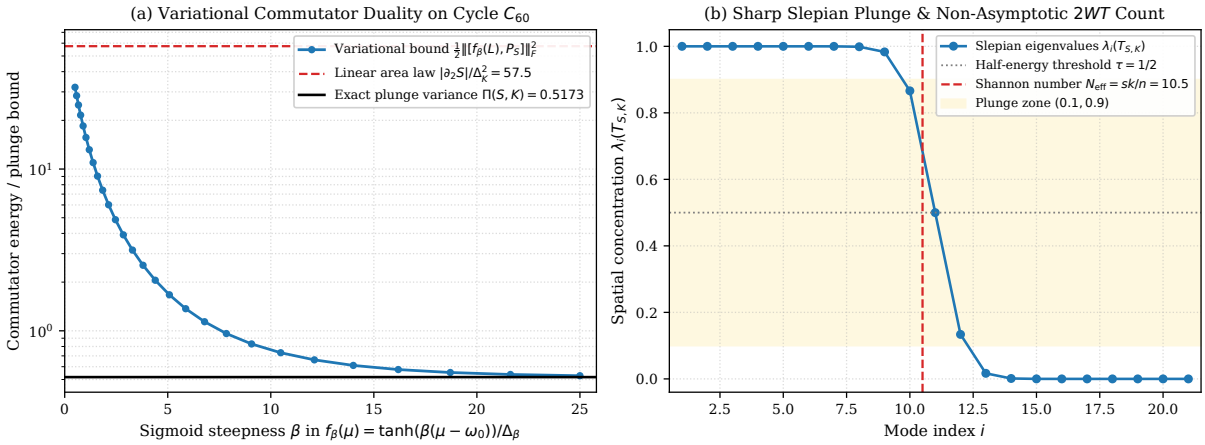

Figure 1: **(a)** Variational commutator duality on C_{60} ($s = 30, k = 21$): normalized sigmoid filters $\tanh(\beta(L - \omega_0))$ tighten the linear area law $|\partial_2 S|/\Delta_K^2 = 57.5$ monotonically down to the exact plunge variance $\Pi(S, K) = 0.5173$ ($1.05\times$ at $\beta = 20$). **(b)** Sharp Slepian eigenvalue plunge on C_{60} centered at the Shannon number $N_{\text{eff}} = sk/n = 10.5$.

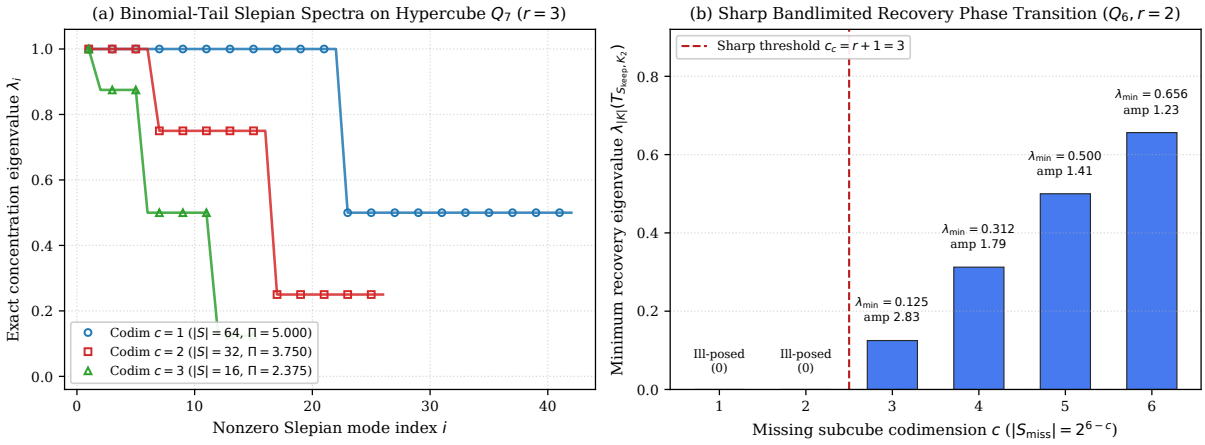

Figure 2: **(a)** Exact binomial-tail Slepian spectra on Q_7 ($r = 3$) for subcube codimensions $c \in \{1, 2, 3\}$: analytical staircases (Theorem 4.1(a), solid) match numerical diagonalization (markers) to $< 3 \times 10^{-15}$. **(b)** Sharp bandlimited signal recovery phase transition on punctured hypercubes $Q_6 \setminus S_{\text{miss}}$ ($r = 2$, Theorem 5.1(b)): ill-posed for $c \leq 2$ and well-posed for $c \geq 3$ with exact minimum eigenvalue $\mathbb{P}(\text{Bin}(c, 1/2) \geq 3)$.

bounds, exact hypercube binomial-tail spectra, and sharp bandlimited recovery thresholds. Two natural open problems arise directly from these results:

1. **Logarithmic plunge refinement on discrete D-dimensional grids.** On D-dimensional tori $\mathbb{Z}_N^D$ with macroscopic bands where $\Delta_K = \Theta(1/N)$, the linear area law gives $\Pi(S, K) = O(N^{D-1}/\Delta_K^2) = O(N^{D+1})$ (e.g., $O(N^2)$ in $D = 1$), whereas classical Landau–Widom asymptotics exhibit a sharp $O(N^{D-1} \log N)$ scaling; proving sharp non-asymptotic log-factor bounds for discrete graph Laplacians via smooth rational or sigmoid commutator filters in Theorem 2.2(a) remains open.
2. **Higher-order simplicial Hodge Slepian operators.** Extending Theorem 2.2 to k-form Slepian projections $P_{S_k} Q_K P_{S_k}$ for combinatorial Hodge Laplacians $L_k = \partial_k^\top \partial_k + \partial_{k+1} \partial_{k+1}^\top$ on simplicial complexes would link cochain localization directly to simplicial coboundary area laws.

Reproducibility and Open-Source Code

All numerical benchmarks, figures, and automated tests (4/4 pytest modules) are openly archived under the MIT license at `https://github.com/lluiseriksson/graph-slepian-area-laws`.

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