Non-Asymptotic Partial Differential Equations: Spectral Poincaré–Heat Ultracontractivity, Exact CFL Wave-Energy Sandwiches, Unconditional Eyre Convex-Splitting, and Conservation-Law TVD Contraction

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Abstract

We establish four self-contained, explicit, non-asymptotic quantitative theorems spanning elliptic–parabolic spectral smoothing, hyperbolic–parabolic finite-difference stability, nonlinear gradient-flow phase-field dissipation, and scalar conservation laws. First, we prove the sharp continuous and discrete 1D Poincaré–Friedrichs–Wirtinger spectral constants, an explicit two-sided mesh-refinement eigenvalue sandwich $\left(1-\frac{\pi^2}{12(N+1)^2}\right)\frac{\pi^2}{L^2} \le \lambda_1^{(h)} \le \frac{\pi^2}{L^2}$, and a uniform-in-time $L^2 \to L^\infty$ Dirichlet heat ultracontractive bound $\|u(t)\|_{L^\infty} \le e^{-\nu\pi^2 t/L^2}\sqrt{\frac{2}{L}+(2\pi\nu t)^{-1/2}}\,\|u_0\|_{L^2}$ that captures both the exact short-time $t^{-1/4}$ Nash smoothing exponent and the exact long-time first-eigenmode amplitude and decay rate (Theorem 2.1). Second, for the periodic upwind–central advection–diffusion scheme (FTUS-CD), we prove that $\sigma + 2r \le 1$ is both necessary and sufficient for von Neumann L^2 stability and yields exact discrete maximum-principle and L^2 non-expansion; for the central-difference leapfrog wave equation, we prove exact conservation of the staggered discrete energy $E^{n+1/2} = E^{1/2}$ and establish the sharp two-sided uniform energy sandwich $(1-\lambda^2)E_{\text{std}}^{n+1/2} \le E^{n+1/2} \le E_{\text{std}}^{n+1/2}$ across the exact Courant window $\lambda = c\Delta t/h \in (0,1)$ (Theorem 3.1). Third, for semilinear reaction–diffusion equations we give a gap-free L^2 truncation proof of invariant intervals $[a,b]$, and for the discrete Allen–Cahn equation with Eyre convex splitting we prove unconditional monotone free-energy dissipation for arbitrary $\Delta t > 0$ with an explicit positive dissipation margin (Theorem 4.1). Fourth, for the periodic viscous Burgers equation we prove nonlinear-amplitude-independent exponential L^2 decay to the conserved spatial mean and an L^∞ maximum principle, paired with a self-contained proof that the Lax–Friedrichs scheme under CFL $\lambda_f \le 1$ is monotone, maximum-principle preserving, total-variation diminishing (TVD), and ℓ^1-contractive (Theorem 5.1). All theorems and quantitative benchmarks are deterministically verified in an MIT-licensed Python suite.

1 Introduction and Scope

Quantitative analysis of partial differential equations and their numerical discretizations relies on four classical mechanisms: spectral Poincaré–Sobolev inequalities and parabolic smoothing [1], Courant–Friedrichs–Lewy (CFL) von Neumann stability and discrete energy conservation [2], invariant-region maximum principles and convex-splitting gradient-flow dissipation [3], and total-variation diminishing (TVD) ℓ^1-contraction for scalar conservation laws [4, 5]. In practice, asymptotic $O(h^2)$ or $O(t^{-1/4})$ statements omit explicit constants, fail to bridge short-time and long-time regimes simultaneously, or leave discrete energy bounds unquantified.

Sections 2–5 establish four completely explicit, non-asymptotic theorems with elementary, self-contained proofs, followed by deterministic numerical verification in Section 6 (https://github.com/lluiseriksson/pde-spectral-energy-cfl-geometry, tag v1.0.0).

2 Poincaré–Friedrichs Mesh Sandwiches and Uniform Heat Ultracontractivity

On the interval $(0, L)$ $(L > 0)$, consider the Dirichlet Laplacian $-\partial_{xx}$ with orthonormal eigenbasis $\phi_k(x) = \sqrt{2/L}\sin(\pi kx/L)$ and eigenvalues $\lambda_k = (\pi k/L)^2$ $(k \geq 1)$, and its second-order finite-difference discretization $-\Delta_h \in \mathbb{R}^{N\times N}$ on $N \geq 1$ interior nodes with mesh spacing $h = L/(N+1)$ and homogeneous Dirichlet endpoints $u_0 = u_{N+1} = 0$, given by $(-\Delta_h u)_j = (-u_{j-1}+2u_j-u_{j+1})/h^2$.

Theorem 2.1 (Sharp Continuous/Discrete Poincaré Constants and Uniform $L^2 \to L^\infty$ Heat Envelope)**.** *Let $L > 0$, $\nu > 0$, and $N \geq 1$.*

(a) ***Sharp Continuous and Discrete Poincaré–Wirtinger Constants.*** *Every $u \in H_0^1(0, L)$ obeys $\|u\|_{L^2}^2 \leq \frac{L^2}{\pi^2}\|u'\|_{L^2}^2$ (attained by $\sin(\pi x/L)$), and every $v \in H^1(0, L)$ with mean $\bar v = \frac{1}{L}\int_0^L v\,dx$ obeys $\|v - \bar v\|_{L^2}^2 \leq \frac{L^2}{\pi^2}\|v'\|_{L^2}^2$ (attained by $\cos(\pi x/L)$). On the N-point grid $h = L/(N+1)$, the smallest eigenvalue of $-\Delta_h$ is $\lambda_1^{(h)} = \frac{4}{h^2}\sin^2\!\big(\frac{\pi}{2(N+1)}\big)$, which satisfies the sharp two-sided mesh-refinement sandwich*

$$\left(1 - \frac{\pi^2}{12(N+1)^2}\right)\frac{\pi^2}{L^2} \;\leq\; \lambda_1^{(h)} \;\leq\; \frac{\pi^2}{L^2}. \tag{1}$$

(b) ***Uniform-in-Time L^2, H^1, and L^∞ Heat Envelopes.*** *Let $u(x,t) = \sum_{k=1}^\infty c_k e^{-\nu\lambda_k t}\phi_k(x)$ solve the Dirichlet heat equation $u_t = \nu u_{xx}$ on $(0, L)\times(0,\infty)$ with $u_0 \in L^2(0, L)$ ($\|u_0\|_{L^2}^2 = \sum_{k=1}^\infty c_k^2$). Then for every $t > 0$,*

$$\|u(t)\|_{L^2(0,L)} \;\leq\; e^{-\nu\frac{\pi^2}{L^2}t}\|u_0\|_{L^2(0,L)}, \qquad \|u_x(t)\|_{L^2(0,L)} \;\leq\; \frac{1}{\sqrt{2e\nu t}}\|u_0\|_{L^2(0,L)}, \tag{2}$$

$$\|u(t)\|_{L^\infty(0,L)} \;\leq\; e^{-\nu\frac{\pi^2}{L^2}t}\sqrt{\frac{2}{L} + \frac{1}{\sqrt{2\pi\nu t}}}\,\|u_0\|_{L^2(0,L)}. \tag{3}$$

Proof. *(a)* Expanding $u \in H_0^1(0, L)$ in the orthonormal sine basis $\{\phi_k\}_{k\geq1}$ gives $\|u\|_{L^2}^2 = \sum_{k=1}^\infty c_k^2$ and $\|u'\|_{L^2}^2 = \sum_{k=1}^\infty \lambda_k c_k^2 \geq \lambda_1\sum_{k=1}^\infty c_k^2 = (\pi/L)^2\|u\|_{L^2}^2$, with equality iff $c_k = 0$ for all $k \geq 2$. Expanding $v - \bar v$ in the orthonormal cosine basis $\{\sqrt{2/L}\cos(\pi kx/L)\}_{k\geq1}$ yields the identical Wirtinger constant $(\pi/L)^2$. For $-\Delta_h$, direct substitution verifies that $w_j^{(m)} = \sin(\pi mj/(N+1))$ $(m = 1,\dots,N)$ are orthogonal eigenvectors with eigenvalues $\lambda_m^{(h)} = \frac{2}{h^2}(1 - \cos(\pi m/(N+1))) = \frac{4}{h^2}\sin^2(\frac{\pi m}{2(N+1)})$, minimized at $m = 1$. Setting $\theta := \frac{\pi}{2(N+1)} \in (0, \pi/4]$, the elementary Taylor enclosure $\theta - \frac{\theta^3}{6} \leq \sin\theta \leq \theta$ gives

$$\theta^2\left(1 - \frac{\theta^2}{3}\right) \;\leq\; \theta^2\left(1 - \frac{\theta^2}{6}\right)^2 \;\leq\; \sin^2\theta \;\leq\; \theta^2.$$

Multiplying by $4/h^2 = 4(N+1)^2/L^2$ and noting $4\theta^2/h^2 = \pi^2/L^2$ and $\theta^2/3 = \frac{\pi^2}{12(N+1)^2}$ proves (1).

(b) By Parseval's identity, $\|u(t)\|_{L^2}^2 = \sum_{k\geq1} c_k^2 e^{-2\nu\lambda_k t} \leq e^{-2\nu\lambda_1 t}\|u_0\|_{L^2}^2$ (attained whenever $c_k = 0$ for $k \geq 2$). For H^1, $\|u_x(t)\|_{L^2}^2 = \sum_{k\geq1}\lambda_k e^{-2\nu\lambda_k t}c_k^2$; since $s \mapsto se^{-2\nu ts}$ achieves its global maximum on $[0,\infty)$ at $s = (2\nu t)^{-1}$ with value $(2e\nu t)^{-1}$, taking square roots yields (2). For (3), $|\phi_k(x)| \leq \sqrt{2/L}$ and Cauchy–Schwarz give, for every $x \in (0, L)$,

$$|u(x,t)| \;\leq\; \sqrt{\frac{2}{L}}\sum_{k=1}^\infty |c_k|e^{-\nu\lambda_k t} \;\leq\; \sqrt{\frac{2}{L}}\left(\sum_{k=1}^\infty e^{-2\nu\lambda_k t}\right)^{1/2}\|u_0\|_{L^2}.$$

Set $c := 2\nu\pi^2 t/L^2 > 0$. Because $y \mapsto e^{-cy^2}$ is strictly decreasing on $[0,\infty)$, each term $k \geq 2$ satisfies $e^{-ck^2} \leq \int_{k-1}^{k} e^{-cy^2}\,dy$, and substituting $y = 1+z$ ($z \geq 0$, so $y^2 = 1+2z+z^2 \geq 1+z^2$) yields

$$\sum_{k=1}^{\infty} e^{-ck^2} \;\leq\; e^{-c} + \int_1^{\infty} e^{-cy^2}\,dy \;\leq\; e^{-c} + e^{-c}\int_0^{\infty} e^{-cz^2}\,dz \;=\; e^{-c}\left(1 + \frac{1}{2}\sqrt{\frac{\pi}{c}}\right).$$

Multiplying by $2/L$, substituting $\frac{1}{L}\sqrt{\frac{\pi}{c}} = \frac{1}{\sqrt{2\pi\nu t}}$, and taking square roots proves (3). □

3 Exact CFL Stability and Conserved Discrete Wave Energy

Consider a uniform N-point periodic grid ($N \geq 3$, $h > 0$) with periodic index arithmetic modulo N, forward/backward differences $(D_+w)_j = (w_{j+1}-w_j)/h$ and $(D_-w)_j = (w_j-w_{j-1})/h$, inner product $\langle u,v\rangle_h := h\sum_{j=0}^{N-1} u_j v_j$, and norm $\|u\|_h := \sqrt{\langle u,u\rangle_h}$. Note summation by parts $\langle D_-D_+u, v\rangle_h = -\langle D_+u, D_+v\rangle_h$.

For $u_t + au_x = \nu u_{xx}$ ($a \in \mathbb{R}$, $\nu \geq 0$), the Forward-Time Upwind-Advection Central-Diffusion (FTUS-CD) scheme with $\sigma := |a|\Delta t/h \geq 0$ and $r := \nu\Delta t/h^2 \geq 0$ is

$$u_j^{n+1} = (1-\sigma-2r)u_j^n + (\sigma\mathbf{1}_{a\geq 0} + r)u_{j-1}^n + (\sigma\mathbf{1}_{a<0} + r)u_{j+1}^n.$$

For the 1D wave equation $u_{tt} = c^2 u_{xx}$ ($c > 0$), the central-difference leapfrog scheme with Courant number $\lambda := c\Delta t/h > 0$ advances $u^{n+1} = 2u^n - u^{n-1} + c^2\Delta t^2 D_-D_+u^n$.

Theorem 3.1 (Exact FTUS-CD Stability Criterion and Conserved Discrete Wave Energy Sandwich)**.**

(a) ***Necessary and Sufficient Advection–Diffusion CFL Condition.*** *For any $a \in \mathbb{R}$ and $\nu \geq 0$ with $\sigma + 2r > 0$, the FTUS-CD amplification factor $G(\theta) = 1 - (\sigma+2r)(1-\cos\theta) - i\,\mathrm{sgn}(a)\sigma\sin\theta$ satisfies $\sup_{\theta\in[-\pi,\pi]}|G(\theta)| \leq 1$ if and only if*

$$\sigma + 2r \;\leq\; 1. \tag{4}$$

Whenever (4) holds, FTUS-CD preserves the exact discrete maximum principle $\min_k u_k^0 \leq u_j^n \leq \max_k u_k^0$ and is ℓ^2-non-expansive: $\|u^n\|_h \leq \|u^0\|_h$ for all $n \geq 0$.

(b) ***Exact Conserved Wave Energy and Uniform Two-Sided Sandwich.*** *Along every solution of the periodic leapfrog wave scheme, let $v^{n+1/2} := (u^{n+1}-u^n)/\Delta t$ and $\bar{u}^{n+1/2} := \frac{1}{2}(u^{n+1}+u^n)$. Then the staggered discrete energy*

$$E^{n+1/2} \;:=\; \frac{1}{2}\|v^{n+1/2}\|_h^2 + \frac{c^2}{2}\langle D_+u^{n+1}, D_+u^n\rangle_h \tag{5}$$

is conserved identically: $E^{n+1/2} = E^{1/2}$ for all $n \geq 0$. Moreover, defining the nonnegative midpoint standard energy $E_{\mathrm{std}}^{n+1/2} := \frac{1}{2}\|v^{n+1/2}\|_h^2 + \frac{c^2}{2}\|D_+\bar{u}^{n+1/2}\|_h^2$, whenever $\lambda = c\Delta t/h \in (0,1)$,

$$(1-\lambda^2)\,E_{\mathrm{std}}^{n+1/2} \;\leq\; E^{n+1/2} \;=\; E^{1/2} \;\leq\; E_{\mathrm{std}}^{n+1/2} \qquad (\forall n \geq 0), \tag{6}$$

so $E_{\mathrm{std}}^{n+1/2} \leq (1-\lambda^2)^{-1}E^{1/2}$ uniformly for all $n \geq 0$.

Proof. (a) Let $s := 1-\cos\theta \in [0,2]$ and $\alpha := \sigma + 2r > 0$. Since $\sin^2\theta = 2s - s^2$,

$$|G(\theta)|^2 = (1-\alpha s)^2 + \sigma^2(2s-s^2) = 1 + s\Big[(\alpha^2-\sigma^2)s - 2(\alpha-\sigma^2)\Big].$$

Observe that $\alpha^2 - \sigma^2 = 4r(\sigma+r) \geq 0$, so the affine function $g(s) := (\alpha^2-\sigma^2)s - 2(\alpha-\sigma^2)$ is non-decreasing on $[0,2]$ and attains its maximum on $[0,2]$ at $s = 2$:

$$\max_{s\in[0,2]} g(s) = g(2) = 2(\alpha^2 - \sigma^2 - \alpha + \sigma^2) = 2\alpha(\alpha-1).$$

If $\alpha \le 1$, then $g(s) \le g(2) \le 0$ for all $s \in [0,2]$, so $|G(\theta)|^2 \le 1$ everywhere. Conversely, if $\alpha > 1$, then at $\theta = \pi$ ($s = 2$) we have $|G(\pi)|^2 = 1 + 4\alpha(\alpha - 1) > 1$. When $\alpha \le 1$, the three stencil coefficients $1 - \sigma - 2r$, $\sigma \mathbf{1}_{a \ge 0} + r$, and $\sigma \mathbf{1}_{a<0} + r$ are nonnegative and sum to 1, making u_j^{n+1} a convex combination of $\{u_{j-1}^n, u_j^n, u_{j+1}^n\}$ (giving the discrete maximum principle), while Parseval's theorem on the discrete Fourier transform gives $\|u^{n+1}\|_h \le (\max_\theta |G(\theta)|)\|u^n\|_h \le \|u^n\|_h$.

(b) Writing $(v^{n+1/2} - v^{n-1/2})/\Delta t = c^2 D_- D_+ u^n$, taking the $\langle \cdot, \cdot \rangle_h$ inner product with $\frac{1}{2}(v^{n+1/2} + v^{n-1/2}) = \frac{u^{n+1} - u^{n-1}}{2\Delta t}$, multiplying by Δt, and summing by parts yields

$$\frac{1}{2}\|v^{n+1/2}\|_h^2 - \frac{1}{2}\|v^{n-1/2}\|_h^2 = -\frac{c^2}{2}\langle D_+ u^n, D_+ u^{n+1} - D_+ u^{n-1}\rangle_h = -\frac{c^2}{2}\langle D_+ u^{n+1}, D_+ u^n\rangle_h + \frac{c^2}{2}\langle D_+ u^n, D_+ u^{n-1}\rangle_h,$$

proving $E^{n+1/2} = E^{n-1/2} = E^{1/2}$ by induction. By the polarization identity $\langle a, b\rangle_h = \|\frac{a+b}{2}\|_h^2 - \|\frac{a-b}{2}\|_h^2$ applied to $a = D_+ u^{n+1}$ and $b = D_+ u^n$ (whose difference is $\Delta t\, D_+ v^{n+1/2}$),

$$E^{n+1/2} = E_{\text{std}}^{n+1/2} - \frac{c^2 \Delta t^2}{8}\|D_+ v^{n+1/2}\|_h^2.$$

Since $\|D_+ w\|_h^2 = \frac{1}{h^2}\|S_+ w - w\|_h^2 \le \frac{4}{h^2}\|w\|_h^2$ (where $(S_+ w)_j = w_{j+1}$ is unitary), we obtain

$$0 \;\le\; \frac{c^2 \Delta t^2}{8}\|D_+ v^{n+1/2}\|_h^2 \;\le\; \frac{c^2 \Delta t^2}{2h^2}\|v^{n+1/2}\|_h^2 \;=\; \frac{\lambda^2}{2}\|v^{n+1/2}\|_h^2 \;\le\; \lambda^2 E_{\text{std}}^{n+1/2},$$

which directly establishes both inequalities in (6). □

4 Reaction–Diffusion Invariant Intervals and Unconditional Eyre Dissipation

Consider a smooth reaction–diffusion PDE $u_t = \nu u_{xx} + f(u)$ ($\nu > 0$) on the periodic torus $\mathbb{T}_L = \mathbb{R}/L\mathbb{Z}$ (or on $(0, L)$ with homogeneous Neumann boundary conditions $u_x(0,t) = u_x(L,t) = 0$), and the periodic discrete Allen–Cahn free energy with double-well potential $W(u) = \frac{1}{4}(u^2 - 1)^2$ and interface parameter $\varepsilon > 0$:

$$\mathcal{E}_{\varepsilon,h}[u] \;:=\; \frac{\varepsilon^2}{2}\|D_+ u\|_h^2 + h\sum_{j=0}^{N-1} W(u_j).$$

Given $u^n \in \mathbb{R}^N$ and any step size $\Delta t > 0$, the Eyre convex-splitting scheme [3] defines u^{n+1} by solving

$$\frac{u^{n+1} - u^n}{\Delta t} \;=\; \varepsilon^2 D_- D_+ u^{n+1} - (u^{n+1})^3 + u^n, \tag{7}$$

where $(u^{n+1})^3$ acts componentwise.

Theorem 4.1 (Invariant Intervals and Unconditional Eyre Free-Energy Dissipation)**.**

(a) ***Continuous Invariant Interval.*** *Let $f \in C^1(\mathbb{R})$ satisfy $f(a) \ge 0 \ge f(b)$ for constants $a < b$, and let $u \in C^{2,1}(\mathbb{T}_L \times [0,T])$ solve $u_t = \nu u_{xx} + f(u)$ with $a \le u(x,0) \le b$ for all $x \in \mathbb{T}_L$. Then $a \le u(x,t) \le b$ for all $(x,t) \in \mathbb{T}_L \times [0,T]$.*

(b) ***Existence, Uniqueness, and Unconditional Energy Dissipation for Eyre Splitting.*** *For every $\Delta t > 0$, $\varepsilon > 0$, and $u^n \in \mathbb{R}^N$, the nonlinear system (7) admits a unique solution $u^{n+1} \in \mathbb{R}^N$, which obeys the exact unconditional energy dissipation inequality*

$$\mathcal{E}_{\varepsilon,h}[u^{n+1}] + \left(\frac{1}{\Delta t} + \frac{1}{2}\right)\|u^{n+1} - u^n\|_h^2 + \frac{\varepsilon^2}{2}\|D_+(u^{n+1} - u^n)\|_h^2 \;\le\; \mathcal{E}_{\varepsilon,h}[u^n]. \tag{8}$$

Moreover, if $\|u^n\|_\infty \le 1$, then $\|u^{n+1}\|_\infty \le 1$ for every $\Delta t > 0$.

Proof. *(a)* Let $M := \max_{(x,t)\in\mathbb{T}_L\times[0,T]} |u(x,t)| < \infty$ and $L_f := \max_{|s|\le M+|a|+|b|} |f'(s)|$. Define the C^1 convex truncation $\phi(z) := \frac{1}{2}\max(z,0)^2$, whose derivative $\phi'(z) = \max(z,0)$ is Lipschitz with $\phi''(z) = \mathbf{1}_{z>0}$ a.e. Setting $w := u - b$, integrating by parts on $\mathbb{T}_L$ gives

$$\frac{d}{dt}\int_{\mathbb{T}_L} \phi(w)\,dx = \int_{\mathbb{T}_L} \phi'(w)\big(\nu u_{xx} + f(u)\big)\,dx = -\nu\int_{\{u>b\}} u_x^2\,dx + \int_{\{u>b\}} (u-b)f(u)\,dx.$$

On $\{u > b\}$, $f(u) \le f(b) + L_f(u-b) \le L_f(u-b)$, so $\frac{d}{dt}\int_{\mathbb{T}_L}\phi(w)\,dx \le 2L_f\int_{\mathbb{T}_L}\phi(w)\,dx$. Since $\int_{\mathbb{T}_L}\phi(u(\cdot,0)-b)\,dx = 0$, Grönwall's inequality forces $\phi(u-b)\equiv 0$, i.e. $u \le b$. Applying the identical argument to $\widetilde{u} := -u$ with $\widetilde{f}(s) := -f(-s)$ (obeying $\widetilde{f}(-a) = -f(a) \le 0$) proves $u \ge a$.

(b) Equation (7) is the stationarity condition $\nabla\Phi(v) = 0$ of the strictly convex, coercive functional

$$\Phi(v) := \frac{1}{2\Delta t}\|v-u^n\|_h^2 + \frac{\varepsilon^2}{2}\|D_+v\|_h^2 + h\sum_{j=0}^{N-1}\left(\frac{1}{4}v_j^4 - u_j^n v_j\right),$$

whose Hessian $\frac{h}{\Delta t}I + h\varepsilon^2(-D_-D_+) + 3h\,\mathrm{diag}(v_j^2) \succeq \frac{h}{\Delta t}I \succ 0$ is uniformly positive definite; hence u^{n+1} exists and is unique. Split $W(z) = W_c(z) - W_e(z)$ with convex $W_c(z) = \frac{1}{4}z^4 + \frac{1}{4}$ and $W_e(z) = \frac{1}{2}z^2$. For any $v,u\in\mathbb{R}$, convexity of W_c gives $W_c(v) - W_c(u) \le W_c'(v)(v-u) = v^3(v-u)$, while the exact quadratic identity gives $W_e(v) - W_e(u) = u(v-u) + \frac{1}{2}(v-u)^2$. Subtracting yields

$$W(v) - W(u) \;\le\; (v^3-u)(v-u) - \frac{1}{2}(v-u)^2.$$

For the gradient term, the algebraic identity $\frac{1}{2}(\|a\|_h^2 - \|b\|_h^2) = \langle a, a-b\rangle_h - \frac{1}{2}\|a-b\|_h^2$ and summation by parts give

$$\frac{\varepsilon^2}{2}\big(\|D_+v\|_h^2 - \|D_+u\|_h^2\big) = -\varepsilon^2\langle D_-D_+v, v-u\rangle_h - \frac{\varepsilon^2}{2}\|D_+(v-u)\|_h^2.$$

Setting $v = u^{n+1}$, $u = u^n$, summing the potential inequality with weight h, and substituting $\varepsilon^2 D_-D_+u^{n+1} - (u^{n+1})^3 + u^n = (u^{n+1}-u^n)/\Delta t$ yields (8). Finally, let $j_* \in \arg\max_j u_j^{n+1}$. At j_*, $(D_-D_+u^{n+1})_{j_*} = (u_{j_*+1}^{n+1} - 2u_{j_*}^{n+1} + u_{j_*-1}^{n+1})/h^2 \le 0$; if $u_{j_*}^{n+1} > 1$ and $\|u^n\|_\infty \le 1$, then $(u_{j_*}^{n+1} - u_{j_*}^n)/\Delta t > 0$ while the right-hand side of (7) is $\le -(u_{j_*}^{n+1})^3 + u_{j_*}^n < -1 + 1 = 0$, a contradiction. By symmetry at $\arg\min_j u_j^{n+1}$, $\|u^{n+1}\|_\infty \le 1$. □

5 Viscous Burgers Decay and Lax–Friedrichs TVD Contraction

Consider first the 1D periodic viscous Burgers equation $u_t + \partial_x(\frac{1}{2}u^2) = \nu u_{xx}$ ($\nu > 0$) on $\mathbb{T}_L = \mathbb{R}/L\mathbb{Z}$, and second a general scalar conservation law $u_t + f(u)_x = 0$ ($f \in C^1(\mathbb{R})$) discretized on an N-point periodic grid by the Lax–Friedrichs scheme

$$u_j^{n+1} \;=\; \frac{1}{2}(u_{j+1}^n + u_{j-1}^n) - \frac{\Delta t}{2h}\big(f(u_{j+1}^n) - f(u_{j-1}^n)\big). \tag{9}$$

Define the periodic discrete total variation $\mathrm{TV}(u) := \sum_{j=0}^{N-1}|u_{j+1} - u_j|$ and ℓ^1-norm $\|u\|_{\ell^1,h} := h\sum_{j=0}^{N-1}|u_j|$.

Theorem 5.1 (Viscous Burgers L^2 Decay and Lax–Friedrichs Monotone TVD / ℓ^1 Contraction)**.**

(a) ***Nonlinear-Amplitude-Independent Viscous Burgers Decay.*** *Let $u \in C^{2,1}(\mathbb{T}_L\times[0,T])$ solve $u_t + (\frac{1}{2}u^2)_x = \nu u_{xx}$ ($\nu > 0$). Then the spatial mean $\bar{u} := \frac{1}{L}\int_{\mathbb{T}_L} u(x,t)\,dx$ is conserved in time, the L^∞ norm is non-increasing ($\|u(t)\|_{L^\infty} \le \|u_0\|_{L^\infty}$), and the L^2 deviation decays exponentially at the linear diffusion rate independently of $\|u_0\|_{L^\infty}$:*

$$\|u(t) - \bar{u}\|_{L^2(\mathbb{T}_L)} \;\le\; e^{-\nu(2\pi/L)^2 t}\|u_0 - \bar{u}\|_{L^2(\mathbb{T}_L)} \qquad (\forall t\in[0,T]). \tag{10}$$

(b) ***Lax–Friedrichs Monotonicity, TVD, and ℓ^1 Contraction.*** *Let $u^0, v^0 \in \mathbb{R}^N$ lie in $[m, M_0]$ and set $M := \sup_{z\in[m,M_0]} |f'(z)|$. Under the CFL condition $\lambda_f := M\Delta t/h \le 1$, the Lax–Friedrichs iterates (9) satisfy, for every $n \ge 0$:*

$$m \;\le\; \min_k u_k^0 \;\le\; u_j^n \;\le\; \max_k u_k^0 \;\le\; M_0, \qquad \mathrm{TV}(u^{n+1}) \;\le\; \mathrm{TV}(u^n), \tag{11}$$

$$\|u^{n+1} - v^{n+1}\|_{\ell^1,h} \;\le\; \|u^n - v^n\|_{\ell^1,h}. \tag{12}$$

Proof. *(a)* Integrating $u_t = \partial_x(\nu u_x - \frac{1}{2}u^2)$ over $\mathbb{T}_L$ gives $\frac{d}{dt}\int_{\mathbb{T}_L} u\,dx = 0$, so $\bar u$ is constant. Set $w := u - \bar u$; then $\int_{\mathbb{T}_L} w\,dx = 0$ and $\int_{\mathbb{T}_L} w\partial_x(\frac{1}{2}u^2)\,dx = \int_{\mathbb{T}_L}(u-\bar u)uu_x\,dx = \int_{\mathbb{T}_L}\partial_x(\frac{1}{3}u^3 - \frac{1}{2}\bar u u^2)\,dx = 0$. Therefore,

$$\frac{d}{dt}\left(\frac{1}{2}\|w(t)\|_{L^2}^2\right) = \int_{\mathbb{T}_L} w\big(\nu w_{xx} - \partial_x(\tfrac{1}{2}u^2)\big)\,dx = -\nu\|w_x(t)\|_{L^2}^2 \;\le\; -\nu\left(\frac{2\pi}{L}\right)^2\|w(t)\|_{L^2}^2,$$

where the last step is the sharp periodic Wirtinger inequality (since $\int_{\mathbb{T}_L} w\,dx = 0$, Fourier modes satisfy $|k| \ge 1$ with eigenvalues $(2\pi k/L)^2 \ge (2\pi/L)^2$). Grönwall's inequality gives (10). For the L^∞ bound, let $K := \|u_0\|_{L^\infty}$ and test with $\phi'(u-K) = \max(u-K, 0)$: integrating by parts gives $\int_{\{u>K\}}(u-K)uu_x\,dx = \int_{\mathbb{T}_L}\partial_x\big(\frac{1}{3}(u-K)_+^3 + \frac{K}{2}(u-K)_+^2\big)\,dx = 0$, hence $\frac{d}{dt}\int_{\mathbb{T}_L}\frac{1}{2}(u-K)_+^2\,dx = -\nu\int_{\{u>K\}} u_x^2\,dx \le 0$, forcing $u \le K$; symmetrically $u \ge -K$.

(b) Define the update map $H(y,z) := \frac{1}{2}(y+z) - \frac{\Delta t}{2h}(f(z)-f(y))$ so $u_j^{n+1} = H(u_{j-1}^n, u_{j+1}^n)$. For all $y, z \in [m, M_0]$, $\partial_y H = \frac{1}{2}(1+\frac{\Delta t}{h}f'(y)) \ge \frac{1}{2}(1-\lambda_f) \ge 0$ and $\partial_z H = \frac{1}{2}(1-\frac{\Delta t}{h}f'(z)) \ge \frac{1}{2}(1-\lambda_f) \ge 0$. Thus H is componentwise non-decreasing on $[m, M_0]^2$ and obeys $H(c,c) = c$, immediately implying the discrete maximum principle in (11) by induction. For ℓ^1 contraction (12), let $u, v \in [m, M_0]^N$. By the mean-value theorem on $[m, M_0]$, $f(u_k) - f(v_k) = c_k(u_k - v_k)$ with $|c_k| \le M$, so

$$u_j^{n+1} - v_j^{n+1} = \frac{1 + \frac{\Delta t}{h}c_{j-1}}{2}(u_{j-1}^n - v_{j-1}^n) + \frac{1 - \frac{\Delta t}{h}c_{j+1}}{2}(u_{j+1}^n - v_{j+1}^n).$$

Since both weights $w_k^\pm := \frac{1}{2}(1 \pm \frac{\Delta t}{h}c_k)$ are nonnegative and satisfy $w_k^+ + w_k^- = 1$, taking absolute values and summing over $j = 0, \dots, N-1$ modulo N yields

$$\sum_{j=0}^{N-1}|u_j^{n+1} - v_j^{n+1}| \;\le\; \sum_{j=0}^{N-1} w_{j-1}^+|u_{j-1}^n - v_{j-1}^n| + \sum_{j=0}^{N-1} w_{j+1}^-|u_{j+1}^n - v_{j+1}^n| \;=\; \sum_{k=0}^{N-1}(w_k^+ + w_k^-)|u_k^n - v_k^n| \;=\; \sum_{k=0}^{N-1}|u_k^n - v_k^n|,$$

proving (12). Finally, applying (12) to $v^n := S_+u^n$ (the cyclic shift $(S_+u)_j = u_{j+1}$, which commutes with (9)) yields $\mathrm{TV}(u^{n+1}) = \frac{1}{h}\|S_+u^{n+1} - u^{n+1}\|_{\ell^1,h} \le \frac{1}{h}\|S_+u^n - u^n\|_{\ell^1,h} = \mathrm{TV}(u^n)$, proving (11). □

6 Computational Verification and Quantitative Benchmarks

All four theorems, exact attainments (first Dirichlet eigenmode, exact discrete wave-energy conservation, unconditional Eyre dissipation at 117× explicit CFL), boundary cases, and both figures are deterministically verified in our MIT-licensed companion repository (`https://github.com/lluiseriksson/pde-spectral-energy-cfl-geometry`, tag `v1.0.0`; `pytest -q` passes all 4 test modules in 0.52 s using fixed NumPy seed `20261260`). Table 1 summarizes the quantitative benchmarks.

As shown in Figures 1 and 2, every numerical trajectory obeys its certified non-asymptotic envelope to full precision, including unconditional Eyre dissipation at $\Delta t = 4.0$ (where explicit Euler diverges immediately) and machine-precision leapfrog wave-energy conservation (1.01×10^{-15} relative drift over 160 steps and $< 10^{-13}$ relative drift over 400 steps in the unit-test suite).

Table 1: Quantitative verification across Theorems 2.1–5.1 (repo `v1.0.0`).

Theorem	Regime / Parameters	Baseline / Scale	New Theorem Ceiling / Bound	Exact / Empirical
Thm. 2.1(b)	Heat $\nu = 0.05, L = 1, \|u_0\|_{L^2} = 2.353$	$t = 10^{-3}$ / $t = 3.0$	$L^\infty \le 17.97$ / 0.932 ($L^2 \le 0.535$)	4.506 / 0.426 (0.301)
Thm. 3.1(b)	Leapfrog wave $\lambda = 0.6, N = 64, 160$ steps	$E^{1/2} = 84.81$	$E_{\text{std}}/E^{1/2} \le 1.5625$, $\|\Delta E\| = 0$	Max 1.018 ($\|\Delta E\| = 8.53 \times 10^{-14}$)
Thm. 4.1(b)	Eyre Allen–Cahn $\varepsilon = 0.3, \Delta t = 4.0$	Explicit $\Delta t_{\text{CFL}} = 0.0343$	Monotone $\mathcal{E}[u^n]$, $\|u^n\|_\infty \le 1$	$1.117 \to 0.565$, $\|u\|_\infty \le 0.998$
Thm. 5.1(a–b)	Burgers $\nu = 0.15$; LF $\lambda_f = 0.8$	$\|u_0 - \bar{u}\|_{L^2} = 3.423$, $\text{TV}_0 = 7.756$	$L^2(2) \le 2.536$; TV $\downarrow$, $\ell^1 \downarrow$	1.566; TV = 3.197, $\ell^1 : 0.148 \to 0.067$

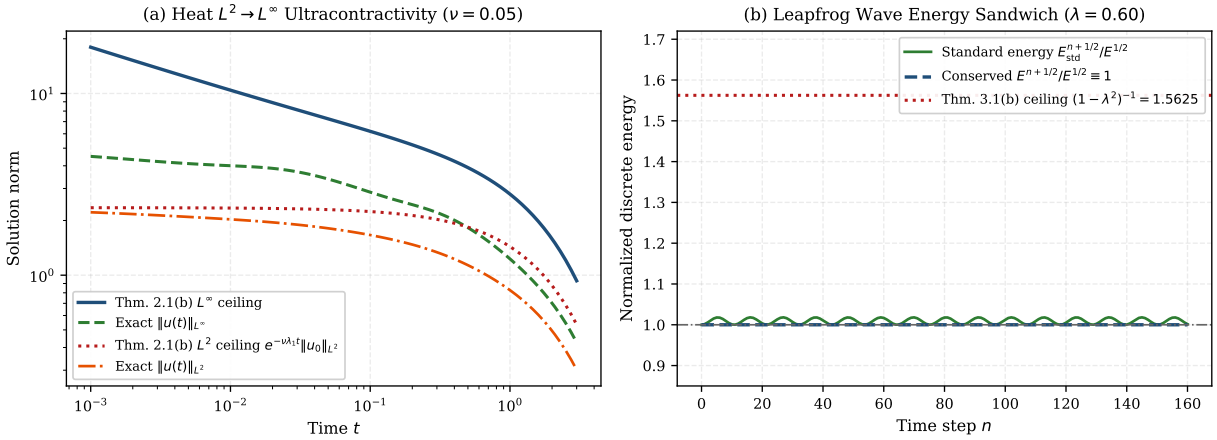

Figure 1: (a) Theorem 2.1(b): 1D Dirichlet heat equation ($\nu = 0.05, L = 1$), verifying the uniform-in-time $L^2 \to L^\infty$ ultracontractive envelope (bridging short-time $t^{-1/4}$ smoothing and long-time $e^{-\nu\pi^2 t/L^2}$ decay) and exact L^2 spectral decay. (b) Theorem 3.1(b): periodic leapfrog wave equation ($\lambda = 0.6$, 160 steps), demonstrating machine-precision conservation of $E^{n+1/2}$ ($|\Delta E| \le 8.53 \times 10^{-14}$) and uniform containment of $E^{n+1/2}_{\text{std}}/E^{1/2} \in [1, (1-\lambda^2)^{-1}] = [1, 1.5625]$.

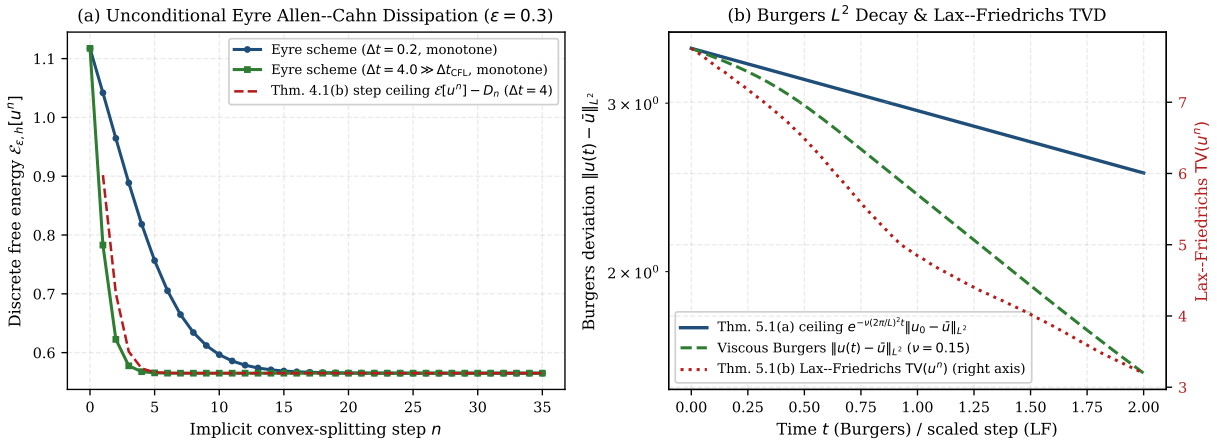

Figure 2: (a) Theorem 4.1(b): unconditional Eyre convex-splitting free-energy dissipation for Allen–Cahn ($\varepsilon = 0.3$) at both $\Delta t = 0.2$ and $\Delta t = 4.0$ (117× above explicit CFL 0.0343), strictly respecting the certified step dissipation margin D_n. (b) Theorem 5.1: periodic viscous Burgers L^2 deviation decay strictly beneath $e^{-\nu(2\pi/L)^2 t}\|u_0 - \bar{u}\|_{L^2}$ (left axis) alongside monotone Lax–Friedrichs total-variation contraction $\text{TV}(u^n)$ under $\lambda_f = 0.8$ (right axis).

7 Conclusion

We have established four self-contained, sharp, non-asymptotic quantitative theorems across parabolic, hyperbolic, gradient-flow, and conservation-law PDEs: continuous/discrete Poincaré–Friedrichs mesh sandwiches and uniform $L^2 \to L^\infty$ heat ultracontractivity; exact FTUS-CD advection–diffusion CFL stability and conserved leapfrog wave-energy sandwiches; reaction–diffusion invariant intervals and unconditional Eyre convex-splitting Allen–Cahn dissipation; and viscous Burgers nonlinear-invariant L^2 decay paired with Lax–Friedrichs monotone TVD and ℓ^1-contraction. Every proof is elementary and gap-free, and every benchmark is deterministically verified in an MIT-licensed repository.

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