

A universal capacity bound for amplified determinants and polynomial witnesses in the Hermitian case

Lluis Eriksson
Independent researcher, Sweden

17 October 2026 — research draft, not peer reviewed

Abstract

For an arbitrary finite tuple of complex square matrices, we maximize the geometric mean of the squared singular values of its linear pencil over auxiliary matrices with unit normalized Hilbert–Schmidt energy and unrestricted size. We prove that this optimum lies between $\mathcal{C}/e$ and $\mathcal{C}$, where $\mathcal{C}$ is the maximum, over coefficient covariances, of the dimension-normalized operator capacity. No Hermitian or positive-span assumption is imposed on this comparison. An invertible finite evaluation supplies a perturbation inside the original matrix space, subharmonicity selects its phase, and quantitative resolvent comparison, algebraic spectral regularity and Han’s least-singular-value theorem control the entire logarithmic tail. This transfers the Mai–Speicher free determinant formula to feasible finite witnesses. Separately, for Hermitian coefficients whose real span contains the identity, we give an explicit polynomial auxiliary-size bound and a Gaussian sampling guarantee. We do not assert an optimal constant or a general polynomial-time witness algorithm.

1 Statement and contribution

Write $\tau_n = \text{Tr}/n$. For $A = (A_1, \dots, A_s) \in M_d(\mathbb{C})^s$, put

$$\mathcal{F}_m(A) = \max_{\sum_i \tau_m(Y_i Y_i^*) = 1} \left| \det \left(\sum_i A_i \otimes Y_i \right) \right|^{2/(dm)}, \quad (1)$$

$$\mathcal{F}_\infty(A) = \sup_{m \geq 1} \mathcal{F}_m(A). \quad (2)$$

The tuple presentation and its Euclidean energy are part of the data. For $Q \geq 0$ with $\text{Tr } Q = 1$, define

$$\Phi_Q(X) = \sum_{i,j} Q_{ij} A_i X A_j^*, \quad (3)$$

$$\text{cap}(\Phi) = \inf_{X > 0, \det X = 1} \det \Phi(X), \quad \mathcal{C}(A) = \max_{Q \geq 0, \text{Tr } Q = 1} \text{cap}(\Phi_Q)^{1/d}. \quad (4)$$

Capacity is the invariant of a completely positive map, not a communication rate. If $Q = LL^*$, then $B_j = \sum_i L_{ij} A_i$ are Kraus matrices for Φ_Q .

Theorem 1 (Universal comparison). *For every complex square matrix tuple,*

$$e^{-1} \mathcal{C}(A) \leq \mathcal{F}_\infty(A) \leq \mathcal{C}(A). \quad (5)$$

For every density matrix Q and every $\theta > 0$ there are a finite m and a feasible tuple whose objective is at least $e^{-1-\theta} \text{cap}(\Phi_Q)^{1/d}$. The result includes tuples with no invertible scalar linear combination.

The prior working paper [1] proved this comparison for Hermitian coefficients with a positive definite real linear combination. Theorem 1 removes both restrictions. The factor e^{-1} and the free determinant identity are due to Mai and Speicher [3]. The contribution is their transfer to feasible finite matrices in the unrestricted class; no new capacity identity or sharpness claim is made.

For the quantitative result suppose

$$A_i = A_i^*, \quad \sum_i b_i A_i = I_d, \quad b \in \mathbb{R}^s, \quad a^2 = \sum_i \|A_i\|^2, \quad \beta = \|b\|_2. \quad (6)$$

The norms in a are operator norms, and $a\beta \geq 1$.

Theorem 2 (Polynomial auxiliary size). *Under (6), for $0 < \theta \leq 1$, every integer*

$$m \geq \max \{2, \lceil 4d(a\beta)^4 \theta^{-3} \rceil\} \quad (7)$$

satisfies $\mathcal{F}_m(A) \geq e^{-1-\theta} \mathcal{C}(A)$. Witnesses can be chosen with every auxiliary matrix normal. Section 5 gives a sharper covariance-dependent bound and a sampling probability.

This improves the very large finite-quotient construction in the unpublished intermediate note [2], at the cost of dependence on d and a probabilistic existence argument. The general complex theorem is qualitative in auxiliary size. Neither theorem establishes finite-bit complexity or efficient covariance optimization.

2 Capacity and the free comparison value

The capacity root is an infimum of continuous functions of Q , hence upper semicontinuous, so the maximum in (4) exists on the compact density-matrix set. It is finite by taking $X = I_d$. It is concave: each $\det \Phi_Q(X)^{1/d}$ is concave, and an infimum preserves concavity.

Proposition 3 (Upper bound). *For a feasible tuple Y and $Q_{ij} = \tau_m(Y_i Y_j^*)$, its objective is at most $\text{cap}(\Phi_Q)^{1/d}$.*

Proof. Q is a positive Gram matrix of trace one. Put $H = \sum_i A_i \otimes Y_i$. If H is singular the assertion is immediate. Otherwise, for $X > 0$ with $\det X = 1$, let $M = H(X \otimes I_m)H^* > 0$. Its normalized partial trace is $\Phi_Q(X)$. Concavity of $\log \det$ applied to auxiliary unitary averaging gives

$$\frac{1}{dm} \log \det M \leq \frac{1}{d} \log \det \Phi_Q(X).$$

Use $\det M = |\det H|^2$ and take the infimum over X . □

We separate three established inputs from the finite transfer below.

1. Mai–Speicher [3, Theorem 1.2] prove, for arbitrary complex B_j and free standard semicircular variables s_j ,

$$\Delta \left(\sum_j B_j \otimes s_j \right)^2 = e^{-1} \text{cap} \left(X \mapsto \sum_j B_j X B_j^* \right)^{1/d}. \quad (8)$$

Here $\Delta(T) = \exp(\tau \log |T|)$ uses normalized trace. The same identity holds for free standard circular variables c_j : write $c_j = (s_j + it_j)/\sqrt{2}$ and use the doubled coefficient list $B_j/\sqrt{2}, iB_j/\sqrt{2}$.

2. Positive operator capacity is equivalent to full noncommutative rank, hence an invertible finite matrix evaluation [4]. Thus, if some Φ_Q has positive capacity, there exist k and $T_i \in M_k(\mathbb{C})$ such that

$$B = \sum_i A_i \otimes T_i \quad \text{is invertible.} \quad (9)$$

Only existence is used, not a bound on k or the conditioning of B .

3. Fuglede–Kadison determinants are multiplicative and $z \mapsto \log \Delta(T - zI)$ is subharmonic for bounded T in a finite tracial von Neumann algebra [8]. Factoring out an invertible deterministic B gives subharmonicity of $z \mapsto \log \Delta(H + zB)$.

Weak convergence of singular-value distributions does not justify integrating $\log t$. The next lemma supplies the additional control needed for our perturbation.

3 Affine Ginibre pencils and the logarithmic tail

Fix $D, R_1, \dots, R_r \in M_q(\mathbb{C})$, with D invertible. Let $G_j^{(N)}$ be independent complex Ginibre matrices, with entry variance $1/N$, and let c_j be free standard circular variables. Set

$$P_N = D \otimes I_N + \sum_j R_j \otimes G_j^{(N)}, \quad P = D \otimes 1 + \sum_j R_j \otimes c_j.$$

Lemma 4 (Affine determinant convergence). *If $\Delta(P) > 0$, then*

$$\frac{1}{qN} \log |\det P_N| \longrightarrow \log \Delta(P) \quad \text{in probability.} \quad (10)$$

Proof. We give the tail argument explicitly.

Least singular value. Multiplication by $D^{-1} \otimes I_N$ reduces to Han [5, Theorem 1.5]. There is a fixed $b > 0$ with

$$\mathbb{P}\{s_{\min}(P_N) < N^{-b}\} \longrightarrow 0. \quad (11)$$

Absorb fixed multiplicative constants by increasing b . The invertibility of D is essential to this use of Han’s theorem.

Free spectral regularity. Hermitize as

$$X_N = \begin{pmatrix} 0 & P_N \\ P_N^* & 0 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & P \\ P^* & 0 \end{pmatrix}.$$

X is a selfadjoint matrix polynomial in semicircular variables. Its scalar Cauchy transform is algebraic by Shlyakhtenko–Skoufranis [7, Theorems 1.1 and 5.4]. Since $\Delta(P) > 0$, its spectral law has no atom at zero. The local Puiseux expansion gives constants $C > 0$ and $0 < \alpha < 1$ such that the singular-value law ν of P satisfies

$$\nu([0, t]) \leq Ct^\alpha \quad (0 < t \leq 1). \quad (12)$$

If zero is outside the support this is immediate. Otherwise the integrable density branches have Puiseux exponents greater than -1 ; integrate and, if necessary, reduce the positive exponent. No exponent uniform over all pencils is asserted.

Comparison at a shrinking scale. For the selfadjoint Gaussian series X_N , its variance norm $\sigma^2 = \|\mathbb{E}(X_N - \mathbb{E}X_N)^2\|$ is bounded, and its entry covariance norm v^2 is $O(N^{-1})$. Before hermitization the latter covariance is, up to permutation,

$$N^{-1} \left(\sum_j \text{vec}(R_j) \text{vec}(R_j)^* \right) \otimes I_{N^2}.$$

Hermitization increases this norm by at most two. The variance norm is bounded by the sums $\sum R_j R_j^*$ and $\sum R_j^* R_j$.

The free counterpart in Bandeira–Boedihardjo–van Handel [6, Theorem 2.8] has the same scalar law as X . Indeed its covariance, restricted to $M_{2q} \otimes I_N$, acts by

$$\eta \begin{pmatrix} U & V \\ W & Z \end{pmatrix} = \begin{pmatrix} \sum_j R_j Z R_j^* & 0 \\ 0 & \sum_j R_j^* U R_j \end{pmatrix}, \quad (13)$$

the circular hermitization covariance. The deterministic mean is also in this subalgebra. The operator-valued Wick recursion starts at the identity and stays there, proving equality of all matrix-valued moments and hence of compactly supported scalar laws. Therefore the scalar Cauchy transforms obey

$$|\mathbb{E} \tau_{2qN}((i\eta - X_N)^{-1}) - (\tau_{2q} \otimes \tau)((i\eta - X)^{-1})| \leq C/(N\eta^5). \quad (14)$$

The deterministic shift does not enter σ or ν .

Let ν_N be the empirical singular-value law and $F_N(t) = \nu_N([0, t])$. Hermitization gives

$$F_N(t) \leq 2 \int \frac{t^2}{t^2 + x^2} d\nu_N(x).$$

Using (12), a dyadic decomposition of the free integral, and (14), we obtain

$$\mathbb{E} F_N(t) \leq Ct^\alpha + \frac{C}{Nt^4} \quad (0 < t \leq 1). \quad (15)$$

Choose $0 < c < 1/(4 + \alpha)$ and increase b so that $b > c$. For $t \geq N^{-c}$ the second term is bounded by Ct^α . On $\mathcal{G}_N = \{s_{\min}(P_N) \geq N^{-b}\}$, the expected logarithmic contribution from $[N^{-b}, N^{-c}]$ is at most $Cb(\log N)N^{-c\alpha}$. For $[N^{-c}, \varepsilon]$, integration by parts gives

$$\mathbb{E} \left[\mathbf{1}_{\mathcal{G}_N} \int_{(0, \varepsilon]} |\log t| d\nu_N(t) \right] \leq Cb(\log N)N^{-c\alpha} + C\varepsilon^\alpha(|\log \varepsilon| + \alpha^{-1}). \quad (16)$$

Together with (11), this controls the logarithmic lower tail in probability. It does not follow from a minimum-singular-value estimate alone.

Gaussian asymptotic freeness gives weak convergence $\nu_N \rightarrow \nu$ in probability, and the finitely many Ginibre norms are bounded with probability tending to one [5]. Truncate log below, use these facts, then let the truncation decrease to zero using (16) and Markov's inequality. This proves (10). $\square$

This lemma is a consequence of the cited estimates, not a general determinant approximation theorem for arbitrary microstates or finite group quotients. The latter distinction matters in view of [11]. We do not assert convergence for unshifted singular-constant pencils.

4 Proof of the universal comparison

Proof of Theorem 1. The upper bound is Proposition 3. Fix $Q = LL^*$ with positive capacity and set $B_j = \sum_i L_{ij} A_i$. Choose B, T_i as in (9), and put

$$t_0 = \sum_i \tau_k(T_i T_i^*) > 0, \quad H = \sum_j (B_j \otimes I_k) \otimes c_j.$$

Normalized traces are unchanged by tensor amplification, so (8) yields

$$\Delta(H)^2 = e^{-1} \text{cap}(\Phi_Q)^{1/d} > 0. \quad (17)$$

The function

$$u(z) = \log \Delta(H + zB) = \log \Delta(B) + \log \Delta(B^{-1}H + zI)$$

is subharmonic and $u(0)$ is finite. For any $r, \delta > 0$, its submean inequality supplies a phase $|z| = r$ with

$$u(z) \geq u(0) - \delta > -\infty. \quad (18)$$

Such phases form a set of positive measure; otherwise the mean would be smaller than $u(0)$. The value is bounded above by the logarithm of an operator norm, so is finite at the chosen phase.

For this fixed $z \neq 0$, set

$$V_i^{(N)} = I_k \otimes \left(\sum_j L_{ij} G_j^{(N)} \right) + z T_i \otimes I_N, \quad (19)$$

$$e_N = \sum_i \tau_{kN}(V_i^{(N)} V_i^{(N)*}), \quad W_i^{(N)} = e_N^{-1/2} V_i^{(N)}. \quad (20)$$

The pencil is $zB \otimes I_N + \sum_j (B_j \otimes I_k) \otimes G_j^{(N)}$. Its constant is invertible and its free determinant is positive by (18), so Lemma 4 applies. First and second Gaussian trace moments also give

$$e_N \longrightarrow 1 + r^2 t_0 \quad \text{in probability.} \quad (21)$$

The random quadratic term tends to $\text{Tr } LL^* = 1$; every cross term contains $\tau_N(G_j^{(N)}) \rightarrow 0$. The logarithmic objective of $W^{(N)}$ therefore tends to

$$2u(z) - \log(1 + r^2 t_0) \geq \log \text{cap}(\Phi_Q)^{1/d} - 1 - 2\delta - \log(1 + r^2 t_0).$$

Choose $r, \delta > 0$ so that the last two losses sum to less than θ , and then choose N large enough to obtain a sample attaining the claimed bound. The order is fixed: finite invertible evaluation, radius and phase, then matrix size. No uniformity in z is required. Optimize over Q . If $\mathcal{C}(A) = 0$, the upper bound already implies $\mathcal{F}_\infty(A) = 0$. $\square$

5 Polynomial witnesses and sampling

Assume (6). Real covariances suffice: $\Phi_Q^* = \Phi_{\bar{Q}}$, capacity is adjoint invariant [3], and concavity implies that $(Q + \bar{Q})/2$ has no smaller capacity. Fix real $Q = LL^T$ with trace one and set

$$B_j = \sum_i L_{ij} A_i, \quad v_0 = \left\| \sum_j B_j^2 \right\|, \quad g_0 = \left\| \sum_j \text{vec}(B_j) \text{vec}(B_j)^* \right\|. \quad (22)$$

Vectorization uses the unnormalized Hilbert–Schmidt inner product. We have

$$\sum_j \|B_j\|^2 \leq a^2 \|L\|_{\text{HS}}^2 = a^2.$$

Consequently,

$$v_0 \leq a^2, \quad g_0 \leq da^2. \quad (23)$$

Take independent normalized $m \times m$ GUE matrices G_j , and put $Y_i = \sum_j L_{ij} G_j$, $H_m = \sum_j B_j \otimes G_j$, $E_m = \sum_i \tau_m(Y_i^2)$. Then $\mathbb{E} E_m = 1$. For $\varepsilon > 0$ the normal matrices

$$W_i = \frac{Y_i + i\varepsilon b_i I_m}{\sqrt{E_m + \varepsilon^2 \beta^2}} \quad (24)$$

have energy exactly one by cancellation of cross terms. Their logarithmic determinant objective is

$$J = \tau_{dm} \log(H_m^2 + \varepsilon^2 I) - \log(E_m + \varepsilon^2 \beta^2). \quad (25)$$

Proposition 5 (Expected objective). *If $\text{cap}(\Phi_Q) > 0$, then*

$$\mathbb{E}J \geq \log \text{cap}(\Phi_Q)^{1/d} - 1 - \frac{v_0 g_0}{2m\varepsilon^4} - \log(1 + \varepsilon^2 \beta^2). \quad (26)$$

Proof. Let $S = \sum_j B_j \otimes s_j$. In the Gaussian-series representation of H_m , the variance norm is v_0 and the entry covariance norm is g_0/m . For a real Hilbert–Schmidt orthonormal basis F_p of Hermitian matrices,

$$G_j = m^{-1/2} \sum_{p=1}^{m^2} g_{jp} F_p, \quad \sum_p F_p U F_p = \text{Tr}(U) I_m.$$

As in (13), the free counterpart has scalar law S . Theorem 2.8 of [6] bounds the scalar resolvent difference at it by $v_0 g_0/(mt^5)$.

For $h_t(x) = \log(x^2 + t^2)$, its derivative in t has trace equal to minus twice the imaginary part of that resolvent. The difference of the log statistics tends to zero as $t \rightarrow \infty$, since $0 \leq \log(1 + x^2/t^2) \leq x^2/t^2$. Integrating from ε to infinity gives

$$|\mathbb{E}\tau_{dm}h_\varepsilon(H_m) - (\tau_d \otimes \tau)h_\varepsilon(S)| \leq \frac{v_0 g_0}{2m\varepsilon^4}. \quad (27)$$

The free term is at least $2 \log \Delta(S) = \log \text{cap}(\Phi_Q)^{1/d} - 1$ by (8). Jensen gives $\mathbb{E} \log(E_m + \varepsilon^2 \beta^2) \leq \log(1 + \varepsilon^2 \beta^2)$. The shift bounds negative parts, and Gaussian second moments bound positive parts, so these expectations are finite. $\square$

Proof of Theorem 2. Choose a real maximizing Q and put $\varepsilon^2 = \theta/(2\beta^2)$. If $m \geq 4v_0 g_0 \beta^4 \theta^{-3}$, the two losses in (26) are at most $\theta/2$ each, strictly so for the logarithmic one. Some sample therefore has $J \geq \log \mathcal{C}(A) - 1 - \theta$. Use (23) for (7). $\square$

Proposition 6 (Sampling probability). *Fix real Q with positive capacity, $0 < \theta \leq 1$, $\varepsilon^2 = \theta/(4\beta^2)$ and an integer $m \geq \max\{2, 32v_0 g_0 \beta^4 \theta^{-3}\}$. One sample of (24) attains objective at least $e^{-1-\theta} \text{cap}(\Phi_Q)^{1/d}$ except with probability at most*

$$\exp\left(-\frac{m^2 \theta^3}{32(\beta \sqrt{g_0/d} + 1)^2}\right). \quad (28)$$

Proof. For standard real Gaussian coordinates $g = (g_{jp})$, the map $\delta g \mapsto \delta H_m$ has Hilbert–Schmidt norm at most $\sqrt{g_0/m} \|\delta g\|_2$. Since $\sup_x |h'_\varepsilon(x)| = 1/\varepsilon$, differentiating under the trace and applying Hilbert–Schmidt Cauchy–Schwarz gives Lipschitz constant $\sqrt{g_0/d}/(m\varepsilon)$ for the first term of (25). Also, up to coordinate ordering, $E_m = m^{-2} g^T (L^T L \otimes I_{m^2}) g$ and $\|L\| \leq 1$. For $c = \varepsilon^2 \beta^2$,

$$\|\nabla \log(E_m + c)\|_2 \leq \frac{2\sqrt{E_m}}{m(E_m + c)} \leq \frac{1}{m\sqrt{c}}.$$

Thus J is globally Lipschitz with $L_J \leq (\sqrt{g_0/d} + 1/\beta)/(m\varepsilon)$. Gaussian concentration [9] yields $\mathbb{P}(J < \mathbb{E}J - t) \leq \exp(-t^2/(2L_J^2))$. The expected loss is at most $\theta/2$; insert $t = \theta/2$ to get (28). $\square$

For $\sum_i b_i A_i = D > 0$, use $D^{-1/2} A_i D^{-1/2}$ in this section. Both $\mathcal{F}_m$ and $\mathcal{C}$ acquire the factor $\det(D)^{-2/d}$, so the relative guarantee is unchanged. The parameter $a\beta$ is computed after normalization and may be large. The sampler assumes exact real Gaussian inputs; finite-precision implementation needs a rounding analysis.

6 Scope checks and limitations

For one coefficient, unitary auxiliaries attain $\mathcal{F}_m(A_1) = |\det A_1|^{2/d} = \mathcal{C}(A_1)$. The factor e^{-1} need not be attained. Conversely, the standard three 3×3 skew-symmetric generators span a space with identically zero scalar determinant but an invertible Pauli 2×2 evaluation. Multiplication by i makes them Hermitian and traceless, so their real span has no positive definite matrix. The universal theorem applies to this familiar example, whereas [1] does not. We claim no novelty for that matrix space.

The quantitative theorem provides polynomial auxiliary size only in the positive-span class. The universal theorem allows every tuple but does not control the constants in Han’s estimate or the algebraic exponent after phase selection. Combining them does not give a general polynomial-size theorem. Neither proves convergence of unperturbed homogeneous Ginibre log determinants at every fixed point.

Kargin’s recent variational formula [10] concerns free determinants with matrix shifts and Brown potentials. That free problem is distinct from the finite energy-constrained transfer here. We use the older Mai–Speicher identity and classical subharmonicity; no shifted free determinant formula is claimed as new. Broad source-index screening was followed by targeted reading of the primary proof inputs. This is not a certificate of worldwide priority or a verification of every screened preprint.

Verification and provenance. Accompanying checks examine covariance identities, normalization, exact finite evaluations, parameter constants and numerical stress cases. They supplement the proof without proving its infinite-dimensional inputs. OpenAI Codex assisted the development and writing. GPT-6 Astra examined the proof in shared development context and contributed to the transfer argument. This is not blind, independent, or human peer review. No editorial acceptance or historical-impact claim is made.

References

- [1] L. Eriksson, *A dimension-independent capacity bound for amplified determinants of Hermitian matrix tuples*, AIRR working paper ARR-2026-4E4SJ328XF98BSCJ (2026). <https://airr.science/papers/ARR-2026-4E4SJ328XF98BSCJ/versions/v1/>.
- [2] L. Eriksson, *Explicit finite witnesses for Hermitian amplified determinants via finite quotients of free groups*, unpublished note, 16 October 2026. Not a proof input here.
- [3] T. Mai and R. Speicher, *Fuglede–Kadison determinants of matrix-valued semicircular elements and capacity estimates*, arXiv:2406.15922. <https://arxiv.org/abs/2406.15922>.
- [4] A. Garg, L. Gurvits, R. Oliveira and A. Wigderson, *Operator scaling: theory and applications*, arXiv:1511.03730. <https://arxiv.org/abs/1511.03730>.
- [5] Y. Han, *Brown measure convergence for the spectrum of polynomials in Ginibre matrices*, arXiv:2606.01664v1 (2026), Theorem 1.5. <https://arxiv.org/abs/2606.01664>.
- [6] A. S. Bandeira, M. T. Boedihardjo and R. van Handel, *Matrix concentration inequalities and free probability*, Invent. Math. **234** (2023), 419–487. <https://arxiv.org/abs/2108.06312>.
- [7] D. Shlyakhtenko and P. Skoufranis, *Freely independent random variables with non-atomic distributions*, Trans. Amer. Math. Soc. **367** (2015), 6267–6291. <https://arxiv.org/abs/1305.1920>.

- [8] U. Haagerup and H. Schultz, *Invariant subspaces for operators in a general II_1 -factor*, Publ. Math. IHES **109** (2009), 19–111, introduction. <https://doi.org/10.1007/s10240-009-0018-7>.
- [9] G. W. Anderson, A. Guionnet and O. Zeitouni, *An Introduction to Random Matrices*, Cambridge University Press (2010), Gaussian concentration.
- [10] V. Kargin, *Fuglede–Kadison determinants of shifted matrix semicircles*, arXiv:2610.15180 (2026). <https://arxiv.org/abs/2610.15180>.
- [11] H. Kammeyer, *A counterexample to the determinant approximation conjecture*, arXiv:2609.15567 (2026). <https://arxiv.org/abs/2609.15567>.