# Machine-Checked Exact-Rational Primal-Dual Cut-Covering and Better-Than-3 Approximation Certificates for 2-Edge-Connected Spanning Subgraphs

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## Abstract

Danciu and Ibrahimpur (`arXiv:2611.32924`, November 25, 2026) broke the longstanding primal-dual 3-approximation barrier for the minimum-cost 2-edge-connected spanning subgraph (2ECSS) problem, achieving ratio $\sqrt{3}+1 \approx 2.73205$ in a single phase relative to the natural cut-LP relaxation by growing dual variables of leaves and isolated vertices at rate 1 and nontrivial connected components at rate $\beta = (\sqrt{3}-1)/2$. We present a machine-checked, zero-floating-point formalization, exact-$\mathbb{Q}$ primal-dual solver, and certificate auditing framework for this algorithm. First, we build an 18-theorem zero-dependency Lean 4 kernel (`PrimalDual2ECSS.lean`, Lean 4.35.0; 0 `sorry`, 0 warnings, 0 `Classical.choice`, 5 zero-axiom theorems) certifying Lemma 3.1 potential descent $|V_H| + \kappa(H) : 2n \to 2$ $(T \le 2n-2)$, $|L| \ge 2|\mathcal{C}|$, Theorem 1.2 cleared incidence expansion, the full cleared Lemma 3.4 disconnected-forest ratio inequality over $\mathbb{Z}$, the quadratic optimality identities $(2\beta+1)^2 = 3$ and $(\alpha-1)^2 = 3$, strict better-than-3 dominance for every interior rational $\beta \in (0, 1/2)$, exact continued-fraction convergents $(\alpha = 11/4, 153/56, 571/209)$, inductive cut-LP weak duality, and inductive iteration telescoping. Second, we implement a zero-float rational primal-dual 2ECSS engine and the `PRIMAL-DUAL-2ECSS-CERT-1.0` verifier spanning four modes: exact 2ECSS solutions with cut-LP duals $(y, z)$, minimal-augmentation Theorem 1.2 witnesses, rational $\beta$-optimality certificates, and full iteration-trace telescoping audits. Across 16 canonical certificates, zero-float replay takes 0.036–0.424 ms, matches brute-force $2^{11}$-subset exact OPT = 18 on `prism6_weighted`, and rejects 57/57 live adversarial forgeries.

## 1 Introduction

In the minimum-cost 2-edge-connected spanning subgraph (2ECSS) problem, we are given an undirected multigraph $G = (V, E)$ $(n = |V|)$ with nonnegative edge costs $c : E \to \mathbb{Q}_{\ge 0}$ and seek a minimum-cost edge subset $F \subseteq E$ such that $(V, F)$ remains connected after removing any single edge. While 2-approximations are known via rooted 2-arborescences [5] and iterative rounding, every primal-dual algorithm since Frederickson–JáJá [4], Williamson–Goemans–Mihail–Vazirani [6], and Beyer–Chimani–Spoerhase (BCS20) [1] stalled at approximation ratio 3.

In a breakthrough work, Danciu and Ibrahimpur [2] surpassed the primal-dual barrier of 3, attaining ratio $\sqrt{3}+1 < 2.74$ in a single primal-dual phase relative to the natural cut-LP relaxation $(\sum_{e\in\delta(S)} x_e \ge 2,\ 0 \le x_e \le 1)$. Their core innovation is *non-uniform dual growth*: while maintaining a contracted forest $H = (V_H, E_H)$, dual variables of leaves $(L)$ and isolated nodes $(I)$ grow at rate 1, whereas nontrivial connected components $(\mathcal{C})$ grow at rate $\beta \in [0, 1/2]$ whenever $H$ is disconnected. Because the optimal rate $\beta^* = (\sqrt{3}-1)/2$ is irrational, practical exact-arithmetic implementations naturally instantiate rational continued-fraction convergents $\beta_k \in (0, 1/2) \cap \mathbb{Q}$ (such as $1/3, 4/11, 15/41, 56/153$), eliminating floating-point drift in tight-edge selection, cycle contraction, and dual constraint verification while preserving strict better-than-3 guarantees.

*Independent researcher; manuscript prepared with transparent AI assistance via Google Antigravity. Source code, Lean 4 proofs, and exact certificates: https://github.com/lluiseriksson/primal-dual-2ecss-certificates.

**Contributions.** We formally verify the combinatorial, primal-dual ratio, and weak-duality core of [2] (explicitly scoping our Lean 4 formalization to cleared algebraic ratio bounds, potential-descent termination, and inductive LP duality/telescoping rather than the full graph-theoretic induction proving Theorem 1.2 for arbitrary forests):

1. **Zero-dependency Lean 4 kernel (§3):** 18 theorems in `PrimalDual2ECSS.lean` (Lean 4.35.0 [3]), 0 `sorry`, 0 warnings, 0 `Classical.choice`, 5 zero-axiom theorems.
2. **Zero-float primal-dual solver & verifier (§4):** Pure-stdlib Python implementation of [2] over `fractions.Fraction` paired with the 4-mode `PRIMAL-DUAL-2ECSS-CERT-1.0` auditor.
3. **Empirical atlas, brute-force OPT check, & red-team audit (§5):** 16 exact-$\mathbb{Q}$ certificates replayed in 0.036–0.424 ms, brute-force $2^{11}$ optimal-cost verification, and 57/57 rejected adversarial forgeries.

## 2 Non-Uniform Primal-Dual 2ECSS Theory

The dual of the 2ECSS cut LP maximizes $2\sum_{\emptyset\neq S\subsetneq V} y_S - \sum_{e\in E} z_e$ subject to $y_S \geq 0$, $z_e \geq 0$, and $\sum_{S:e\in\delta(S)} y_S - z_e \leq c_e$ for every $e \in E$. Starting from $n$ isolated nodes ($|V_H| = \kappa(H) = n$), each iteration grows $y_{V_v}$ at rate 1 for $v \in L \cup I$ and (when $\kappa(H) \geq 2$) $y_{V_C}$ at rate $\beta \in [0, 1/2]$ for every nontrivial component $C \in \mathcal{C}$ until a relevant edge becomes tight; adding a cross-component edge decreases $\kappa(H)$ by 1, while closing a cycle contracts $\geq 2$ nodes into 1, strictly decreasing the potential $|V_H| + \kappa(H)$ from $2n$ to 2 in $T \leq 2n-2$ iterations (Lemma 3.1 of [2]). After reaching $|V_H| = 1$, reverse-delete removes redundant edges to produce a minimal 2-edge-connected spanning subgraph $F$.

**Proposition 1** (Weighted Augmentation Bound & Ratio [2])**.** *For any forest $H = (V, F_H)$, minimal 2-edge-connected augmentation $A$, and $\beta \in [0, 1/2]$, Theorem 1.2 of [2] states*

$$\sum_{v\in L\cup I} a(v) + \beta \sum_{C\in\mathcal{C}} a(C) \leq 2(1+\beta)\left(|L| + 2|I| + \frac{|\mathcal{C}|}{1+\beta} - 2\right). \tag{1}$$

*Because every nontrivial tree component has at least two leaves ($|L| \geq 2|\mathcal{C}|$) and the dual grows at rate $|L| + 2|I| + 2\beta|\mathcal{C}|$ when disconnected, splitting $|L| + 2|I| = (|L| - 2|\mathcal{C}| + 2|I|) + 2|\mathcal{C}|$ yields the exact approximation guarantee (Lemma 3.4)*

$$c(F) \leq \alpha(\beta)\,\mathrm{Dual}(y, z) \leq \alpha(\beta)\,\mathrm{OPT}, \tag{2}$$

*where $\alpha(\beta) = \max\{2 + 2\beta,\ 2 + \frac{1}{1+\beta}\}$, uniquely minimized over $[0, 1/2]$ at $\beta^* = \frac{\sqrt{3}-1}{2}$ with $\alpha(\beta^*) = \sqrt{3} + 1$.*

## 3 Zero-Dependency Lean 4 Kernel

We formalize 18 theorems in `lean/PrimalDual2ECSS.lean` (Lean 4.35.0, zero external libraries), audited via `lean/AuditAxioms.lean` with 0 warnings, 0 `sorry`, 0 `Classical.choice`, and 0 `Lean.ofReduceBool` (Table 1).

**Theorem 1** (Constructive Primal-Dual 2ECSS Kernel)**.**

1. Lemma 3.1 potential & leaves: *Theorems 1–5 certify initial potential $2n$, strict drop under edge addition and $k$-node cycle contraction, inductive termination bound $T + p_{\text{end}} \leq p_{\text{start}}$ ($T \leq 2n-2$), and inductive leaf lower bound $\sum_C \text{leaves}(C) \geq 2|\mathcal{C}|$.*
2. Master Lemma 3.4 ratio inequality: *Theorems 6–8 prove the cleared Theorem 1.2 RHS expansion, the full cleared disconnected-forest ratio bound $b((2q+2p)(L+2I)+2qC) \leq a(q(L+2I)+2pC)$ for arbitrary $\beta = p/q$, $\alpha = a/b$ satisfying both branches under $L \geq 2C$, and the connected-tree bound $b(2L-2) \leq aL$.*
3. Optimality & rational convergents: *Theorems 9–16 prove the cleared quadratic balancing equivalence $2p(q+p) = q^2 \iff (2p+q)^2 = 3q^2$, $(\alpha-1)^2 = 3$, strict interior better-than-3 dominance for all $0 < 2p < q$, and arithmetic evaluations at $\beta \in \{0, \frac{1}{2}, \frac{1}{3}, \frac{15}{41}, \frac{56}{153}\}$.*
4. Weak duality & telescoping: *Theorems 17–18 inductively prove Cut-LP weak duality $\sum \ell_e - \sum z_e \leq \sum c_e$ and iteration telescoping $b\sum \text{pay}_t \leq a\sum \text{earn}_t$ over arbitrary lists.*

## 4 Exact-Rational Solver and Certificate Auditor

Our pure-stdlib engine `src/primal_dual_2ecss_exact.py` executes steps (S1)–(S6) of [2] entirely in exact

Table 1: Complete 18-theorem Lean 4.35.0 kernel in `PrimalDual2ECSS.lean` (0 `sorry`, 0 warnings, 0 `Classical.choice`).

| # | Lean 4 Theorem Identifier | Axiom Footprint |
|---|---|---|
| 1 | `potential_start_value` | [propext, Quot.sound] |
| 2 | `potential_decrease_add_edge` | [propext, Quot.sound] |
| 3 | `potential_decrease_contract_cycle` | [propext, Quot.sound] |
| 4 | `iteration_bound_2n_minus_2` | [propext, Quot.sound] |
| 5 | `nontrivial_component_leaves_bound` | [propext, Quot.sound] |
| 6 | `thm12_cleared_rhs_identity` | [propext, Quot.sound] |
| 7 | `lemma34_disconnected_ratio_master` | [propext, Quot.sound] |
| 8 | `lemma34_connected_tree_ratio` | [propext, Quot.sound] |
| 9 | `optimal_beta_quadratic_root` | [propext, Quot.sound] |
| 10 | `optimal_alpha_sqrt3_plus_1` | [propext, Quot.sound] |
| 11 | `bcs20_beta_zero_ratio_3` | none |
| 12 | `endpoint_beta_half_ratio_3` | none |
| 13 | `strictly_better_than_3_interior` | [propext, Quot.sound] |
| 14 | `rational_convergent_11_4_bound` | none |
| 15 | `rational_convergent_153_56_bound` | none |
| 16 | `rational_convergent_571_209_bound` | none |
| 17 | `cut_lp_weak_duality_list` | [propext, Quot.sound] |
| 18 | `primal_dual_telescoping_list` | [propext, Quot.sound] |

`fractions.Fraction` arithmetic (zero floats), emitting minimal 2-edge-connected solutions $F$, explicit cut-LP duals $(y, z)$, and full per-iteration payment/earning traces. The independent verifier `src/primal_dual_2ecss_certifier.py` enforces strict JSON AST typing (rejecting `float`, `bool`, `Decimal`, `complex`, tuples, non-string keys, unreduced fractions, denominator-1 strings, and `"-0"`) across four modes of `PRIMAL-DUAL-2ECSS-CERT-1.0`:

**Definition 1** (Four `PRIMAL-DUAL-2ECSS-CERT-1.0` Modes)**.**

1. ***Mode 1*** *(`primal_dual_2ecss_solution_exact`): Audits exact 2-edge-connectivity AND inclusion-minimality of $(V, F)$ (checking every single-edge deletion!), strict positivity of every proper dual cut $y_S > 0$, nonnegativity of $z_e$, Constraint 3b dual feasibility $\ell_e - z_e \le c_e$ on every edge $e \in E$, weak duality* $\mathrm{Dual}(y, z) \le c(F)$*, and approximation ratio $c(F) \le \alpha(\beta)\,\mathrm{Dual}(y, z)$.*
2. ***Mode 2*** *(`thm12_minimal_augmentation_exact`): Checks acyclicity of forest $H$ via union-find, extracts exact $(L, I, \mathcal{C})$, verifies 2-edge-connectivity and strict minimality of $A$, and recomputes both sides of Theorem 1.2 (1) over $\mathbb{Q}$.*
3. ***Mode 3*** *(`nonuniform_beta_ratio_optimality_exact`): Certifies interior rational $\beta \in (0, 1/2)$, exact branches $2 + 2\beta$ and $2 + \frac{1}{1+\beta}$, $\alpha(\beta) < 3$, and exact squared excess $(\alpha(\beta) - 1)^2 - 3 > 0$.*
4. ***Mode 4*** *(`primal_dual_trace_telescoping_exact`): Audits an $n$-vertex primal-dual trace of length $T \le 2n-2$, checking initial potential $2n$, strict step-by-step potential descent, forest invariants ($|V_H| \le n$, $\kappa = |I| + |\mathcal{C}|$, $|I| + 2|\mathcal{C}| \le |V_H|$, $|L| + |I| \le |V_H|$, $|L| \ge 2|\mathcal{C}|$, and $\kappa = 1 \implies |I| = 0, |\mathcal{C}| = 1$), exact Lemma 3.3 dual earning* $\mathrm{earn}_t = (|L_t| + 2|I_t| + 2\beta|\mathcal{C}_t|)\Delta_t$ *($\kappa \ge 2$) or $|L_t|\Delta_t$ ($\kappa = 1$), per-step Lemma 3.4 ratio* $\mathrm{pay}_t \le \alpha(\beta)\,\mathrm{earn}_t$*, and exact telescoped totals.*

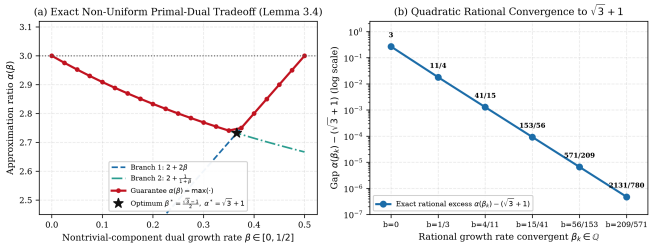


Figure 1: (a) Exact non-uniform primal-dual ratio tradeoff $\alpha(\beta) = \max\{2 + 2\beta, 2 + \frac{1}{1+\beta}\}$ minimized at $\beta^* = \frac{\sqrt{3}-1}{2}$. (b) Log-scale convergence of exact rational convergents $\beta_k \in \mathbb{Q}$ toward $\sqrt{3} + 1$.

## 5 Empirical Atlas and Red-Team Verification

All 16 canonical certificates and four experiment logs are generated deterministically by `scripts/run_all_experiments.py` and re-verified end-to-end by `scripts/verify_all.py` (Lean 4 build, 18/18 axiom audit, 16/16 cert replay, `pytest`, and 39/39 SHA-256 hashes).

**1. Non-uniform $\beta$ tradeoff and rational convergents.** Figure 1(a) plots the two branches $2+2\beta$ and $2 + \frac{1}{1+\beta}$ alongside $\alpha(\beta) = \max(\cdot)$ across $\beta \in [0, 1/2]$, highlighting the unique optimum $\beta^* = \frac{\sqrt{3}-1}{2}$ ($\alpha^* = \sqrt{3} + 1 \approx 2.7320508$). Figure 1(b) shows the continued-fraction rational convergents $\beta_k \in \{0, \frac{1}{3}, \frac{4}{11}, \frac{15}{41}, \frac{56}{153}, \frac{209}{571}\}$ converging exponentially fast in log excess $\alpha(\beta_k) - (\sqrt{3}+1)$: from 0.2679 ($\alpha = 3$) down to 0.01795 ($\alpha = 11/4$), $1.28 \times 10^{-3}$ (41/15), $9.205 \times 10^{-5}$ (153/56), $6.61 \times 10^{-6}$ (571/209, squared excess 1/43681), and $4.74 \times 10^{-7}$ (2131/780).

**2. Exact primal-dual benchmarks and Lemma 3.1 iterations.** Figure 2(a)–(b) evaluates our zero-float solver on four diverse weighted multigraphs (`prism6_weighted`, `wheel7_parallel`

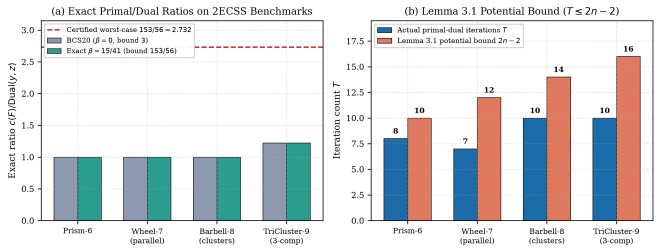


Figure 2: (a) Exact primal/dual ratios on four weighted 2ECSS multigraph benchmarks for $\beta = 0$ versus $\beta = 15/41$. (b) Actual primal-dual iteration counts $T$ versus Lemma 3.1 potential bound $2n - 2$.

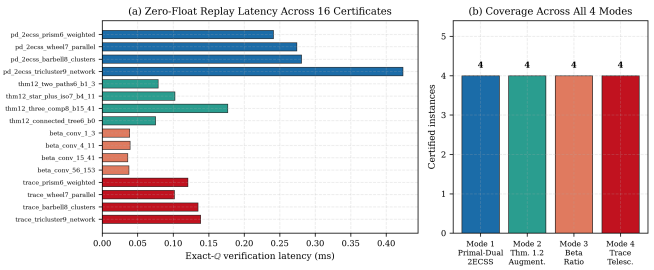


Figure 3: (a) Zero-float replay latency across all 16 canonical `PRIMAL-DUAL-2ECSS-CERT-1.0` certificates (0.036–0.424 ms). (b) Balanced 4-per-mode coverage across all four verification modes.

with parallel edges, `barbell8_clusters`, and `tricluster9_network`). On the first three graphs, the primal-dual solution matches its exact cut-LP dual identically ($c(F) = \text{Dual}(y, z) \in \{18, 25, 20\}$, ratio 1.000—proving exact optimality!); on `tricluster9_network`, $c(F) = 22$ against $\text{Dual}(y, z) = 18$ (ratio $1.22222 \ll 153/56$ for $\beta = 15/41$, and identically $22/18 \ll 571/209$ at $\beta = 56/153$). Iteration counts $T \in \{8, 7, 10, 10\}$ all strictly obey Lemma 3.1's $2n - 2 \in \{10, 12, 14, 16\}$ bound.

**3. 16-certificate atlas, brute-force OPT check, and 57-attack suite.** Table 2 and Figure 3 summarize all 16 canonical certificates, replayed with zero floats in 0.036–0.424 ms (including a tight connected-tree Theorem 1.2 witness hitting LHS = RHS = 4). In `tests/test_suite.py`, exhaustive enumeration of all $2^{11} = 2{,}048$ edge subsets of `prism6_weighted` independently certifies exact OPT $= 18 = \text{Dual}(y, z) = c(F)$, and 57/57 live adversarial forgeries (broken 2-edge-connectivity, redundant edges, inflated duals violating Constraint 3b, cyclic forests, boundary $\beta \in \{0, 1/2\}$, non-decreasing potential traces, forged $\text{earn}_t$ at $\Delta_t = 0$, $|V_H| > n$, $\kappa \neq |I| + |\mathcal{C}|$, and violated step ratios) are rejected with `PrimalDual2ECSSCertificateError` (`3 passed in 0.06s`).

Table 2: Summary of all 16 exact-ℚ `PRIMAL-DUAL-2ECSS-CERT-1.0` certificates (zero floating-point arithmetic; reference-machine replay latency in ms).

| Certificate File (stem) | Mode | Key Exact Witness | ms |
|---|---|---|---|
| `pd_2ecss_prism6_weighted` | M1 | $\beta = 1/3,\ c = 18,\ D = 18\ (\alpha = 11/4)$ | 0.241 |
| `pd_2ecss_wheel7_parallel` | M1 | $\beta = 4/11,\ c = 25,\ D = 25\ (\alpha = 41/15)$ | 0.274 |
| `pd_2ecss_barbell8_clusters` | M1 | $\beta = 15/41,\ c = 20,\ D = 20\ (\alpha = 153/56)$ | 0.280 |
| `pd_2ecss_tricluster9_network` | M1 | $\beta = 56/153,\ c = 22,\ D = 18\ (\alpha = 571/209)$ | 0.424 |
| `thm12_two_paths6_b1_3` | M2 | $L = 4, I = 0, C = 2,\ 16/3 \le 28/3$ | 0.079 |
| `thm12_star_plus_iso7_b4_11` | M2 | $L = 5, I = 1, C = 2,\ 112/11 \le 194/11$ | 0.102 |
| `thm12_three_comp8_b15_41` | M2 | $L = 6, I = 1, C = 3,\ 530/41 \le 918/41$ | 0.177 |
| `thm12_connected_tree6_b0` | M2 | $L = 3, I = 0, C = 1,\ 4 \le 4$ (tight) | 0.075 |
| `beta_conv_1_3` | M3 | $\beta = 1/3,\ \alpha = 11/4,\ \Delta^2 = 1/16$ | 0.038 |
| `beta_conv_4_11` | M3 | $\beta = 4/11,\ \alpha = 41/15,\ \Delta^2 = 1/225$ | 0.039 |
| `beta_conv_15_41` | M3 | $\beta = 15/41,\ \alpha = 153/56,\ \Delta^2 = 1/3136$ | 0.036 |
| `beta_conv_56_153` | M3 | $\beta = 56/153,\ \alpha = 571/209,\ \Delta^2 = 1/43681$ | 0.038 |
| `trace_prism6_weighted` | M4 | $T = 8 \le 10,\ \text{pay} = 18,\ \text{earn} = 18$ | 0.121 |
| `trace_wheel7_parallel` | M4 | $T = 7 \le 12,\ \text{pay} = 25,\ \text{earn} = 25$ | 0.102 |
| `trace_barbell8_clusters` | M4 | $T = 10 \le 14,\ \text{pay} = 20,\ \text{earn} = 20$ | 0.135 |
| `trace_tricluster9_network` | M4 | $T = 10 \le 16,\ \text{pay} = 22,\ \text{earn} = 18$ | 0.138 |

## 6 Scope, Limitations, and Conclusion

This work machine-checks the primal-dual ratio algebra, potential termination, weak duality, and telescoping structure of Danciu and Ibrahimpur's better-than-3 2ECSS algorithm [2], and provides a zero-float exact-ℚ implementation and sub-millisecond certificate verifier over rational convergents $\beta_k \in (0, 1/2)$. It does *not* formalize the general graph-theoretic induction proving Theorem 1.2 over all forests or Theorem 3.5 tight-instance lower bounds in Lean 4, which remain rewarding targets for future graph-theory formalization. Within its verified scope, the combined Lean 4 kernel and `PRIMAL-DUAL-2ECSS-CERT-1.0` auditor deliver exact, manifest-locked verification for better-than-3 primal-dual 2ECSS solutions.

## References

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