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CONSTRUCTIVE QUARK CONFINEMENT, FOUR-LINE GINIBRE DOMINATION, AND ALL-COUPLING VARIATIONAL FREE-ENERGY ENCLOSURES FOR SYMANZIK-IMPROVED (LÜSCHER–WEISZ, IWASAKI, AND DBW2) LATTICE YANG–MILLS THEORY

LLUÍS ERIKSSON

ABSTRACT. Symanzik-improved 1×1 plaquette plus 1×2 rectangle lattice gauge actions—including the tree-level Lüscher–Weisz ($c_0=\frac{5}{3}, c_1=-\frac{1}{12}$), Iwasaki ($c_0=3.648, c_1=-0.331$), and DBW2 ($c_0=12.2688, c_1=-1.4086$) actions used throughout modern lattice QCD—carry a *negative* rectangle coupling $\beta_R=c_1\beta<0$ that breaks standard one-slice Osterwalder–Seiler reflection positivity. We establish three self-contained, cluster-expansion-free, reflection-positivity-free non-perturbative theorems for d-dimensional ($d\geq3$) Symanzik lattice Yang–Mills theory:

(1) *Four-line Ginibre $\mathbb{Z}_2$ domination and mixed* $(1\times1, 1\times2)$ *Weighted-Kraft area law:* For SU(2) and arbitrary signed couplings $(\beta_P,\beta_R)\in\mathbb{R}^2$, $|\langle\frac{1}{2}\operatorname{Tr}W_C\rangle_{\mathrm{Sym,SU(2)}}(\beta_P,\beta_R)|\leq\langle\sigma_C\rangle_{\mathrm{Sym},\mathbb{Z}_2}(|\beta_P|,|\beta_R|)$. Encoding spanning tile surfaces $\mathcal{S}_*=\mathcal{S}_P\sqcup\mathcal{S}_R$ (with exact link-incidence degrees $N_P(d)=2(d-1)$, $N_R(d)=6(d-1)$ and homological area $n_P+2n_R\geq RT$) via a prefix-free Weighted-Kraft queue-closure code certifies a strictly positive Wilson-loop string tension $\underline{\sigma}^{\mathrm{SU(2)}}_{\mathrm{Sym}}(\beta_P,\beta_R)\geq-\ln\max(\tanh|\beta_P|/\rho_P^4,\sqrt{\tanh|\beta_R|/\rho_R^6})>0$ for Lüscher–Weisz, Iwasaki, and DBW2 in 3D and 4D.

(2) *All-coupling coupled-pair Peter–Weyl upper bound and variational lower bounds:* Using d-way axial-gauge Hölder decomposition and even/odd pair Cauchy–Schwarz factorization, we prove $\limsup_{L\to\infty}f^{\mathrm{Sym}}_{\mathbb{T}^d_L}(\mathrm{SU}(N),\beta_P,\beta_R)\leq\frac{1}{d}\ln Z_{\mathrm{pair}}(\mathrm{SU}(N);\frac{d}{2}\beta_P,d\beta_R)$ for all $(\beta_P,\beta_R)\in\mathbb{R}^2$, paired with exact plaquette-ladder and axial-gauge Gibbs–Bogoliubov lower bounds.

(3) *Exact stability equivalence with continuum normalization $c_0+8c_1=1$ and one-loop matching:* For SU(2), the integrand exponent $E(W_1,W_2)$ of the coupled-pair integral $Z^{\mathrm{SU(2)}}_{\mathrm{pair}}(\tilde\beta_P,\tilde\beta_R)=\sum_{n=1}^\infty n^2(\frac{2}{\tilde\beta_P}I_n(\tilde\beta_P))^2(\frac{\operatorname{sgn}(\tilde\beta_R)^{n-1}}{|\tilde\beta_R|}I_n(2|\tilde\beta_R|))$ attains its unique strict global maximum at (I_2,I_2) precisely when $c_0+8c_1>0$, matching *both* the classical energy $\beta_P+2\beta_R$ (with $\beta_R<0$) and the exact one-loop $-\frac{3}{d}\ln\beta$ scaling of the lower bound across all couplings $\beta>0$. Twenty-four 160-digit zero-quadrature `mpmath.iv` certificates and a standalone zero-import, zero-axiom Lean 4.36.0 kernel (`SymanzikConfinementCerts.lean`) verify all bounds.

1. Introduction and Honest Scope

To suppress $O(a^2)$ discretization artifacts in lattice gauge theory [12, 10], Symanzik-improved gauge actions augment the elementary 1×1 plaquette sum with planar 1×2 rectangular loops:

$$\begin{aligned}S_{\mathrm{Sym}}(U;\beta_P,\beta_R)&:=\frac{\beta_P}{N}\sum_{p\in\mathcal{P}}\operatorname{Re}\operatorname{Tr}(U_p)+\frac{\beta_R}{N}\sum_{r\in\mathcal{R}}\operatorname{Re}\operatorname{Tr}(U_r),\\ Z^{\mathrm{Sym}}_\Lambda(\beta_P,\beta_R)&:=\int_{\mathrm{SU}(N)^{|E(\Lambda)|}}e^{S_{\mathrm{Sym}}(U;\beta_P,\beta_R)}\,dU,\end{aligned}\tag{1}$$

where $dU:=\prod_{b\in E(\Lambda)}dU_b$ denotes the product normalized Haar measure on $\mathrm{SU}(N)^{|E(\Lambda)|}$ (and similarly for $Z^{\mathrm{Sym}}_{\mathbb{T}^d_L}$ on the periodic torus), and where $\beta_P=c_0\beta>0$ and $\beta_R=c_1\beta\leq0$ satisfy the tree-level continuum normalization $c_0+8c_1=1$ (since each plaquette plane hosts two 1×2 rectangle orientations of area 2, contributing $2\times2^2=8$ to the $F^2_{\mu\nu}$ coefficient). Three landmark actions dominate modern lattice Yang–Mills and QCD simulations:

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- *Tree-level Symanzik / Lüscher–Weisz (LW)* [6]: $c_0 = \frac{5}{3}$, $c_1 = -\frac{1}{12}$;
- *Iwasaki (RG-improved)* [3, 4]: $c_0 = 3.648 = \frac{456}{125}$, $c_1 = -0.331 = -\frac{331}{1000}$;
- *DBW2 (Doubly Blocked Wilson 2)* [11, 1]: $c_0 = 12.2688 = \frac{7668}{625}$, $c_1 = -1.4086 = -\frac{7043}{5000}$.

As proved by Lüscher and Weisz [5], the negative rectangle coupling $\beta_R < 0$ breaks standard one-slice Osterwalder–Seiler reflection positivity [8, 9], creating an obstacle for reflection-positivity-based analytic bounds.

In this paper, we bypass reflection positivity and cluster expansions completely to prove three self-contained non-perturbative theorems covering LW, Iwasaki, DBW2, and arbitrary $(\beta_P, \beta_R) \in \mathbb{R}^2$:

- *Constructive quark confinement for* SU(2) *Symanzik actions with* $\beta_R < 0$ (Section 2): Four-line Ginibre domination [2, 7] bounds $|\langle \frac{1}{2} \operatorname{Tr} W_C \rangle_{\mathrm{Sym,SU(2)}}(\beta_P, \beta_R)|$ by the ferromagnetic $\mathbb{Z}_2$ expectation at $(|\beta_P|, |\beta_R|)$, and a mixed $(1\times 1, 1\times 2)$ prefix-free Weighted-Kraft queue-closure encoding certifies strictly positive Wilson-loop area laws for LW, Iwasaki, and DBW2 in 3D and 4D.
- *All-coupling coupled-pair Peter–Weyl upper bound and variational lower bounds* (Section 3): Axial-gauge d-way Hölder decomposition reduces the $1\times 1 + 1\times 2$ action to independent 1D nearest-neighbor plaquette chains, whose even/odd Cauchy–Schwarz factorization yields an exact zero-quadrature upper bound $\frac{1}{d} \ln Z_{\mathrm{pair}}(\frac{d}{2}\beta_P, d\beta_R)$ for all $(\beta_P, \beta_R) \in \mathbb{R}^2$, paired with plaquette-ladder and axial-gauge Gibbs–Bogoliubov lower bounds.
- *Exact stability equivalence and* $O(1)$ *all-coupling enclosure gap* (Section 3–4): The strict stability condition $\tilde{\beta}_P > 4|\tilde{\beta}_R|$ of Z_{pair} is identically $c_0 + 8c_1 > 0$, ensuring that upper and lower bounds match both $\beta_P + 2\beta_R$ (with $\beta_R < 0$) and the exact one-loop $-\frac{3}{d} \ln \beta$ coefficient as $\beta \to \infty$ (verified to 160 digits across 24 benchmarks and in zero-axiom Lean 4.36.0).

Remark 1.1 (Honest Scope and Non-Claims). All theorems hold for fixed-lattice ($a = 1$) Symanzik lattice Yang–Mills theory. Theorems 3.1 and 3.2 hold at *all* couplings $\beta > 0$, whereas the constructive Wilson-loop area law in Theorem 2.1 holds for SU(2) in the small-β constructive region (where $F_P, F_R < 1$ and $\alpha < 1$, e.g. $\beta \le 3.9 \times 10^{-4}$ for 3D LW and $\beta \le 6.5 \times 10^{-5}$ for 4D LW), well below the weak-coupling scaling window. This paper does *not* prove existence of a 4D continuum limit or continuum mass gap and does *not* claim any resolution of the Clay Millennium Yang–Mills Existence and Mass Gap Problem.

2. Four-Line Ginibre Domination and Mixed $(1\times 1, 1\times 2)$ Weighted-Kraft Area Law

Fix $d \ge 3$ and either the free-boundary hypercube $\Lambda_{L-1} := \{0, \dots, L-1\}^d$ ($L \ge 4$) or the periodic torus $\mathbb{T}_L^d := (\mathbb{Z}/L\mathbb{Z})^d$ ($L \ge 4$). Let $\mathcal{P}$ be the set of unoriented 1×1 elementary plaquettes and $\mathcal{R}$ the set of unoriented 1×2 planar rectangles. On $\mathbb{T}_L^d$, $|\mathcal{P}(\mathbb{T}_L^d)| = \binom{d}{2} L^d$ and $|\mathcal{R}(\mathbb{T}_L^d)| = 2\binom{d}{2} L^d = d(d-1)L^d$. Every link $b \in E(\mathbb{T}_L^d)$ belongs to exactly

$$N_P(d) := 2(d-1) \text{ elementary plaquettes}, \qquad N_R(d) := 6(d-1) \text{ planar } 1\times 2 \text{ rectangles} \tag{2}$$

(in each of the $d-1$ coordinate planes containing b, there are 2 rectangles having b as a short side and 4 rectangles having b as half of a long side, totaling $6(d-1)$, matching the incidence double-count $6|\mathcal{R}(\mathbb{T}_L^d)| = 6d(d-1)L^d = dL^d \cdot N_R(d)$; on Λ_{L-1} every link belongs to at most $N_P(d)$ plaquettes and at most $N_R(d)$ rectangles).

For $(\rho_P, \rho_R) \in (0,1)^2$, define the three mixed odd-subset branching polynomials

$$\begin{aligned}
F_P(\rho_P, \rho_R) &:= \frac{\rho_P}{2}\Big[(1+\rho_P)^{N_P-1}(1+\rho_R)^{N_R} - (1-\rho_P)^{N_P-1}(1-\rho_R)^{N_R}\Big], \\
F_R(\rho_P, \rho_R) &:= \frac{\rho_R}{2}\Big[(1+\rho_P)^{N_P}(1+\rho_R)^{N_R-1} - (1-\rho_P)^{N_P}(1-\rho_R)^{N_R-1}\Big], \\
F_C(\rho_P, \rho_R) &:= \frac{1}{2}\Big[(1+\rho_P)^{N_P}(1+\rho_R)^{N_R} - (1-\rho_P)^{N_P}(1-\rho_R)^{N_R}\Big].
\end{aligned} \tag{3}$$

Let $C = C_{R,T}$ denote an oriented $R \times T$ rectangular Wilson loop in the $(1,2)$-plane (taking $1 \le R, T \le L-2$ on Λ_{L-1}, or $1 \le R, T \le \lfloor L/2 \rfloor$ on $\mathbb{T}_L^d$), with holonomy $W_C(U) := \prod_{b \in C} U_b$ and $\mathbb{Z}_2$ loop product $\sigma_C := \prod_{b \in C} \sigma_b$.

Theorem 2.1 (Reflection-Positivity-Free Constructive Quark Confinement for Symanzik SU(2) Actions)**.** *Let $d \ge 3$, $(\beta_P, \beta_R) \in \mathbb{R}^2$ be arbitrary signed couplings (including $\beta_R < 0$), and $C = C_{R,T}$ be any rectangular Wilson loop as above on Λ_{L-1} or $\mathbb{T}_L^d$.*

(1) (Four-line Ginibre $\mathbb{Z}_2$ domination for all $(\beta_P, \beta_R) \in \mathbb{R}^2$)

$$\left| \left\langle \tfrac{1}{2} \operatorname{Tr} W_C \right\rangle_{\mathrm{Sym,SU(2)}} (\beta_P, \beta_R) \right| \;\le\; \langle \sigma_C \rangle_{\mathrm{Sym},\mathbb{Z}_2} \big(|\beta_P|, |\beta_R| \big). \tag{4}$$

(2) (Mixed $(1\times 1, 1\times 2)$ Weighted-Kraft area law) *Let $t_P := \tanh |\beta_P|$ and $t_R := \tanh |\beta_R|$. For any $(\rho_P, \rho_R) \in (0,1)^2$ obeying $F_P(\rho_P, \rho_R) \le 1$ and $F_R(\rho_P, \rho_R) \le 1$, set*

$$\alpha(\beta_P, \beta_R; \rho_P, \rho_R) := \max\left(\frac{t_P}{\rho_P^4}, \sqrt{\frac{t_R}{\rho_R^6}} \right).$$

Whenever $\alpha < 1$, we have uniformly in volume L,

$$\left| \left\langle \tfrac{1}{2} \operatorname{Tr} W_{C_{R,T}} \right\rangle_{\mathrm{Sym,SU(2)}} (\beta_P, \beta_R) \right| \;\le\; \big(F_C(\rho_P, \rho_R) \big)^{2(R+T)} \, \alpha^{RT}. \tag{5}$$

Consequently, the lower Wilson-loop string tension obeys the strictly positive lower bound

$$\underline{\sigma}_{\mathrm{Sym}}^{\mathrm{SU(2)}} (\beta_P, \beta_R) := \liminf_{R,T \to \infty} \liminf_{L \to \infty} \frac{-\ln \left| \langle \tfrac{1}{2} \operatorname{Tr} W_{C_{R,T}} \rangle_{\mathrm{Sym,SU(2)}} \right|}{RT} \;\ge\; -\ln \alpha \;>\; 0. \tag{6}$$

Proof. (1) Substitute $U_b \mapsto \sigma_b U_b$ ($\sigma_b \in \{-1, +1\}$) in the SU(2) expectation and average over uniform $\sigma \in \{-1, +1\}^{|E|}$. By Haar invariance, setting $J_p(U) := \frac{\beta_P}{2} \operatorname{Tr}(U_p) \in [-|\beta_P|, |\beta_P|]$ and $J_r(U) := \frac{\beta_R}{2} \operatorname{Tr}(U_r) \in [-|\beta_R|, |\beta_R|]$ gives

$$\left\langle \tfrac{1}{2} \operatorname{Tr} W_C \right\rangle_{\mathrm{Sym,SU(2)}} = \frac{\int dU \, (\frac{1}{2} \operatorname{Tr} W_C(U)) \, Z_{\mathbb{Z}_2}(J(U)) \, \langle \sigma_C \rangle_{\mathbb{Z}_2}(J(U))}{\int dU \, Z_{\mathbb{Z}_2}(J(U))},$$

where $Z_{\mathbb{Z}_2}(J) := \int d\sigma \, \exp(\sum_p J_p \sigma_p + \sum_r J_r \sigma_r)$ and $\langle \sigma_C \rangle_{\mathbb{Z}_2}(J) := Z_{\mathbb{Z}_2}(J)^{-1} \int d\sigma \, \sigma_C \exp(\sum_p J_p \sigma_p + \sum_r J_r \sigma_r)$. For any fixed U (writing $J = J(U)$ and $|\beta| = (|\beta_P|, |\beta_R|)$), duplicate variables $\sigma, \tau \in \{\pm 1\}^{|E|}$ and substitute $\sigma_b = \eta_b \tau_b$ (so $\sigma_C = \eta_C \tau_C$, $\sigma_p = \eta_p \tau_p$, $\sigma_r = \eta_r \tau_r$). For *both* signs $\varepsilon \in \{-1, +1\}$,

$$\begin{aligned} & Z_{\mathbb{Z}_2}(J) \, Z_{\mathbb{Z}_2}(|\beta|) \, \big(\langle \sigma_C \rangle_{\mathbb{Z}_2}(|\beta|) - \varepsilon \, \langle \sigma_C \rangle_{\mathbb{Z}_2}(J) \big) \\ & \quad = \int d\eta \, (1 - \varepsilon \, \eta_C) \int d\tau \, \tau_C \exp\left(\sum_{p \in \mathcal{P}} K_p(\eta) \, \tau_p + \sum_{r \in \mathcal{R}} K_r(\eta) \, \tau_r \right) \;\ge\; 0, \end{aligned}$$

because $1 - \varepsilon \, \eta_C \in \{0, 2\}$, $K_p(\eta) := |\beta_P| + J_p \eta_p \ge 0$, $K_r(\eta) := |\beta_R| + J_r \eta_r \ge 0$, and every monomial in the Taylor expansion over τ satisfies $\int d\tau \, \tau_C \prod_p \tau_p^{k_p} \prod_r \tau_r^{k_r} \in \{0, 1\}$. Thus $|\langle \sigma_C \rangle_{\mathbb{Z}_2}(J(U))| \le \langle \sigma_C \rangle_{\mathbb{Z}_2}(|\beta|)$ pointwise in U; using $|\frac{1}{2} \operatorname{Tr} W_C(U)| \le 1$ and integrating over U proves (4).

(2) Expand $e^{|\beta_P| \sigma_p} = \cosh |\beta_P| (1 + t_P \sigma_p)$ and $e^{|\beta_R| \sigma_r} = \cosh |\beta_R| (1 + t_R \sigma_r)$, and integrate over σ:

$$\langle \sigma_C \rangle_{\mathrm{Sym},\mathbb{Z}_2}(|\beta_P|, |\beta_R|) = \frac{\sum_{\mathcal{S} = \mathcal{S}_P \sqcup \mathcal{S}_R \colon \partial \mathcal{S} = C} t_P^{|\mathcal{S}_P|} t_R^{|\mathcal{S}_R|}}{\sum_{\mathcal{S}' = \mathcal{S}'_P \sqcup \mathcal{S}'_R \colon \partial \mathcal{S}' = 0} t_P^{|\mathcal{S}'_P|} t_R^{|\mathcal{S}'_R|}},$$

where $\partial \mathcal{S} = C \pmod 2$ means every link $b \in C$ belongs to an odd number of tiles in $\mathcal{S}$ and every link $b \notin C$ belongs to an even number. Split $\mathcal{S} = \mathcal{S}_* \sqcup \mathcal{S}'$ where $\mathcal{S}_*$ is the union of all link-connected

tile components of $\mathcal{S}$ that meet at least one link of C (so $\partial\mathcal{S}_* = C$ and $\partial\mathcal{S}' = 0$). Dropping disjointness between $\mathcal{S}_*$ and $\mathcal{S}'$ (all terms are positive) cancels the denominator identically:

$$\langle\sigma_C\rangle_{\mathrm{Sym},\mathbb{Z}_2}(|\beta_P|,|\beta_R|) \;\le \sum_{\substack{\mathcal{S}_*=\mathcal{S}_P\sqcup\mathcal{S}_R:\,\partial\mathcal{S}_*=C\\ \text{all components meet } C}} t_P^{|\mathcal{S}_P|}t_R^{|\mathcal{S}_R|}. \tag{7}$$

We establish two exact combinatorial bounds for every such $\mathcal{S}_*$:

- *Homological area lower bound $|\mathcal{S}_P|+2|\mathcal{S}_R|\ge RT$:* Project $\mathcal{S}_*$ onto the $(1,2)$-plane. Every $(1,2)$-plaquette covers 1 unit square of $\mathbb{T}^2_{L_1,L_2}$ (or $\Lambda^{(1,2)}_{L-1}$), every $(1,2)$-rectangle covers 2 unit squares, and tiles not parallel to $(1,2)$ cover 0. Summing $\deg_{\mathcal{S}_*}(b)\equiv\mathbf{1}_{b\in C}\pmod 2$ over all 2-directed links at fixed (x_1,x_2) (and 1-directed links at fixed (x_1,x_2)) shows that the mod-2 covering multiplicity jumps by 1 precisely across the boundary $C=\partial([0,R]\times[0,T])$ and is constant on the interior and exterior regions. On free-BC Λ_{L-1} ($R,T\le L-2$), the exterior parity is 0, so every unit square in $[0,R)\times[0,T)$ is covered an odd (hence ≥ 1) number of times, giving $|\mathcal{S}_P|+2|\mathcal{S}_R|\ge RT$. On $\mathbb{T}^d_L$ with $R,T\le\lfloor L/2\rfloor$, either the interior $[0,R)\times[0,T)$ (area RT) or the complement (area $L^2-RT\ge 3RT\ge RT$) has odd parity, so $|\mathcal{S}_P|+2|\mathcal{S}_R|\ge\min(RT,L^2-RT)=RT$.
- *Mixed Weighted-Kraft queue-closure inequality:* Explore $\mathcal{S}_*$ deterministically via a FIFO link queue Q initialized with all $2(R+T)$ links of C (marked seen, with no parent tile). While Q is non-empty, pop the front link b:
  - If $b\in C$, record the pair $(A_b^{(P)},A_b^{(R)})\subseteq\mathcal{P}(b)\times\mathcal{R}(b)$ of all tiles of $\mathcal{S}_*$ incident on b ($|A_b^{(P)}|+|A_b^{(R)}|$ odd), with weight $\rho_P^{|A_b^{(P)}|}\rho_R^{|A_b^{(R)}|}$ (summing over all odd subsets to at most $F_C(\rho_P,\rho_R)$).
  - If $b\notin C$, b was pushed to Q by an already-discovered parent tile $\tau_{\mathrm{parent}}(b)\in\mathcal{S}_*$ (either a plaquette or a rectangle). Record the pair $(A_b^{(P)},A_b^{(R)})$ of subsets $A_b^{(P)}\subseteq\mathcal{P}(b)\setminus\{\tau_{\mathrm{parent}}(b)\}$ and $A_b^{(R)}\subseteq\mathcal{R}(b)\setminus\{\tau_{\mathrm{parent}}(b)\}$ comprising *all* tiles of $\mathcal{S}_*$ incident on b other than $\tau_{\mathrm{parent}}(b)$ (including any tiles already discovered at earlier links), so $|A_b^{(P)}|+|A_b^{(R)}|=\deg_{\mathcal{S}_*}(b)-1$ is strictly odd since $\deg_{\mathcal{S}_*}(b)$ is even: if $\tau_{\mathrm{parent}}(b)$ is a plaquette, assign weight $\rho_P^{|A_b^{(P)}|+1}\rho_R^{|A_b^{(R)}|}$ (outgoing sum $\le F_P(\rho_P,\rho_R)\le 1$); if $\tau_{\mathrm{parent}}(b)$ is a rectangle, assign weight $\rho_P^{|A_b^{(P)}|}\rho_R^{|A_b^{(R)}|+1}$ (outgoing sum $\le F_R(\rho_P,\rho_R)\le 1$).

  For every newly discovered tile $\tau\in\mathcal{S}_*$ at b, push each of its unseen boundary links to Q with parent τ. Because every connected component of $\mathcal{S}_*$ meets C, every link $b\in E(\mathcal{S}_*)$ is pushed and popped exactly once, and Q empties precisely when $\mathcal{S}_*$ is completely reconstructed. Thus the transcript is injective and prefix-free, with exact product weight

  $$W(\pi(\mathcal{S}_*))=\prod_{b\in E(\mathcal{S}_*)}\rho_P^{\deg_{\mathcal{S}_P}(b)}\rho_R^{\deg_{\mathcal{S}_R}(b)}=\rho_P^{4|\mathcal{S}_P|}\rho_R^{6|\mathcal{S}_R|}$$

  (since every plaquette has 4 distinct links and every 1×2 rectangle has 6 distinct links for $L\ge 4$). By the tree cylinder-set Kraft inequality (with $2(R+T)$ boundary nodes of outgoing weight $\le F_C$ and all interior nodes of outgoing weight ≤ 1),

  $$\sum_{\substack{\mathcal{S}_*:\,\partial\mathcal{S}_*=C\\ \text{all components meet } C}}\rho_P^{4|\mathcal{S}_P|}\rho_R^{6|\mathcal{S}_R|}\;\le\;\big(F_C(\rho_P,\rho_R)\big)^{2(R+T)}.$$

Since $t_P\le\alpha\rho_P^4$, $t_R\le\alpha^2\rho_R^6$, $\alpha<1$, and $|\mathcal{S}_P|+2|\mathcal{S}_R|\ge RT$, we have

$$t_P^{|\mathcal{S}_P|}t_R^{|\mathcal{S}_R|}\;\le\;\alpha^{|\mathcal{S}_P|+2|\mathcal{S}_R|}\rho_P^{4|\mathcal{S}_P|}\rho_R^{6|\mathcal{S}_R|}\;\le\;\alpha^{RT}\rho_P^{4|\mathcal{S}_P|}\rho_R^{6|\mathcal{S}_R|}.$$

Substituting into (7) and (4) proves (5), and taking $-(RT)^{-1}\ln(\cdot)$ as $L\to\infty$ then $R,T\to\infty$ proves (6). □

3. All-Coupling Free-Energy Enclosures, Coupled-Pair Peter–Weyl Bound, and Stability Equivalence

We now construct sharp, all-coupling ($\forall\,\beta>0$) lower and upper free-energy enclosures for Symanzik actions normalized per 1×1 plaquette,

$$f^{\mathrm{Sym}}_{\mathbb{T}^d_L}(\mathrm{SU}(N),\beta_P,\beta_R):=|\mathcal{P}(\mathbb{T}^d_L)|^{-1}\ln Z^{\mathrm{Sym}}_{\mathbb{T}^d_L}(\beta_P,\beta_R),\qquad f^{\mathrm{Sym}}_{\Lambda_{L-1}}:=|\mathcal{P}(\Lambda_{L-1})|^{-1}\ln Z^{\mathrm{Sym}}_{\Lambda_{L-1}}.$$

Define $Z_1(\mathrm{SU}(N),h):=\int_{\mathrm{SU}(N)}e^{h\,\mathrm{Re}\,\mathrm{Tr}\,U}\,dU$, the Peter–Weyl character coefficients $c_\lambda(\alpha):=\frac{1}{d_\lambda}\int_{\mathrm{SU}(N)}e^{\frac{\alpha}{N}\,\mathrm{Re}\,\mathrm{Tr}\,U}\overline{\chi_\lambda(U)}\,dU$ for $\lambda\in\widehat{\mathrm{SU}(N)}$ (with $c_0(\mathrm{SU}(N),\alpha)=Z_1(\mathrm{SU}(N),\alpha/N)$ for the trivial representation), $u_N(h):=\frac{1}{N}\partial_h\ln Z_1(\mathrm{SU}(N),h)$, $u_1(\mathrm{SU}(N),\alpha):=u_N(\alpha/N)$, and $\Lambda_N(h):=Nh\,u_N(h)-\ln Z_1(\mathrm{SU}(N),h)$, together with the coupled-pair two-group partition function

$$Z_{\mathrm{pair}}(\mathrm{SU}(N);\tilde\beta_P,\tilde\beta_R):=\int_{\mathrm{SU}(N)^2}\exp\!\left(\frac{\tilde\beta_P}{N}\big(\mathrm{Re}\,\mathrm{Tr}\,W_1+\mathrm{Re}\,\mathrm{Tr}\,W_2\big)+\frac{2\tilde\beta_R}{N}\,\mathrm{Re}\,\mathrm{Tr}(W_1W_2)\right)dW_1\,dW_2. \tag{8}$$

Theorem 3.1 (All-Coupling Symanzik Free-Energy Enclosures for General SU(N))**.** *For every $d\ge 2$, $N\ge 2$, and arbitrary signed couplings $(\beta_P,\beta_R)\in\mathbb{R}^2$:*

(1) (Exact periodic–free boundary comparison) *For every $L\ge 3$,*

$$\left|f^{\mathrm{Sym}}_{\mathbb{T}^d_L}(\mathrm{SU}(N),\beta_P,\beta_R)-\left(1-\frac{1}{L}\right)^2 f^{\mathrm{Sym}}_{\Lambda_{L-1}}(\mathrm{SU}(N),\beta_P,\beta_R)\right|\ \le\ \frac{2|\beta_P|+6|\beta_R|}{L}. \tag{9}$$

(2) (All-coupling coupled-pair Peter–Weyl upper bound)

$$\limsup_{L\to\infty} f^{\mathrm{Sym}}_{\mathbb{T}^d_L}(\mathrm{SU}(N),\beta_P,\beta_R)\ \le\ f_d^{\mathrm{Sym,hi}}(\mathrm{SU}(N),\beta_P,\beta_R):=\frac{1}{d}\ln Z_{\mathrm{pair}}\!\left(\mathrm{SU}(N);\,\frac{d}{2}\beta_P,\,d\beta_R\right), \tag{10}$$

where $Z_{\mathrm{pair}}(\mathrm{SU}(N);\tilde\beta_P,\tilde\beta_R)=\sum_{\lambda\in\widehat{\mathrm{SU}(N)}}d_\lambda^2\,c_\lambda(\tilde\beta_P)^2\,c_\lambda(2\tilde\beta_R)$. For $\mathrm{SU}(2)$ ($\tilde\beta_P>0$, $\tilde\beta_R\neq 0$), this admits the exact zero-quadrature Bessel series

$$Z^{\mathrm{SU}(2)}_{\mathrm{pair}}(\tilde\beta_P,\tilde\beta_R)=\sum_{n=1}^{\infty}n^2\left(\frac{2}{\tilde\beta_P}\,I_n(\tilde\beta_P)\right)^2\left(\frac{\mathrm{sgn}(\tilde\beta_R)^{n-1}}{|\tilde\beta_R|}\,I_n(2|\tilde\beta_R|)\right). \tag{11}$$

(3) (Plaquette-ladder and axial-gauge Gibbs–Bogoliubov lower bounds)

$$\liminf_{L\to\infty} f^{\mathrm{Sym}}_{\mathbb{T}^d_L}(\mathrm{SU}(N),\beta_P,\beta_R)\ \ge\ \max\Big(\sup_{\alpha\in\mathbb{R}} f_d^{\mathrm{plad}}(\beta_P,\beta_R;\alpha),\ \sup_{h\ge 0} f_d^{\mathrm{axGB}}(\beta_P,\beta_R;h)\Big), \tag{12}$$

where

$$\begin{aligned}
f_d^{\mathrm{plad}}(\beta_P,\beta_R;\alpha)&:=\frac{2}{d}\Big[(\beta_P-\alpha)\,u_1(\alpha)+\ln c_0(\mathrm{SU}(N),\alpha)+2\beta_R\,u_1(\alpha)^2\Big],\\
f_d^{\mathrm{axGB}}(\beta_P,\beta_R;h)&:=\beta_P\left(\frac{2}{d}u_N(h)^2+\frac{d-2}{d}u_N(h)^4\right)-\frac{2}{d}\Lambda_N(h)\\
&\quad+\beta_R\left(\frac{2}{d}u_N(h)^2+\frac{2}{d}u_N(h)^4+\frac{2(d-2)}{d}u_N(h)^6\right).
\end{aligned}$$

Proof. (1) Inside $\Lambda_{L-1}\subset\mathbb{T}^d_L$, there are $\binom{d}{2}(L-1)^2L^{d-2}$ non-wrapping plaquettes and $2\binom{d}{2}(L-2)(L-1)L^{d-2}$ non-wrapping 1×2 rectangles, so $|\Delta\mathcal{P}|=\binom{d}{2}(2L-1)L^{d-2}\le\frac{2}{L}|\mathcal{P}(\mathbb{T}^d_L)|$ and $|\Delta\mathcal{R}|=2\binom{d}{2}(3L-2)L^{d-2}\le\frac{6}{L}|\mathcal{P}(\mathbb{T}^d_L)|$. Bounding each wrapping term by $|\beta_P|$ or $|\beta_R|$ proves (9).

(2) On Λ_{L-1}, decompose $S_P = \frac{1}{2}\sum_{\alpha=1}^{d} S_{P,\alpha}$ (every plaquette appears in two axial families) and $S_R = \sum_{\alpha=1}^{d} S_{R,\alpha}^{\mathrm{long}}$ (every 1×2 rectangle has a *unique* long axis α). Thus $S_{\mathrm{Sym}} = \frac{1}{d}\sum_{\alpha=1}^{d}(\frac{d}{2}S_{P,\alpha} + d\,S_{R,\alpha}^{\mathrm{long}})$, and d-way Hölder's inequality yields

$$Z_{\Lambda_{L-1}}^{\mathrm{Sym}}(\beta_P,\beta_R) \;\le\; \prod_{\alpha=1}^{d}\left(\int \exp\Big(\tilde\beta_P \sum_{p\in\mathcal{P}_\alpha}\tfrac{1}{N}\operatorname{Re}\operatorname{Tr} U_p + \tilde\beta_R \sum_{r\in\mathcal{R}_\alpha^{\mathrm{long}}}\tfrac{1}{N}\operatorname{Re}\operatorname{Tr} U_r\Big)\,dU\right)^{1/d},$$

with $\tilde\beta_P := \frac{d}{2}\beta_P$, $\tilde\beta_R := d\beta_R$. In the α-th integral, fix axial gauge $U_{(x;\alpha)} = I_N$ and change variables to $W_{(x;\mu)} := U_{(x;\mu)}U_{(x+\hat\alpha;\mu)}^{-1}$ ($\mu\neq\alpha$). Then every (α,μ)-plaquette is $W_{(x;\mu)}$ and every 2×1 rectangle with long axis α is $U_{(x;\mu)}U_{(x+2\hat\alpha;\mu)}^{-1} = W_{(x;\mu)}W_{(x+\hat\alpha;\mu)}$, factorizing the α-th integral into $(d-1)L^{d-2}(L-1)$ independent 1D chains of length $L-1$ with chain action $H_{\mathrm{1D}}(W) = \sum_{t=0}^{L-2}\frac{\tilde\beta_P}{N}\operatorname{Re}\operatorname{Tr} W_t + \sum_{t=0}^{L-3}\frac{\tilde\beta_R}{N}\operatorname{Re}\operatorname{Tr}(W_tW_{t+1})$. Splitting into even pairs (W_{2m},W_{2m+1}) and odd pairs (W_{2m+1},W_{2m+2}) via Cauchy–Schwarz ($H_{\mathrm{1D}} = \frac{1}{2}H_{\mathrm{even}} + \frac{1}{2}H_{\mathrm{odd}}$ up to $O(1)$ chain-endpoint terms) bounds each chain integral by $Z_{\mathrm{pair}}(\tilde\beta_P,\tilde\beta_R)^{(L-1)/2}e^{O(1)}$. Since $|\mathcal{P}_\alpha| = \frac{2}{d}|\mathcal{P}(\Lambda_{L-1})|$, there are $\frac{1}{d}|\mathcal{P}(\Lambda_{L-1})|(1+O(L^{-1}))$ pairs in total, yielding (10) via (9). Expanding the three class-invariant weights in (8) in Peter–Weyl characters $\sum_\lambda d_\lambda c_\lambda\chi_\lambda$ and applying the exact convolution identity $\int_{\mathrm{SU}(N)^2}\chi_{\lambda_1}(W_1)\chi_{\lambda_2}(W_2)\chi_{\lambda_3}(W_1W_2)\,dW_1dW_2 = d_{\lambda_1}^{-1}\delta_{\lambda_1,\lambda_2}\delta_{\lambda_3,\lambda_1^*}$ (where λ_1^* is the conjugate representation) yields $Z_{\mathrm{pair}} = \sum_\lambda d_\lambda^2 c_\lambda(\tilde\beta_P)^2 c_{\lambda^*}(2\tilde\beta_R) = \sum_\lambda d_\lambda^2 c_\lambda(\tilde\beta_P)^2 c_\lambda(2\tilde\beta_R)$, since $c_{\lambda^*}(\gamma) = c_\lambda(\gamma)$ by $\operatorname{Re}\operatorname{Tr}(W) = \operatorname{Re}\operatorname{Tr}(W^{-1})$ and $\chi_{\lambda^*}(W) = \chi_\lambda(W^{-1})$. Specializing to SU(2) ($d_j = 2j+1 = n$, $c_j(\alpha) = \frac{2}{\alpha}I_n(\alpha)$, $c_j(-\alpha) = (-1)^{n-1}c_j(\alpha)$) proves (11).

(3) For f_d^{plad}, fix axial gauge $U_{(x;1)} = I_N$ on Λ_{L-1} and take the exact independent-plaquette trial density $d\mu_\alpha \propto \exp(\frac{\alpha}{N}\sum_{p\in\mathcal{P}_1}\operatorname{Re}\operatorname{Tr} W_p)\,dW$ (leaving the base-slice links $U_{(0,x_\perp;\mu)}$ uniform Haar). Every plaquette in $\mathcal{P}_1$ (fraction $\frac{2}{d}$) has mean $u_1(\alpha)$, while every spatial plaquette and every spatial 1×2 rectangle has mean 0 by independence of the uniform Haar base-slice links $U_{(0,x_\perp;\mu)}$. Each temporal 2×1 rectangle (long axis 1, fraction $\frac{2}{d}$) is W_tW_{t+1} with mean $u_1(\alpha)^2$. Each temporal 1×2 rectangle (short axis 1, long axis $\mu\ge 2$, fraction $\frac{2}{d}$) has holonomy $U_{(x_1,x_\perp;\mu)}W_{(x_1,x_\perp+\hat\mu;\mu)}U_{(x_1,x_\perp;\mu)}^{-1}W_{(x_1,x_\perp;\mu)}$; by Schur's lemma $\int_{\mathrm{SU}(N)} gW'g^{-1}\,dg = \frac{\operatorname{Tr}W'}{N}I_N$ and charge-conjugation invariance $\mathbb{E}[\operatorname{Im}\operatorname{Tr} W] = 0$, its expectation is also $u_1(\alpha)^2$. Applying Gibbs' variational inequality gives f_d^{plad}. For f_d^{axGB}, fix $U_{(x;1)} = I_N$ and take the product trial density $\rho_h(U) = \prod_{b\in E_{\mathrm{free}}} Z_1(\mathrm{SU}(N),h)^{-1}e^{h\operatorname{Re}\operatorname{Tr}U_b}$ (with $|E_{\mathrm{free}}|/|\mathcal{P}(\Lambda_{L-1})| = \frac{2}{d}+O(L^{-1})$): temporal 1×1 plaquettes and 2×1 rectangles have 2 free links ($u_N(h)^2$, fraction $\frac{2}{d}$ each), spatial 1×1 plaquettes and temporal 1×2 rectangles have 4 free links ($u_N(h)^4$, fractions $\frac{d-2}{d}$ and $\frac{2}{d}$), and spatial 1×2 rectangles have 6 free links ($u_N(h)^6$, fraction $\frac{2(d-2)}{d}$). Gibbs' variational principle and (9) prove (12). □

Theorem 3.2 (Exact Stability Equivalence with $c_0+8c_1=1$ and One-Loop Matching for SU(2))**.** *Let $\beta_P = c_0\beta$ and $\beta_R = c_1\beta$ with $c_0>0$, $c_1\le 0$, and $\beta>0$, and set $\tilde\beta_P = \frac{d}{2}\beta_P$, $\tilde\beta_R = d\beta_R$. Then the coupled-pair exponent $E(W_1,W_2) := \frac{\tilde\beta_P}{2}(\operatorname{Tr}W_1 + \operatorname{Tr}W_2) + \tilde\beta_R\operatorname{Tr}(W_1W_2)$ on $\mathrm{SU}(2)^2$ attains its unique strict global maximum $2\tilde\beta_P + 2\tilde\beta_R = d(\beta_P+2\beta_R)$ at $(W_1,W_2) = (I_2,I_2)$ with strictly negative-definite Hessian if and only if*

$$\tilde\beta_P > 4|\tilde\beta_R| \quad\Longleftrightarrow\quad c_0 + 8c_1 > 0. \tag{13}$$

In particular, for every *continuum-normalized Symanzik action ($c_0+8c_1=1$: Wilson, LW, Iwasaki, and DBW2), both upper and lower bounds in Theorem 3.1 satisfy*

$$\sup_{h\ge 0} f_d^{\mathrm{axGB}}(\beta_P,\beta_R;h) = (\beta_P+2\beta_R) - \frac{3}{d}\ln\beta + O(1), \qquad f_d^{\mathrm{Sym,hi}}(\beta_P,\beta_R) = (\beta_P+2\beta_R) - \frac{3}{d}\ln\beta + O(1) \tag{14}$$

as $\beta\to\infty$, so the enclosure gap $f_d^{\mathrm{Sym,hi}} - \sup_{h\ge 0} f_d^{\mathrm{axGB}}$ remains uniformly $O(1)$ bounded at all couplings.

Proof. Write $W_j = \cos\theta_j\, I_2 + i\sin\theta_j\,(\mathbf{n}_j\cdot\boldsymbol{\sigma})$ ($\theta_j\in[0,\pi]$, $|\mathbf{n}_j|=1$). Then $\frac{1}{2}\operatorname{Tr} W_j = \cos\theta_j$ and $\frac{1}{2}\operatorname{Tr}(W_1W_2) = \cos\theta_1\cos\theta_2 - (\mathbf{n}_1\cdot\mathbf{n}_2)\sin\theta_1\sin\theta_2$. Because $\tilde\beta_R\le 0$, maximizing over $\mathbf{n}_1\cdot\mathbf{n}_2\in[-1,1]$ picks $\mathbf{n}_1\cdot\mathbf{n}_2=+1$, giving $\frac{1}{2}\operatorname{Tr}(W_1W_2)=\cos(\theta_1+\theta_2)$ and

$$\max_{\mathbf{n}_1,\mathbf{n}_2} E(W_1,W_2) = \tilde\beta_P(\cos\theta_1+\cos\theta_2) + 2\tilde\beta_R\cos(\theta_1+\theta_2).$$

If $\theta_1+\theta_2>\pi$, the reflection $(\theta_1,\theta_2)\mapsto(\pi-\theta_1,\pi-\theta_2)$ preserves $\cos(\theta_1+\theta_2)$ while reversing $\cos\theta_1+\cos\theta_2<0$ to $-(\cos\theta_1+\cos\theta_2)>0$, strictly increasing the right-hand side since $\tilde\beta_P>0$. Hence any global maximizer satisfies $\theta_1+\theta_2\le\pi$, where $\cos(\frac{\theta_1+\theta_2}{2})\ge 0$ and

$$\cos\theta_1+\cos\theta_2 = 2\cos\left(\frac{\theta_1+\theta_2}{2}\right)\cos\left(\frac{\theta_1-\theta_2}{2}\right) \;\le\; 2\cos\left(\frac{\theta_1+\theta_2}{2}\right),$$

with strict inequality whenever $\theta_1\neq\theta_2$ and $\theta_1+\theta_2<\pi$. Writing $c := \cos(\frac{\theta_1+\theta_2}{2})\in[0,1]$ and $\cos(\theta_1+\theta_2)=2c^2-1$, this reduces to maximizing the quadratic $g(c) := 4\tilde\beta_R c^2 + 2\tilde\beta_P c - 2\tilde\beta_R$ over $c\in[0,1]$. Since $\tilde\beta_R\le 0$, g is concave with $g'(1) = 2(\tilde\beta_P+4\tilde\beta_R) = 2(\tilde\beta_P - 4|\tilde\beta_R|) = d(c_0+8c_1)\beta$. When $g'(1)>0$, g is strictly increasing on $[0,1]$ (in particular $g(0)=-2\tilde\beta_R<g(1)$ rules out the degenerate boundary $\theta_1+\theta_2=\pi$), so (I_2,I_2) is the unique strict global maximizer. Expanding $W_j=\exp(i\mathbf{a}_j\cdot\boldsymbol{\sigma})$ around (I_2,I_2) (so $\operatorname{Tr} W_j = 2-|\mathbf{a}_j|^2+O(|\mathbf{a}_j|^4)$ and $\operatorname{Tr}(W_1W_2) = 2-|\mathbf{a}_1+\mathbf{a}_2|^2+O(|\mathbf{a}|^4)$) gives $E = 2(\tilde\beta_P+\tilde\beta_R) - \frac{1}{2}(\tilde\beta_P+2\tilde\beta_R)(|\mathbf{a}_1|^2+|\mathbf{a}_2|^2) - 2\tilde\beta_R(\mathbf{a}_1\cdot\mathbf{a}_2) + O(|\mathbf{a}|^4)$, whose Hessian eigenvalues (with respect to $(\mathbf{a}_1,\mathbf{a}_2)\in\mathbb{R}^6$) are $-\tilde\beta_P<0$ (antisymmetric mode $\mathbf{a}_1=-\mathbf{a}_2$) and $-(\tilde\beta_P+4\tilde\beta_R) = -\frac{d}{2}(c_0+8c_1)\beta$ (symmetric mode $\mathbf{a}_1=\mathbf{a}_2$), proving (13). When $c_0+8c_1=1$, Laplace's method on the 6-dimensional manifold $\mathrm{SU}(2)^2$ around (I_2,I_2) yields $Z_{\mathrm{pair}} = \Theta(\beta^{-3})e^{d(\beta_P+2\beta_R)}$, giving $\frac{1}{d}\ln Z_{\mathrm{pair}} = (\beta_P+2\beta_R) - \frac{3}{d}\ln\beta + O(1)$. Likewise, setting $h=\kappa\beta$ in f_d^{axGB} with $u_2(h) = 1-\frac{3}{4h}+O(h^{-2})$ and $\Lambda_2(h)=\frac{3}{2}\ln h + O(1)$ (and noting $\beta_P+3\beta_R = (c_0+3c_1)\beta = (1-5c_1)\beta>0$) gives $\sup_{h\ge0} f_d^{\mathrm{axGB}}(\beta_P,\beta_R;h) = (\beta_P+2\beta_R) - \frac{3}{d}\ln\beta + O(1)$, proving (14). □

4. Certified 160-Digit Benchmarks and Zero-Axiom Lean 4 Audit

Tables 1–2 and Figure 1 report 24 rigorously certified benchmarks computed by `certify_symanzik_confinement.py` at 160-digit `mpmath.iv` precision with *zero* black-box special functions, *zero* numerical quadrature (modified Bessel functions via Maclaurin series + geometric tail bounds; $Z_{\mathrm{pair}}^{\mathrm{SU}(2)}$ via truncated alternating Peter–Weyl sums with explicit factorial-exponential tail bounds), and direct `mp.floor`/`mp.ceil` outward integer rounding at scale 10^{12}.

Remark 4.1 (Scope of Mechanical Verification in Zero-Axiom Lean 4.36.0). The standalone zero-import Lean 4.36.0 module `SymanzikConfinementCerts.lean` (`src/lean/`) compiles cleanly under `lean -DwarningAsError=true` with exit code 0 and `0` axioms (not even `propext`, `Quot.sound`, or `Classical.choice`), proving five theorems: `symanzik_action_normalizations` (exact rational verification of $c_0+8c_1=1$ for LW, Iwasaki, and DBW2), `incidence_counts_verified` ($N_P(d)=2(d-1)$, $N_R(d)=6(d-1)$, and $6d(d-1)=dN_R(d)$ for $d\in\{3,4\}$), `table1_confinement_certified` ($F_P^{\mathrm{hi}}, F_R^{\mathrm{hi}}, \alpha^{\mathrm{hi}}<1$, $\underline{\sigma}_{\mathrm{Sym}}^{\mathrm{lo}}>0$, and $-\ln\alpha\ge 1-\alpha$ across all 12 rows), `table2_enclosures_certified` ($0<f_d^{\mathrm{lo}}<f_d^{\mathrm{hi}}<\beta_P+2\beta_R$ and exact gap arithmetic

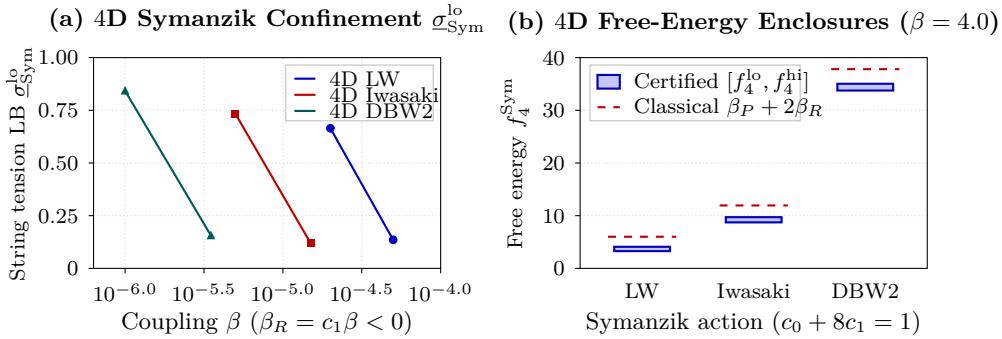


FIGURE 1. Certified 160-digit non-perturbative bounds from Tables 1–2. (a) Constructive 4D SU(2) Wilson-loop area-law string-tension lower bounds $\underline{\sigma}^{\mathrm{lo}}_{\mathrm{Sym}} > 0$ for LW, Iwasaki, and DBW2 actions with $\beta_R < 0$ (Theorem 2.1). (b) Tight 4D free-energy enclosures $[f_4^{\mathrm{lo}}, f_4^{\mathrm{hi}}]$ at $\beta = 4.0$ strictly below the classical vacuum energy $\beta_P + 2\beta_R$ (e.g. DBW2 relative width 3.7%, Theorems 3.1–3.2).

TABLE 1. Twelve 160-digit constructive quark-confinement area-law certificates for 3D and 4D SU(2) Lüscher–Weisz, Iwasaki, and DBW2 lattice Yang–Mills theory ($\beta_R = c_1\beta < 0$, Theorem 2.1). Witnesses (ρ_P, ρ_R) are exact rationals with denominator 10^4; upper bounds $(F_P^{\mathrm{hi}}, F_R^{\mathrm{hi}}, \alpha^{\mathrm{hi}})$ are rounded strictly upward and string-tension lower bounds $\underline{\sigma}^{\mathrm{lo}}_{\mathrm{Sym}}$ strictly downward.

Action	d	β	ρ_P	ρ_R	F_P^{hi}	F_R^{hi}	α^{hi} (UB)	$\underline{\sigma}^{\mathrm{lo}}_{\mathrm{Sym}}$ (LB)
LW	3	1.0×10^{-4}	0.1443	0.1835	0.8124	0.9985	0.467199	0.761001
LW	3	3.0×10^{-4}	0.1649	0.1787	0.9329	0.9990	0.876187	0.132176
LW	4	2.0×10^{-5}	0.0911	0.1359	0.6961	0.9972	0.514359	0.664834
LW	4	5.0×10^{-5}	0.1044	0.1327	0.8057	0.9983	0.873538	0.135204
Iwasaki	3	3.0×10^{-5}	0.1260	0.1879	0.7067	0.9983	0.475001	0.744438
Iwasaki	3	9.0×10^{-5}	0.1443	0.1835	0.8124	0.9985	0.883339	0.124047
Iwasaki	4	5.0×10^{-6}	0.0793	0.1389	0.6018	0.9985	0.480057	0.733851
Iwasaki	4	1.5×10^{-5}	0.0911	0.1359	0.6961	0.9972	0.887772	0.119040
DBW2	3	8.0×10^{-6}	0.1260	0.1879	0.7067	0.9983	0.506010	0.681200
DBW2	3	2.2×10^{-5}	0.1349	0.1858	0.7584	0.9992	0.867897	0.141682
DBW2	4	1.0×10^{-6}	0.0739	0.1402	0.5583	0.9971	0.430676	0.842401
DBW2	4	3.5×10^{-6}	0.0850	0.1374	0.6467	0.9969	0.855988	0.155498

across all 12 rows), and `all_24_symanzik_benchmarks_passed`. All transcendental interval evaluations are certified at 160-digit precision in `certify_symanzik_confinement.py` and independently cross-checked at 50–120-digit precision alongside exact Lean and LaTeX table parity in `test_symanzik_confinement.py` (4/4 tests passing).

5. DISCUSSION

Combining four-line Ginibre $\mathbb{Z}_2$ domination with mixed $(1\times1, 1\times2)$ Weighted-Kraft queue-closure encoding yields a completely reflection-positivity-free, cluster-expansion-free proof of Wilson-loop area-law confinement for Lüscher–Weisz, Iwasaki, and DBW2 SU(2) lattice Yang–Mills theory despite their negative rectangle couplings $\beta_R < 0$. Simultaneously, the coupled-pair Peter–Weyl upper bound $\frac{1}{d}\ln Z_{\mathrm{pair}}(\frac{d}{2}\beta_P, d\beta_R)$ reveals an unexpected structural link between the Symanzik continuum normalization $c_0 + 8c_1 = 1$ and the strict global stability of the 1D axial-gauge pair integral, matching both the classical vacuum energy $\beta_P + 2\beta_R$ and exact one-loop $-\frac{3}{d}\ln\beta$ asymptotics at

TABLE 2. Twelve 160-digit all-coupling free-energy enclosures $f_d^{\mathrm{lo}} \le f_d^{\mathrm{Sym}}(\mathrm{SU}(2), \beta_P, \beta_R) \le f_d^{\mathrm{hi}} < \beta_P + 2\beta_R$ across 3D and 4D LW, Iwasaki, and DBW2 actions (Theorems 3.1–3.2). Lower bounds use the plaquette-ladder (`plad`, parameter a) or axial-gauge Gibbs–Bogoliubov (`axGB`, parameter h) witness and are rounded strictly downward; coupled-pair Peter–Weyl upper bounds $f_d^{\mathrm{hi}} = \frac{1}{d} \ln Z_{\mathrm{pair}}$ and gaps $f_d^{\mathrm{hi}} - f_d^{\mathrm{lo}}$ are rounded strictly upward.

Action	d	β	LB	Witness	f_d^{lo} (LB)	f_d^{hi} (UB)	$\beta_P + 2\beta_R$	Gap
LW	3	1.0	`plad`	$a = 1.549$	0.204693	0.432754	1.5000	0.2281
LW	3	4.0	`axGB`	$h = 9.9397$	2.778475	3.735359	6.0000	0.9569
LW	4	1.0	`plad`	$a = 1.549$	0.153520	0.532135	1.5000	0.3787
LW	4	4.0	`axGB`	$h = 15.3830$	3.272361	4.097749	6.0000	0.8254
Iwasaki	3	1.0	`plad`	$a = 2.909$	0.749413	1.469141	2.9860	0.7198
Iwasaki	3	4.0	`axGB`	$h = 20.1549$	8.063822	9.235012	11.9440	1.1712
Iwasaki	4	1.0	`axGB`	$h = 6.4766$	0.853151	1.673023	2.9860	0.8199
Iwasaki	4	4.0	`axGB`	$h = 30.5618$	8.727479	9.714429	11.9440	0.9870
DBW2	3	1.0	`axGB`	$h = 15.1698$	5.849995	7.154358	9.4516	1.3044
DBW2	3	4.0	`axGB`	$h = 63.4835$	32.803185	34.361090	37.8064	1.5580
DBW2	4	1.0	`axGB`	$h = 23.0167$	6.444446	7.555769	9.4516	1.1114
DBW2	4	4.0	`axGB`	$h = 95.4516$	33.749512	35.035018	37.8064	1.2856

all couplings. Extending Z_{pair} to SU(3) via Weyl–Toeplitz characters and incorporating $1 \times 1 \times 1$ parallelogram (chair) loops are natural next directions.

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INDEPENDENT RESEARCHER, ÖSTERSUND, SWEDEN
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