The Action–Memory Diamond: Exact Resource Regions for Passive Matrix Networks

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10 August 2026

Abstract

A passive lossless network pays two different costs when it rotates a signal subspace over a frequency arc: a continuous Wigner–Smith action and an integer McMillan memory. Known Grassmannian length bounds and rational-inner degree identities constrain these resources separately. For the stated finite rational-inner class, we determine their joint attainable region exactly. Let two rank-k subspaces have principal angles β_j, put $B = \sum_j \beta_j$ and let r be the number of nonzero angles. For a square finite rational inner transfer matrix of degree at most d, define the arc action

$$A = \int_I \operatorname{tr} Q(\theta)\,\mathrm{d}\theta, \qquad Q = -\mathrm{i}S^* \partial_\theta S \succeq 0.$$

If $B > 0$, the exact feasible set is

$$\boxed{d \geq r, \qquad 2B \leq A \leq 2\pi d - 2B}.$$

Both faces are attained. Equivalently,

$$d_{\min}(A) = \max\left\{ r, \left\lceil \frac{A + 2B}{2\pi} \right\rceil \right\}.$$

The upper face is a return cost: the complementary arc must rotate the subspace back, while the full-circle trace action is exactly 2π times the degree. We prove sufficiency by an explicit compiler of normalized rank-one Blaschke–Potapov gates; it realizes every interior point and both faces, including rank-deficient and orthogonal cases. When the endpoint subspaces coincide, the region changes discontinuously to $A \in [0, 2\pi d)$, with the upper endpoint open.

We derive fail-closed noisy degree certificates and give a multi-frequency warning using classical boundary Nevanlinna–Pick theory: three pairwise degree-one routing tasks can require degree two jointly. For orthogonal-line data the obstruction is a frame-independent cycle phase, exhibited by an exact positive semidefinite Pick completion with spectrum $(2, 1, 0)$. Deterministic code compiles random points of the diamond, constructs the degree-two colligation, and is replayed by an independent verifier.

Keywords: rational inner matrix; Wigner–Smith delay; McMillan degree; Blaschke–Potapov product; Grassmannian; passive network; boundary Nevanlinna–Pick interpolation; resource theory.

1 One routing event, two coupled resources

For a lossless causal finite-dimensional network, the boundary transfer matrix $S(\mathrm{e}^{\mathrm{i}\theta})$ is unitary and its proper-delay matrix

$$Q(\theta) = -\mathrm{i}S(\mathrm{e}^{\mathrm{i}\theta})^* \frac{\mathrm{d}}{\mathrm{d}\theta} S(\mathrm{e}^{\mathrm{i}\theta}) \tag{1}$$

is positive semidefinite. The integral of $\operatorname{tr} Q$ over an arc measures a continuous dwell-time or phase-action resource. The McMillan degree counts the dimension of a minimal conservative realization and is integer valued [9, 7].

These resources cannot be chosen independently. A subspace rotation costs action on the observed arc. Less obviously, the same network must close its unitary loop on the complementary arc. The

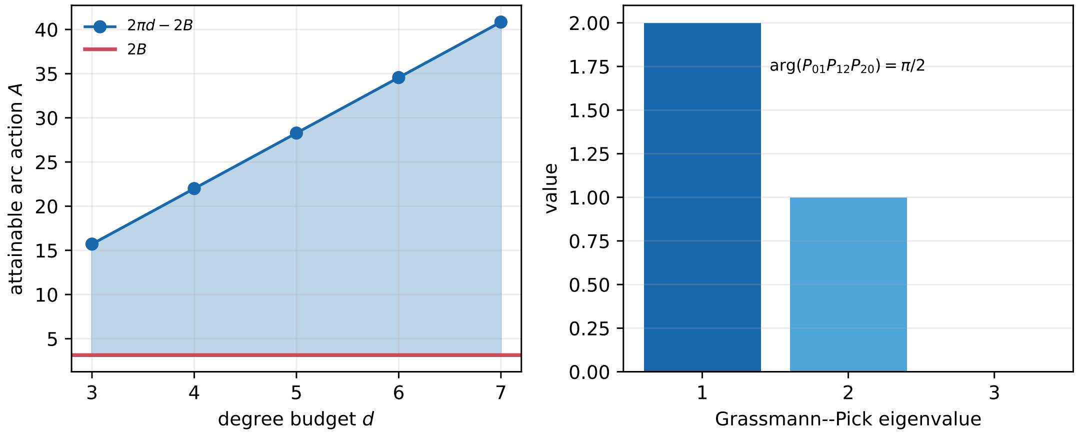

Figure 1: Left: the exact action–memory region for fixed endpoint geometry. The lower face is the forward Grassmann cost and the upper face reserves the same cost for the complementary return. Right: a three-node Grassmann–Pick completion has rank two although every pair has rank one; its cycle product has phase $\pi/2$.

Precedent	Established ingredient	Not addressed there
Qiu–Zhang–Li [11]	Symmetric-gauge Grassmann metrics	Positive delay, rational degree and complementary-arc budget
Antezana–Larotonda–Varela [2]	Optimal unitary and Grassmann paths	Inner analyticity, winding quantization and converse synthesis
Potapov; Alpay et al. [10, 1]	Rank-one factorization and degree	Exact action region under endpoint subspace data
Bolotnikov–Dym [5]	Matrix tangential boundary interpolation	Arc-integrated action and forward/return geometry
Ball–Kim [3]	Stability and McMillan degree for rational interpolants	Inner subspace endpoints and a prescribed arc-action region
Bharath et al. [4]	All-pass design with nodewise group-delay optimization	Closed action–degree diamond and its two boundary compilers

Table 1: Closest precedents and scope distinction. Each cited work supplies the listed ingredient; we are not aware of one containing the present joint attainable-region theorem.

trace action available to pay both journeys is quantized by the determinant winding. This observation turns two one-sided lower bounds into the closed diamond in Fig. 1.

Grassmann metrics and optimal unitary/Grassmann paths are classical [11, Thm. 1] [2, Thms. 16 and 20]; see also the recent projector formulation [8, Sec. IV.4]. Rank-one Blaschke–Potapov factorization and boundary all-pass interpolation are likewise classical [1, Thm. 3.1] [5, Thms. 4.7, 5.1 and 8.2] [4, Thm. 1 and Sec. 4.4]. General rational interpolation with stability and minimum-degree constraints is also classical [3]. Within the sources compared in Table 1, we did not find an exact joint attainable region for arc-integrated trace delay and McMillan degree under a prescribed subspace transfer. That conjunction, including both converses and all boundary cases, is the contribution here; this is a scoped literature statement, not a claim that the surrounding interpolation theories are new.

What is and is not claimed. The theorem concerns square finite rational inner matrices. It does not cover loss, gain, singular inner factors, nonrational continuous-time propagation delays or a non-Markovian bath. The quantity A is trace delay, not thermodynamic work. The multi-node rank formula used later is a gauge-quotient corollary of classical tangential boundary interpolation, not a new Pick theorem.

Question	Preceding paper [6]	Present paper
Continuous two-node data	Exact modal region $2\beta \prec_w q$ for absolutely continuous positive-generator paths	No new modal-profile claim; scalar trace action in the finite rational-inner class
Rational two-node memory	Minimum degree r and a construction on the lower face $A = 2B$	Complete iff region in (A, d), including the upper return face, all interior points and the discontinuous $B = 0$ case
Multi-node information	Pick–Wigner–Smith Gram/rank theorem for specified unitary values and local derivatives	Gauge-quotiented subspace corollary and an exact three-line cyclic obstruction; no new general Pick theorem
Robust conclusion	Separate lower certificates for modal action and hidden-state rank	Joint feasibility/falsification for a proposed (A, d) budget and a cycle-product null control

Table 2: Theorem-level non-overlap with the immediately preceding paper. The earlier modal theorem is finer in spectral profile; the present theorem is stronger only in its exact coupling of total arc action to integer degree.

2 Setup

Let $I = [\theta_0, \theta_1]$ be a proper positively oriented arc, let $d \in \mathbb{N}_0$, and suppose $0 < \theta_1 - \theta_0 < 2\pi$, and write $\zeta_j = e^{i\theta_j}$. Let S be an N-by-N rational inner matrix regular on $\mathbb{T}$. Its McMillan degree is denoted $n = \deg_{\mathrm{McM}} S$.

Fix $\mathcal{X} \in \mathrm{Gr}(k, N)$ and set

$$\mathcal{Y}_0 = S(\zeta_0)\mathcal{X}, \qquad \mathcal{Y}_1 = S(\zeta_1)\mathcal{X}.$$

Let

$$\frac{\pi}{2} \geq \beta_1 \geq \cdots \geq \beta_k \geq 0$$

be their canonical angles and define

$$B = \sum_{j=1}^{k} \beta_j, \qquad r = \#\{j : \beta_j > 0\}. \tag{2}$$

The arc action is

$$A_I(S) = \int_{\theta_0}^{\theta_1} \operatorname{tr} Q(\theta)\, \mathrm{d}\theta. \tag{3}$$

For a finite rational inner matrix, $Q(\theta) \succeq 0$ on $\mathbb{T}$ and

$$\int_0^{2\pi} \operatorname{tr} Q(\theta)\, \mathrm{d}\theta = 2\pi \ \mathrm{wind}(\det S) = 2\pi \deg_{\mathrm{McM}} S. \tag{4}$$

The last equality follows directly from a rank-one Blaschke–Potapov factorization [10, 1].

3 The exact action–memory diamond

Theorem 3.1 (Exact action–memory diamond)**.** *Let I be a proper arc and let $\mathcal{X}, \mathcal{Y} \in \mathrm{Gr}(k, N)$ have principal-angle data (B, r) from (2).*

If $B > 0$, there exists an N-by-N finite rational inner matrix S, regular on $\mathbb{T}$, such that

$$S(\zeta_0)\mathcal{X} = \mathcal{X}, \qquad S(\zeta_1)\mathcal{X} = \mathcal{Y}, \qquad \deg_{\mathrm{McM}} S \leq d, \qquad A_I(S) = A,$$

if and only if

$$\boxed{d \geq r, \qquad 2B \leq A \leq 2\pi d - 2B.} \tag{5}$$

Constants on the left allow an arbitrary prescribed initial image in place of $\mathcal{X}$.

If $B = 0$, the exact region is

$$A = 0 \quad \text{or} \quad d \geq 1 \text{ and } 0 < A < 2\pi d. \tag{6}$$

Thus for $d \geq 1$ it is $[0, 2\pi d)$; for $d = 0$ it is the singleton $\{0\}$.

Corollary 3.2 (Exact least memory at prescribed action). *For $B > 0$ and $A \geq 2B$,*

$$d_{\min}(A) = \max\left\{r, \left\lceil \frac{A + 2B}{2\pi} \right\rceil\right\}. \tag{7}$$

For $B = 0$, $d_{\min}(0) = 0$ and

$$d_{\min}(A) = \left\lfloor \frac{A}{2\pi} \right\rfloor + 1, \qquad A > 0.$$

Three features are worth isolating. First, the two faces in (5) are inclusive even when some angles are orthogonal. Second, an arbitrarily small but positive angle forces one unit of memory, whereas at $B = 0$ the degree can drop to zero. Third, the open upper endpoint in (6) is real: every nonconstant finite inner function has strictly positive trace delay on every nonempty complementary arc.

4 Necessity: the return arc spends the missing action

We give a self-contained trace version of the geometric speed limit.

Lemma 4.1 (Positive-generator subspace speed). *Let $U(t)$ be a differentiable unitary path with $-\mathrm{i}U^*\dot{U} = H(t) \succeq 0$, and let $\mathcal{X}(t) = U(t)\mathcal{X}(0)$. If the endpoint principal angles are β_j, then*

$$2\sum_j \beta_j \leq \int \operatorname{tr} H(t)\,\mathrm{d}t. \tag{8}$$

Proof. Let P be the moving orthogonal projector and write H in blocks relative to $P \oplus (I - P)$ as $\left(\begin{smallmatrix} H_{11} & K \\ K^* & H_{22} \end{smallmatrix}\right)$. The projector velocity has trace norm $\|\dot{P}\|_1 = 2\|K\|_*$. Positivity of the block matrix implies

$$2\|K\|_* \leq \operatorname{tr} H_{11} + \operatorname{tr} H_{22} = \operatorname{tr} H;$$

for example, test positivity against the polar partial isometry of K. The trace-norm Grassmann length dominates twice the sum of endpoint principal angles [11, 2]. Integration gives (8). $\square$

Applying Theorem 4.1 on I gives $A \geq 2B$. On the complementary arc, the same periodic transfer matrix returns $\mathcal{Y}$ to $\mathcal{X}$. Its action is $2\pi n - A$ by (4), so

$$2\pi n - A \geq 2B. \tag{9}$$

Since $n \leq d$, this yields the upper face in (5).

It remains to see why $n \geq r$. Factor S, up to a constant unitary, into n rank-one Blaschke–Potapov factors F_j. Put $P_j(z) = F_1(z) \cdots F_j(z)$ and $R_j = P_j(\zeta_1)P_j(\zeta_0)^*$. Then

$$R_j = [P_{j-1}(\zeta_1)F_j(\zeta_1)F_j(\zeta_0)^*P_{j-1}(\zeta_1)^*]\,R_{j-1}.$$

The bracket is a unitary gate $I + (\lambda - 1)vv^*$ after conjugating its rank-one projector. Such a gate can lower the dimension of the intersection of two k-planes by at most one. After n gates,

$$\dim(\mathcal{X} \cap \mathcal{Y}) \geq k - n.$$

Because $r = k - \dim(\mathcal{X} \cap \mathcal{Y})$, one has $n \geq r$.

When $B = 0$, (4) only gives $A \leq 2\pi n$. Equality cannot hold for $n > 0$: the complementary arc is nonempty and each scalar Poisson kernel in a Potapov factorization is strictly positive there. Hence $A < 2\pi n \leq 2\pi d$. The constant case gives $A = 0$.

5 Two exact one-gate lemmas

The converse rests on two elementary facts. We state them explicitly to fix phases and endpoints.

Lemma 5.1 (Any proper-arc phase)**.** *For a proper arc I and every $\delta \in (0, 2\pi)$, there is a scalar degree-one Blaschke factor b such that*

$$b(\zeta_0) = 1, \qquad b(\zeta_1) = \mathrm{e}^{\mathrm{i}\delta}, \qquad \int_I -\mathrm{i}\bar{b}\,\partial_\theta b\,\mathrm{d}\theta = \delta.$$

Proof. Rotate the arc to $[-\gamma, \gamma]$, where $\gamma = (\theta_1 - \theta_0)/2 \in (0, \pi)$. For $b_\rho(z) = (z - \rho)/(1 - \rho z)$, $-1 < \rho < 1$, its phase advance is

$$\delta_\rho = 4 \arctan\left(\frac{1+\rho}{1-\rho} \tan\frac{\gamma}{2}\right). \tag{10}$$

It increases continuously from 0 to 2π. More explicitly, the desired zero is

$$\rho = \frac{\tan(\delta/4) - \tan(\gamma/2)}{\tan(\delta/4) + \tan(\gamma/2)}. \tag{11}$$

Multiplication by a unimodular constant normalizes the value at the first endpoint. The integral equals the unwrapped phase advance. □

Lemma 5.2 (Exact rank-one line gate)**.** *Let two lines have angle $\alpha \in (0, \pi/2]$. A rank-one unitary*

$$W = I + (\mathrm{e}^{\mathrm{i}\delta} - 1)vv^*$$

maps the first line to the second, after choosing v, if and only if

$$2\alpha \le \delta \le 2\pi - 2\alpha. \tag{12}$$

Proof. In the plane of the two lines, write $v = \cos\eta\, x + \mathrm{e}^{\mathrm{i}\varphi} \sin\eta\, x_\perp$. Then

$$|\langle x, Wx\rangle|^2 = 1 - \sin^2(\delta/2)\sin^2(2\eta). \tag{13}$$

Thus $|\langle x, Wx\rangle| = \cos\alpha$ is possible exactly when $\sin\alpha \le |\sin(\delta/2)|$, which is (12). For a constructive choice, set

$$t = \cos^2\eta = \frac{1 - \sqrt{1 - \sin^2\alpha / \sin^2(\delta/2)}}{2}$$

and choose φ to match the phase of the prescribed second line. This also covers both equalities in (12). □

6 Sufficiency: compiling every point

Assume first that $B > 0$. Choose mutually orthogonal principal planes with vectors $x_j \in \mathcal{X}$ and $x_j^\perp \in \mathcal{X}^\perp$ such that the target principal line is

$$y_j = \cos\beta_j\, x_j + \sin\beta_j\, x_j^\perp.$$

Only the r positive angles need gates.

Fix an integer n with $r \le n \le d$ and

$$2B \le A \le 2\pi n - 2B. \tag{14}$$

Choose positive integers q_j summing to n and subdivide each principal geodesic into positive increments

$$\beta_j = \alpha_{j1} + \cdots + \alpha_{jq_j}.$$

By Theorem 5.2, the phase assigned to step (j, ℓ) may be any value in

$$[2\alpha_{j\ell},\, 2\pi - 2\alpha_{j\ell}].$$

The Minkowski sum of these n intervals is exactly the interval in (14). Hence phases $\delta_{j\ell}$ can be chosen to sum to A.

Use Theorem 5.1 to produce normalized scalar factors $b_{j\ell}$ with these arc phases. At each intermediate principal line, use Theorem 5.2 to choose a rank-one projector $v_{j\ell}v_{j\ell}^*$, and define

$$B_{j\ell}(z) = I + (b_{j\ell}(z) - 1)v_{j\ell}v_{j\ell}^*.$$

Multiply the factors in the physical order of the intermediate lines. Every factor is the identity at ζ_0, while at ζ_1 the gates carry each x_j along its principal plane to y_j. The untouched intersection is fixed. Thus the product maps $\mathcal{X}$ to $\mathcal{Y}$.

The product is rational inner. Its delay is a sum of unitary conjugates of the positive rank-one scalar delays, so it is positive. Its determinant is the product of the n scalar factors; consequently its degree is exactly n, its full action is $2\pi n$, and its arc action is $\sum_{j,\ell} \delta_{j\ell} = A$. This proves sufficiency for a fixed n. The union over $r \le n \le d$ is the entire diamond (5): the intervals are nested apart from their already shared lower endpoint, which is attained at degree r.

If $B = 0$ and $A > 0$, choose $n = \lfloor A/(2\pi) \rfloor + 1 \le d$. Split A into n numbers in $(0, 2\pi)$ and place all corresponding normalized scalar factors on a fixed line inside $\mathcal{X}$. The endpoint unitary preserves $\mathcal{X}$ and the action is A. The constant transfer handles $A = 0$. This completes the proof of Theorem 3.1 and Theorem 3.2.

Remark 6.1 (An explicit compiler). The proof is algorithmic. Equal subdivisions $\alpha_{j\ell} = \beta_j/q_j$ suffice. Start every phase at its lower value $2\alpha_{j\ell}$ and distribute the surplus $A - 2B$ across capacities $2\pi - 4\alpha_{j\ell}$. Equations (11) and (13) then return the Blaschke zero and projector without optimization.

7 Sharp examples and discontinuities

A collapsed diamond. For one orthogonal pair, $\beta = \pi/2$, $r = d = 1$. The lower and upper faces coincide, so $A = \pi$ is forced. More generally, $r = d$ mutually orthogonal principal lines force $A = \pi r$. Half the winding is spent on the forward arc and half on the return.

Extra memory widens by 2π. For $\beta = (\pi/3, \pi/6, 0)$, $B = \pi/2$ and $r = 2$. Degree two gives $A \in [\pi, 3\pi]$; degree three gives $A \in [\pi, 5\pi]$. The extra state does not reduce the geometric minimum, but it adds one full turn to the available upper action.

Topological jump at zero motion. For every $\beta > 0$, however small, $d_{\min}(2\beta) = 1$. At $\beta = 0$ and $A = 0$, it is zero. Continuous action can vanish along a sequence while the integer memory remains fixed. Conversely, when $B = 0$ and A approaches $2\pi d$, the required poles concentrate near the boundary but the endpoint subspace still does not move.

8 Fail-closed inference from noisy data

Suppose a projector experiment supplies certified lower bounds $\underline{\beta}_j \le \beta_j$, and an action measurement supplies $A \in [\underline{A}, \overline{A}]$. Put

$$\underline{B} = \sum_j \underline{\beta}_j, \qquad \underline{r} = \#\{j : \underline{\beta}_j > 0\}.$$

Corollary 8.1 (Robust memory certificate)**.** *Every compatible rational inner model satisfies*

$$\deg_{\mathrm{McM}} S \ge \max\left\{\underline{r},\, \left\lceil \frac{\underline{A} + 2\underline{B}}{2\pi} \right\rceil\right\}, \tag{15}$$

whenever $\underline{B} > 0$. A claimed degree budget d is falsified if

$$\overline{A} < 2\underline{B} \quad \text{or} \quad \underline{A} > 2\pi d - 2\underline{B}.$$

The certificate is deliberately one-sided. It never substitutes point estimates for uncertainty sets, and therefore cannot overstate the required degree when the stated error bounds hold. Standard perturbation bounds for projectors turn operator-norm error bars into lower principal-angle bounds; the preceding paper gives one such implementation [6].

9 Beyond two nodes: a cyclic memory invisible pairwise

The diamond completely solves one arc with two endpoint subspaces. It does not say that pairwise costs determine a multi-frequency interpolation. The right classical object is a boundary Pick completion.

Fix an input frame $X \in \mathbb{C}^{N\times k}$, boundary nodes ζ_i, target frames Y_i, and gauges $V_i \in U(k)$. Put $W_i = Y_i V_i$ and prescribe off-diagonal blocks

$$P_{ij} = \frac{I_k - W_i^* W_j}{1 - \overline{\zeta}_i \zeta_j}, \qquad i \neq j, \tag{16}$$

while leaving Hermitian diagonal blocks free. Tangential boundary Nevanlinna–Pick theory implies that the least rational-inner degree equals the least rank of a positive semidefinite completion, minimized also over the gauges [5, Thms. 4.7, 5.1 and 8.2]. Equivalently, this follows from a lurking-isometry construction and the compressed de Branges–Rovnyak kernel. We use this classical corollary only to expose a strict global effect.

Proposition 9.1 (Three-node cyclic memory)**.** *Let $N = 3$, $k = 1$, $\omega = \mathrm{e}^{2\pi \mathrm{i}/3}$, and take nodes $1, \omega, \omega^2$ with three mutually orthogonal target lines. Every two-node subproblem has minimum degree one, but the joint problem has minimum degree two.*

Proof. All off-diagonal target overlaps vanish, independently of scalar frame phases. With

$$a = \frac{1}{1-\omega} = \frac{1}{2} + \frac{\mathrm{i}}{2\sqrt{3}},$$

the diagonal choice $P_{00} = P_{11} = P_{22} = 1$ gives

$$P = \begin{pmatrix} 1 & a & \overline{a} \\ \overline{a} & 1 & a \\ a & \overline{a} & 1 \end{pmatrix}, \qquad \operatorname{spec} P = (2, 1, 0). \tag{17}$$

Thus a degree-two interpolant exists. If a positive semidefinite rank-one completion existed, $P_{ij} = \overline{f}_i f_j$ would force $P_{01}P_{12}P_{20} = |f_0 f_1 f_2|^2$ to be nonnegative real. Instead,

$$P_{01}P_{12}P_{20} = a^3 = \frac{\mathrm{i}}{3\sqrt{3}}. \tag{18}$$

Hence rank one is impossible. For any pair, choosing both diagonal entries as $|P_{ij}|$ gives a positive semidefinite rank-one completion. □

For the present orthogonal-line data the cross blocks, and hence their cycle product, are independent of all frame gauges because every off-diagonal target overlap vanishes. The phase is a holonomy-like cycle invariant and, in this example, records information not visible in pairwise canonical angles. The underlying Pick theorem and its rank formulation are classical; here the example demonstrates the strict separation

$$d_{\min}^{(3\text{ nodes})} = 2 > \max_{i\neq j} d_{\min}^{(i,j)} = 1.$$

A robust cycle witness. For these orthogonal data, or for fixed phase-calibrated tangential frames, let three measured nonzero cross blocks be scalars $\widehat{p}_1, \widehat{p}_2, \widehat{p}_3$, each within ε of its exact value, and let $M = \max_j |\widehat{p}_j|$. Telescoping the product gives

$$\left| \prod_j p_j - \prod_j \widehat{p}_j \right| \le 3\varepsilon(M+\varepsilon)^2.$$

Therefore

$$\left| \operatorname{Im} \prod_j \widehat{p}_j \right| > 3\varepsilon(M+\varepsilon)^2$$

certifies rank at least two without selecting diagonal blocks or solving a rank minimization.

10 Reproducible adversarial audit

The proofs are analytic; computation tests the formulas at their most likely failure interfaces. The producer and verifier are separate programs and the verifier imports no producer functions. The numerical compiler never rounds a positive angle into the $B = 0$ branch; inputs below 10^{-10} radians are reported as numerically indeterminate. The frozen campaign includes:

- 128 randomized feasible diamonds with ranks one through four, prescribed actions, analytic derivatives and adaptive quadrature, plus a separate noncommuting two-gate order-reversal control;
- exact lower and upper faces, orthogonal collapsed diamonds, zero angles, a degree obstruction and the open $B = 0$ endpoint;
- 512 independent scalar phase inversions and 2,048 independent rank-one gate identities;
- the exact Pick spectrum $(2, 1, 0)$, cycle product $\mathrm{i}/(3\sqrt{3})$, a synthesized degree-two unitary colligation and direct boundary interpolation checks;
- 1,024 noisy trials around the rank-two cycle and 1,024 rank-one null controls for the fail-closed product bound.

For the flagship point

$$\beta = (0.92, 0.47, 0.18), \quad B = 1.57, \quad A = 15.25,$$

the compiler uses degree three. The observed maximum principal-angle error is below 3×10^{-16}, the arc-action error is below 2×10^{-14}, and the full-circle action differs from 6π by less than 2×10^{-14}. Across the randomized campaign the frozen maximum arc-action error is below 10^{-10} and no positivity, endpoint or winding test fails. These figures are diagnostics, not evidence replacing the proof.

11 Physical reading and falsifiers

The lower face $A = 2B$ is a direct routing cost. The upper face is equally physical: a finite passive network has a quantized total phase budget, and the unobserved frequencies must retain enough action to close the subspace loop. Adding one internal state enlarges the allowed upper action by exactly 2π. The theorem therefore predicts a staircase in minimum realization order when arc dwell time is increased at fixed endpoint geometry.

The result is falsifiable in several ways. A finite rational inner network inside the stated class with $A < 2B$, $A > 2\pi d - 2B$, or $d < r$ would refute necessity. A feasible point for which the explicit gate compiler fails would refute sufficiency. Lossy, active or nonrational devices are not counterexamples because they leave the model class. Repeated boundary nodes would require confluent Pick data in the multi-node section and are not covered by (16).

12 Conclusion

Continuous delay and discrete memory form one exact resource region. For a nontrivial subspace transfer, the forward and return arcs each pay the same Grassmann trace cost $2B$, while the determinant winding supplies exactly $2\pi d$. Nothing is lost between these necessary inequalities: normalized rank-one Blaschke–Potapov gates compile every point of the diamond. The least memory at prescribed action follows in closed form.

The three-node example marks the boundary of the two-point story. Pairwise angles see support motion; a cyclic Pick phase can demand additional hidden state. This suggests a next problem with genuine content: characterize which cycle holonomies raise the minimum positive-completion rank for matrix-valued subspace data, rather than merely restating general boundary interpolation.

Data and code availability

Source, deterministic certificates, the independent verifier and the exact PDF are archived at the immutable GitHub release. The principal entry points are `research/exact_action_memory_certificate.py` and `verification/verify_exact_action_memory.py`. The certificate contains deterministic seeds, tolerances and every reported diagnostic.

A Proof details omitted from the main line

A.1 The positive-block trace inequality

For completeness, let

$$H = \begin{pmatrix} A & K \\ K^* & C \end{pmatrix} \succeq 0$$

on finite-dimensional spaces. If $K = V|K|$ is its polar decomposition, the standard contraction factorization gives $K = A^{1/2}TC^{1/2}$ with $\|T\| \leq 1$. Equivalently, apply $2\operatorname{Re}\operatorname{tr}(V^*K) \leq \operatorname{tr} A + \operatorname{tr} C$ after completing the square. Since $\operatorname{Re}\operatorname{tr}(V^*K) = \operatorname{tr}|K|$, one obtains

$$2\|K\|_* \leq \operatorname{tr} A + \operatorname{tr} C. \tag{19}$$

For the projector path $P(t) = U(t)P_0U(t)^*$ with $\dot{U} = \mathrm{i}UH$, direct differentiation gives

$$\dot{P} = \mathrm{i}U[H, P_0]U^*.$$

Relative to $P_0 \oplus (I - P_0)$, the commutator has off-diagonal blocks $-K$ and K^*, hence $\|\dot{P}\|_1 = 2\|K\|_*$. The trace-norm length of a Grassmann path is at least $2\sum_j \beta_j$ [11, Thm. 1] [2, Thm. 20]. Combining this fact with (19) proves Theorem 4.1 without a smoothness assumption beyond absolute continuity.

A.2 Why one Potapov factor changes at most one support direction

Let $R = I + (\lambda - 1)vv^*$ and let $\mathcal{M}$ be a k-plane. Every vector in $\mathcal{M} \cap v^\perp$ is fixed, so

$$\dim(\mathcal{M} \cap R\mathcal{M}) \geq k - 1.$$

For a chain $\mathcal{M}_j = R_j\mathcal{M}_{j-1}$, the codimension inside the initial plane can increase by at most one per step. An induction based on

$$\dim(E \cap G) \geq \dim(E \cap F) + \dim(F \cap G) - \dim F$$

gives $\dim(\mathcal{M}_0 \cap \mathcal{M}_n) \geq k - n$. Thus the number of positive canonical angles is at most the number of rank-one factors. This also shows why a small angle does not mean a fraction of a state.

A.3 Endpoint regularity and strict positivity off the observed arc

A scalar Blaschke factor with zero $a \in \mathbb{D}$ has boundary phase derivative

$$p_a(\theta) = \frac{1-|a|^2}{|e^{i\theta}-a|^2} > 0.$$

For a rank-one Potapov factor its delay is $p_a(\theta)vv^*$, and the delay of a product is a sum of unitary conjugates of these terms. Therefore a nonconstant finite product has $\operatorname{tr} Q(\theta) > 0$ at every boundary point. All poles are outside $\overline{\mathbb{D}}$, so the constructed transfer is regular at both endpoints and on the entire circle. This proves both the open face in (6) and the absence of hidden endpoint singularities in the compiler.

B A gauge-quotient boundary Pick corollary

We record the exact classical result used in Theorem 9.1 so that the degree conclusion can be audited without an appeal to engineering terminology.

Proposition B.1 (Boundary Pick rank after quotienting frames)**.** *Let $\zeta_1, \ldots, \zeta_L \in \mathbb{T}$ be distinct, fix an isometric input frame $X \in \mathbb{C}^{N\times k}$, and let Y_i be isometric frames for prescribed target k-planes. For gauges $V_i \in U(k)$ put $W_i = Y_iV_i$ and form the off-diagonal blocks* (16)*. Then the minimum McMillan degree of a square rational inner S, regular at the nodes and satisfying $S(\zeta_i)\operatorname{ran} X = \operatorname{ran} Y_i$, equals*

$$\min\left\{\operatorname{rank} P : P \succeq 0,\ P_{ij} \text{ obeys (16) for } i \neq j,\ V_i \in U(k)\right\}, \tag{20}$$

where the Hermitian diagonal blocks are free.

This is a quotient formulation of matrix tangential boundary interpolation [5]; the short proof is included to expose exactly where the rank enters. Feasible completions exist after adding sufficiently large positive diagonal blocks, and the minimum rank is attained because feasible ranks lie in $\{0, \ldots, Lk\}$. Necessity follows from the compressed input de Branges–Rovnyak kernel

$$X^* \frac{I - S(z)^*S(w)}{1-\overline{z}w} X.$$

At the sampled boundary nodes it is positive, has the off-diagonal blocks (16), and has rank no larger than $\deg_{\mathrm{McM}} S$.

Conversely, factor a feasible rank-s completion as $P_{ij} = F_i^*F_j$, $F_i \in \mathbb{C}^{s\times k}$. The defining identity is equivalent, including on the diagonal, to equality of the Gram matrices of

$$\begin{pmatrix} \zeta_i F_i u \\ Xu \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} F_i u \\ W_i u \end{pmatrix}. \tag{21}$$

Hence their correspondence extends to a unitary colligation $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ on $\mathbb{C}^s \oplus \mathbb{C}^N$. Its transfer

$$S(z) = D + zC(I - zA)^{-1}B$$

interpolates the frames. Any unit-circle eigenspace of the contraction A is reducing for the colligation and decoupled from B and C; deleting it leaves the transfer unchanged and makes it regular at the nodes. Thus its degree is at most s. Applying necessity to a minimum-rank completion precludes a smaller degree and proves (20).

C Compiler and verification contract

For a fixed feasible degree n, the implementation uses the following deterministic contract.

1. Count positive principal angles and allocate $n - r$ extra subdivisions to the principal planes.

2. Subdivide each β_j into positive $\alpha_{j\ell}$.
3. Initialize $\delta_{j\ell} = 2\alpha_{j\ell}$ and distribute the surplus $A - 2B$ without exceeding $2\pi - 4\alpha_{j\ell}$ per gate.
4. Use (11) for each scalar zero and (13) for each rank-one projector.
5. Multiply in physical order and evaluate analytic derivatives, never finite differences. Integrate regular cases adaptively and use the exact phase antiderivative when a Poisson peak is below numerical resolution.

Interface	Independent invariant	Frozen tolerance
Scalar factor	phase formula vs. adaptive Poisson integral	2×10^{-9}
Rank-one gate	endpoint line angle and unit-vector norm	5×10^{-8}
Potapov product	boundary unitarity and $Q \succeq 0$	2×10^{-8}
Arc compiler	prescribed action and principal angles	2×10^{-7}
Full circle	$\int \operatorname{tr} Q = 2\pi n$	5×10^{-7}
Pick synthesis	colligation unitarity and interpolation	2×10^{-7}

Table 3: Numerical contract. The independent verifier uses separate code and tighter algebraic checks for the scalar, gate and cycle identities.

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