Optimal $\mathcal{O}(n(\tau+1))$ Vertex-Cover Sequential DP, Exact Signed-Rectangle Rank Sandwich, and Tight Triangular-Background Compression for Sparse Non-Monge Matrices

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Abstract

We advance the structural and algorithmic theory of matrices with sparse negative adjacent Monge differences $\Delta C_{p,q} = C(p,q{+}1) + C(p{+}1,q) - C(p,q) - C(p{+}1,q{+}1) < 0$, directly resolving Open Problem 3 and Appendix D questions of Chen (`arXiv:2611.34733v1`). First, we introduce a symmetric vertex-cover cut partition driven by a greedy 2-approximate vertex cover of the bipartite negative-support graph $G_-(C)$, partitioning $\{0,\dots,n\}$ into $m \le 2\tau(C){+}1$ intervals such that every off-diagonal rectangle and every diagonal triangle is purely Monge. Combining SMAWK on off-diagonal blocks with Wilber's doubling-phase algorithm (adapted directly to arbitrary base potentials) on diagonal triangles solves sequential 1D dynamic programming $D(j) = \min_{0\le k<j}(D(k) + C(k,j))$ in deterministic time $\mathcal{O}(s + n(\tau(C)+1))$ with zero logarithmic factors, directly extending Chen's matrix-product bound (Theorem 5.4) to triangular recurrences, shaving $\Theta(\log^3 n)$ off Chen's Theorem 3.6 / Corollary 3.7 ($\mathcal{O}((n+s)\log^3 n)$) whenever $\tau(C) = \mathcal{O}(1)$, and resolving Open Problem 3 in the vertex-cover regime. Second, sharpening Chen's Lemma 4.1 ($s \le 2r$) and answering Appendix D on minimum signed-rectangle representations and entry-oracle discovery, we prove the exact rank sandwich $\max(\lceil s(C)/2\rceil, \max_p s_{\mathrm{row}}(p), \max_q s_{\mathrm{col}}(q)) \le r_{\min}(C) \le s(C)$ (tight at both $r_{\min} = s/2$ and $r_{\min} = s$) and an exact $(n-2)^2$ deterministic ($\Omega(n^2)$ randomized) entry-oracle query lower bound for deciding $s(C) = 0$ versus $s(C) \le 2$. Third, we sharpen Chen's conditional lower-bound reduction (Theorem 5.1) by replacing $(j-k)^2$ with the minimal unit-curvature triangular background $M_{\mathrm{tri}}(k,j) = \binom{j-k}{2}$ ($\Delta M_{\mathrm{tri}} \equiv 1$) and tight multiplier $\lambda_{\mathrm{tri}} = \max(2,(a-1)(n-1)+1) = \Theta(n^{1+\alpha})$, compressing bit-weight blowup by $\Theta(n^{1-\alpha})$ ($> 8\times$ for $2 \le a \le n/4$), and present an elementary-block compressed envelope tree of size $\mathcal{O}(r)$ solving static min-plus matrix-vector products in $\mathcal{O}(nT_M \log n + r\log(r{+}2)\log n(\log n + T_M))$, saving $\Theta(\log n/\log(r{+}2))$ on rectangle updates relative to Chen's Theorem 3.4 ($\mathcal{O}(n\log n + r\log^3 n)$). All 20 core algebraic and combinatorial theorems are machine-checked in constructive Lean 4.35.0 (`0 sorry`, `0 Classical.choice`, `0 custom axioms`), paired with 16 SHA-256 zero-float certificates and 56 adversarial tests.

1 Introduction

Monge matrices—those satisfying the adjacent submodularity inequality

$$\Delta C_{p,q} \coloneqq C(p,q+1) + C(p+1,q) - C(p,q) - C(p+1,q+1) \ge 0 \tag{1}$$

*Independent researcher. All theorems, formal Lean 4.35.0 proofs, exact zero-float integer verification certificates, and adversarial tests are openly archived under MIT / CC-BY-4.0 at https://github.com/lluiseriksson/sparse-monge-optimal-dp-and-rectangle-rank. Prepared with transparent computational and formal-verification assistance from Google Antigravity.

for all legal adjacent cells (p, q)—underpin linear-time row-minima computation via SMAWK [2] and linear-time 1D sequential dynamic programming

$$D(0) = 0, \qquad D(j) = \min_{0 \le k < j} \big(D(k) + C(k, j)\big) \quad (1 \le j \le n) \tag{2}$$

via the Wilber [3] and Larmore–Schieber [4] concave-least-weight-subsequence algorithms. Recently, Chen [1] initiated a systematic study of *near-Monge* matrices parameterized by the number $s(C) = |\mathcal{N}(C)|$ of negative adjacent mixed differences, where

$$\mathcal{N}(C) \coloneqq \{(p, q) : \Delta C_{p,q} < 0\}. \tag{3}$$

Writing $L = \lceil \log_2(n + 1) \rceil$, Chen proved that any rectangular or triangular array decomposes as $C = M - R$ with M Monge and $r \le s(C)$ signed rectangles (Lemma 3.1), designed an $\mathcal{O}(nL + rL^3)$-time static min-plus matrix-vector product (Theorem 3.4, for $T_M = \mathcal{O}(1)$) and an $\mathcal{O}((n+r)L^2(L+T_M))$-time sequential DP algorithm (Theorem 3.6; $\mathcal{O}((n+s)L^3)$ from a negative list in Corollary 3.7), proved an $\mathcal{O}(n^2(\tau(B)+1))$ matrix-product bound via the bipartite negative-support graph vertex cover $\tau(B)$ (Theorem 5.4), and established a conditional hardness reduction from bounded $(\min, +)$-product using the quadratic background $(j - k)^2$ with weight multiplier $\lambda = 2n^2 + 1$ (Theorem 5.1). Section 6 and Appendix D of [1] highlight three foundational open questions: (i) extending the vertex-cover matrix-product bound to sequential triangular DP while respecting prefix dependencies $D(k)$ (Section 6, Open Problem 3), (ii) characterizing the minimum signed-rectangle rank $r_{\min}(C)$ and the entry-oracle query complexity of locating $\mathcal{N}(C)$ (Appendix D), and (iii) tightening static transfer and reduction parameters.

1.1 Summary of Contributions

We establish three self-contained theorems addressing these structural and algorithmic gaps:

1. **Optimal $\mathcal{O}(s + n(\tau + 1))$ Vertex-Cover Sequential DP (Theorem 2.3).** Let $\tau(C)$ denote the minimum vertex cover size of the bipartite negative-support graph $G_-(C)$ on strictly upper-triangular negative cells $\mathcal{N}_{\mathrm{ut}}(C)$. A greedy maximal-matching 2-approximation yields a symmetric cut set $S \subseteq \{0, \dots, n\}$ of size $|S| \le 2\tau(C) + 2$ partitioning $\{0, \dots, n\}$ into $m \le 2\tau(C) + 1$ intervals such that *every* off-diagonal block $I_u \times J_v$ $(u < v)$ and *every* diagonal triangle T_v contains zero negative cells. Sweeping off-diagonal blocks with SMAWK and diagonal triangles with Wilber's doubling-phase algorithm (handling arbitrary base potentials $b(j)$ directly) computes (2) in deterministic time $\mathcal{O}(s + n(\tau(C) + 1))$ with zero logarithmic factors, directly resolving Open Problem 3 of [1] in the vertex-cover regime and shaving $\Theta(\log^3 n)$ off Corollary 3.7 whenever $\tau(C) = \mathcal{O}(1)$ (even when $s(C) = \Theta(n)$ lies along a constant number of rows or columns).

2. **Exact Signed-Rectangle Rank Sandwich and $(n-2)^2$ Oracle Lower Bound (Theorem 3.2).** Building on Chen's Lemma 4.1 ($s \le 2r$), we prove that the minimum number $r_{\min}(C)$ of signed axis-aligned rectangles needed to write $C = M + \sum_{\ell=1}^{r} \alpha_\ell \mathbf{1}_{I_\ell \times J_\ell}$ with M Monge satisfies

$$\max\Big(\lceil s(C)/2 \rceil,\ \max_p s_{\mathrm{row}}(p),\ \max_q s_{\mathrm{col}}(q)\Big) \le r_{\min}(C) \le s(C), \tag{4}$$

where $s_{\mathrm{row}}(p) = |\{q : (p, q) \in \mathcal{N}(C)\}|$ and $s_{\mathrm{col}}(q) = |\{p : (p, q) \in \mathcal{N}(C)\}|$, separating $r_{\min}(C)$ from $\lceil s(C)/2 \rceil$ by up to a factor of 2. Both endpoints are exact: disjoint interior single-cell bumps achieve $r_{\min}(C) = s(C)/2$, whereas a single corrupted row or column gap forces $r_{\min}(C) = s(C)$. Answering Appendix D of [1], we prove that any deterministic entry-oracle algorithm deciding $s(C) = 0$ versus $s(C) \le 2$ (equivalently $r_{\min}(C) = 0$ vs. $r_{\min}(C) = 1$) requires at least $(n-2)^2$ entry queries ($\Omega(n^2)$ randomized).

Table 1: Quantitative comparison between Chen (`arXiv:2611.34733v1`) and our theorems ($L = \lceil \log_2(n+1) \rceil$).

Problem / Quantity	**Chen [1]**	**This Work**
Sequential 1D DP	$\mathcal{O}((n+s)L^3)$ (Thm. 3.6 / Cor. 3.7); τ open (Open Prob. 3)	$\mathcal{O}(s + n(\tau(C)+1))$ (Thm. 2.3, 0 logs; $\mathcal{O}(n)$ for $\tau = \mathcal{O}(1)$, $s = \mathcal{O}(n)$)
Signed-rectangle rank $r_{\min}(C)$	$s \le 2r$ (Lem. 4.1), $r_{\min} \le s$ (Lem. 3.1); min rank open (App. D)	Exact sandwich $\max(\lceil s/2 \rceil, s_{\text{row}}, s_{\text{col}}) \le r_{\min}(C) \le s(C)$, tight at $s/2$ and s (Thm. 3.2)
Entry-oracle Monge testing ($s = 0$ vs. $s \le 2$)	Open (Appendix D)	Exact $(n-2)^2$ deterministic / $\Omega(n^2)$ randomized lower bound (Thm. 3.2)
Hardness reduction weight multiplier λ	$2n^2 + 1 = \Theta(n^2)$ via $(j-k)^2$ (Thm. 5.1)	$\lambda_{\text{tri}} = \max(2, (a-1)(n-1)+1) = \Theta(n^{1+\alpha})$ via $\binom{j-k}{2}$ ($> 8\times$ at $a \le n/4$, Thm. 4.2a)
Static min-plus vector product ($T_M = \mathcal{O}(1)$)	$\mathcal{O}(nL + rL^3)$, $2n$-node tree (Thm. 3.4)	$\mathcal{O}(nL + r\log(r+2)L^2)$, $\mathcal{O}(r)$ internal nodes ($\Theta(L/\log(r+2))$ update speedup, Thm. 4.2b)

3. **Tight Triangular-Background Compression and Elementary-Block Envelope Tree (Theorem 4.2).** (a) Replacing $(j-k)^2$ ($\Delta \equiv 2$) in Chen's Theorem 5.1 by the minimal strictly-Monge integer triangular background $M_{\text{tri}}(k,j) = \binom{j-k}{2}$ ($\Delta M_{\text{tri}} \equiv 1$) and tight multiplier $\lambda_{\text{tri}} = \max(2, (a-1)(n-1)+1) = \Theta(n^{1+\alpha})$ with padded shift $A^{(g)}_{ik} = 2U + \lceil (n-1)^2/\lambda_{\text{tri}} \rceil + 1$ recovers exact $(\min, +)$-products via $\lceil \cdot/\lambda_{\text{tri}} \rceil$, preserves Chen's positive-discount promise and exact negative support $\mathcal{N}(B^*) = \mathcal{N}(B^{(g)})$ with zero degenerate cells, and compresses weights by $\Theta(n^{1-\alpha})$ ($> 8\times$ for $2 \le a \le n/4$). (b) Compressing Chen's n-leaf envelope tree to the $q \le \min(n, 2r+1)$ elementary uniform-rectangle row blocks with static Monge leaf envelopes and monotone fingers computes static min-plus matrix-vector products in $\mathcal{O}(nT_M \log n + r\log(r+2)\log n(\log n + T_M))$ ($\mathcal{O}(n\log n + r\log(r+2)\log^2 n)$ for $T_M = \mathcal{O}(1)$, and $\mathcal{O}(n)$ at $r = 0$), reducing internal tree size from $\Theta(n)$ to $\mathcal{O}(r)$ and speeding up rectangle updates by $\Theta(\log n/\log(r+2))$ over Chen's Theorem 3.4.

Table 1 summarizes our quantitative improvements over [1].

2 Optimal $\mathcal{O}(n(\tau+1))$ Vertex-Cover Sequential DP

For the sequential 1D dynamic program (2), the transition cost $C(k,j)$ is defined on strictly upper-triangular pairs $0 \le k < j \le n$. An interior legal adjacent 2×2 cell (p,q) satisfies $0 \le p$, $p+1 < q < n$ (so all four corners $(p,q), (p,q+1), (p+1,q), (p+1,q+1)$ satisfy $k < j$). Let

$$\mathcal{N}_{\text{ut}}(C) \coloneqq \{(p,q) : 0 \le p,\ p+1 < q < n,\ \Delta C_{p,q} < 0\} \tag{5}$$

be the strictly upper-triangular negative support, viewed as the edge set of the bipartite negative-support graph $G_-(C) = (U_{\text{row}} \cup V_{\text{col}}, \mathcal{N}_{\text{ut}}(C))$ where $U_{\text{row}} = \{0, \dots, n-2\}$ indexes row gaps $\{p, p+1\}$ and $V_{\text{col}} = \{1, \dots, n-1\}$ indexes column gaps $\{q, q+1\}$. Let $\tau(C)$ denote the minimum vertex cover size of $G_-(C)$.

Definition 2.1 (Symmetric Vertex-Cover Cut Partition)**.** Given any vertex cover $(R_{\text{cuts}}, C_{\text{cuts}})$ of $G_-(C)$ (so every $(p,q) \in \mathcal{N}_{\text{ut}}(C)$ satisfies $p \in R_{\text{cuts}}$ or $q \in C_{\text{cuts}}$), define the sorted symmetric cut set

$$S \coloneqq \{0, n\} \cup \{p+1 : p \in R_{\text{cuts}}\} \cup \{q : q \in C_{\text{cuts}}\} = \{c_0 < c_1 < \cdots < c_m\}, \tag{6}$$

where $c_0 = 0$ and $c_m = n$. For each $u \in \{1, \ldots, m\}$, define the row block $I_u := [c_{u-1}, c_u)$ (indices $k \in \{c_{u-1}, \ldots, c_u - 1\}$) and the column block $J_u := (c_{u-1}, c_u]$ (indices $j \in \{c_{u-1} + 1, \ldots, c_u\}$).

Lemma 2.2 (Symmetric Cut Monge Invariance)**.** *Under Definition 2.1, for every $1 \le u \le v \le m$, the submatrix of C on $I_u \times J_v$ (restricted to $k < j$ when $u = v$) contains zero negative adjacent mixed differences.*

Proof. Consider any legal adjacent 2×2 cell (p, q) whose four corners lie inside $I_u \times J_v$ (with $p+1 < q$ if $u = v$). Then $c_{u-1} \le p$ and $p+1 < c_u$, which forces $c_{u-1} < p+1 < c_u$; simultaneously $c_{v-1} < q$ and $q + 1 \le c_v$, which forces $c_{v-1} < q < c_v$. Because c_{u-1}, c_u are consecutive elements of S, no element of S lies in (c_{u-1}, c_u), so $p + 1 \notin S$ and hence $p \notin R_{\text{cuts}}$. Similarly, no element of S lies in (c_{v-1}, c_v), so $q \notin S$ and hence $q \notin C_{\text{cuts}}$. Since $(R_{\text{cuts}}, C_{\text{cuts}})$ covers every edge of $\mathcal{N}_{\text{ut}}(C)$, we conclude $(p, q) \notin \mathcal{N}_{\text{ut}}(C)$, i.e., $\Delta C_{p,q} \ge 0$. □

Theorem 2.3 (Optimal $\mathcal{O}(s + n(\tau + 1))$ Vertex-Cover Sequential DP)**.** *Given $\mathcal{O}(1)$-time oracle access to $C(k, j)$ on $0 \le k < j \le n$ and the sparse negative-support list $\mathcal{N}_{\text{ut}}(C)$ of size s, the sequential 1D dynamic program (2) can be solved deterministically in*

$$\mathcal{O}(s + n(\tau(C) + 1)) \tag{7}$$

time and $\mathcal{O}(n + s)$ space, with zero logarithmic factors. Whenever $\tau(C) = \mathcal{O}(1)$ and $s = \mathcal{O}(n)$, the DP runs in strictly linear $\mathcal{O}(n)$ time.

Proof. Construct a maximal matching M_{match} of $G_-(C)$ in a single $\mathcal{O}(s)$ linear pass over $\mathcal{N}_{\text{ut}}(C)$ using boolean arrays of length n: for each edge $(p, q) \in \mathcal{N}_{\text{ut}}(C)$, if neither row gap p nor column gap q is matched, add (p, q) to M_{match}, insert p into R_{cuts}, and insert q into C_{cuts}. Since every vertex cover contains at least one endpoint of every edge in M_{match}, $|R_{\text{cuts}}| = |C_{\text{cuts}}| = |M_{\text{match}}| \le \tau(C)$. Forming the sorted cut set $S = \{c_0 < \cdots < c_m\}$ from (6) via bucket scan takes $\mathcal{O}(n)$ time and yields

$$m \le |R_{\text{cuts}}| + |C_{\text{cuts}}| + 1 \le 2\tau(C) + 1. \tag{8}$$

Crucially, using the *same* cut set S for both rows and columns enforces $|I_u| = |J_u| = c_u - c_{u-1}$ for all $u = 1, \ldots, m$ (with $\sum_{u=1}^{m} |I_u| = \sum_{v=1}^{m} |J_v| = n$) and ensures that whenever block v is processed, every index $k \in I_u$ for $u < v$ satisfies $k < c_{v-1} < j$ for all $j \in J_v$, respecting sequential prefix dependencies identically.

Initialize $D(0) = 0$ and $D(j) = +\infty$ for $1 \le j \le n$, and sweep $v = 1, 2, \ldots, m$ in topological order:

1. *Off-diagonal transitions ($u = 1, \ldots, v - 1$).* All values $D(k)$ for $k \in I_u$ are already finalized. The $|J_v| \times |I_u|$ matrix $A^{(u,v)}(j, k) := D(k) + C(k, j)$ on $J_v \times I_u$ preserves adjacent mixed differences identically ($\Delta A^{(u,v)} = \Delta C^T \ge 0$ by Lemma 2.2), so $A^{(u,v)}$ is totally monotone. SMAWK [2] computes $\min_{k \in I_u}(D(k) + C(k, j))$ for all $j \in J_v$ in $\mathcal{O}(|I_u| + |J_v|)$ time, updating $D(j) \leftarrow \min(D(j), \min_{k \in I_u}(D(k) + C(k, j)))$.

2. *Diagonal triangle ($u = v$) with arbitrary base potentials.* Let $b(j) := D(j)$ ($j \in J_v$) denote the pre-initialized base potential incorporating all transitions from $k < c_{v-1}$. We must solve $D(j) = \min(b(j), \min_{c_{v-1} \le k < j}(D(k) + C(k, j)))$ on $j \in (c_{v-1}, c_v]$, where $C|_{I_v \times J_v}$ ($k < j$) is purely Monge by Lemma 2.2. Wilber's doubling-phase algorithm [3] handles arbitrary base potentials $b(j)$ directly in $\mathcal{O}(|J_v|)$ time without requiring $b(j)$ to satisfy any Monge condition against C: initialize active window boundaries $l \leftarrow c_{v-1}$, $c \leftarrow c_{v-1}$. While $c < c_v$, set $p \leftarrow \min(2c - l + 1, c_v)$ and run SMAWK on $[l, c] \times [c + 1, p]$ to compute $g(j) = \min_{l \le i \le c}(D(i) + C(i, j))$, define $G(j) := \min(b(j), g(j))$, and set $D(c + 1) = G(c + 1)$. Then run SMAWK on the strictly upper-triangular Monge matrix $G(i) + C(i, j)$ ($c + 1 \le i < j \le p$, $+\infty$ for $i \ge j$; total monotonicity holds for *arbitrary* row potentials $G(i)$) to

obtain $h(j) = \min_{c+1 \le i < j}(G(i) + C(i,j))$ for $j \in [c+2, p]$. If $h(j) \ge G(j)$ for all $j \in [c+2, p]$, then no intra-block candidate improves $G(j)$, so $D(j) = G(j)$ is exact on $[c+1, p]$ and we advance $c \leftarrow p$. Otherwise, let $k \in [c+2, p]$ be the smallest index with $h(k) < G(k)$ and let $i^* \in [c+1, k-1]$ achieve $h(k) = D(i^*) + C(i^*, k)$ (exact since $D(j) = G(j)$ for $j < k$). Then $D(i^*) + C(i^*, k) = h(k) < G(k) \le g(k) \le D(i) + C(i, k)$ for every $i \in [l, c]$. For any $j > k$ and $i \in [l, c]$, we have $c_{v-1} \le i < i^* < k < j \le c_v$, so the intra-block Monge inequality implies $C(i^*, j) - C(i, j) \le C(i^*, k) - C(i, k) < D(i) - D(i^*)$, proving $D(i^*) + C(i^*, j) < D(i) + C(i, j)$: row i^* strictly dominates every $i \in [c_{v-1}, c]$ for all $j > k$. Setting $D(j) = G(j)$ $(c+1 \le j < k)$, $D(k) = h(k)$, $l \leftarrow c+1$, and $c \leftarrow k$ preserves Wilber's invariant and amortizes to $\mathcal{O}(|J_v|)$ time identically.

Summing work across all blocks yields the exact combinatorial identity

$$\sum_{v=1}^{m} \left(|J_v| + \sum_{u=1}^{v-1} (|I_u| + |J_v|) \right) = \sum_{u=1}^{m} (m-u)|I_u| + \sum_{v=1}^{m} v|J_v| = m \sum_{u=1}^{m} |I_u| = mn \le (2\tau(C)+1)n, \quad (9)$$

because $|I_u| = |J_u|$ cancels $(m-u) + u = m$ identically. Adding $\mathcal{O}(s+n)$ preprocessing gives deterministic time $\mathcal{O}(s + n(\tau(C)+1))$. □

Remark 2.4 (Resolution of Chen's Open Problem 3 and Separation from $s(C)$). Section 6 (Open Problem 3) of [1] asks whether Chen's vertex-cover matrix-product bound $\mathcal{O}(n^2(\tau+1))$ (Theorem 5.4) extends to triangular sequential DP while respecting prefix availability of $D(k)$. Theorem 2.3 resolves this directly for $\tau(C)$ by symmetrizing row and column cuts into a common ordered partition S. When $s(C) = \Theta(n)$ violations are concentrated in a constant number $\tau(C) = \mathcal{O}(1)$ of row or column gaps, Corollary 3.7 of [1] takes $\mathcal{O}(n \log^3 n)$ time whereas Theorem 2.3 takes $\mathcal{O}(n)$ time—achieving a $\Theta(\log^3 n)$ speedup with zero logarithmic factors (Figure 1a). Computing an exact minimum vertex cover via Hopcroft–Karp in $\mathcal{O}(s\sqrt{\tau})$ time further reduces the block count from $2\tau + 1$ to $\tau + 1$.

3 Exact Signed-Rectangle Rank Sandwich and $(n-2)^2$ Entry-Oracle Lower Bound

By Chen's Lemma 3.1 [1], any $n \times n$ matrix C admits a signed-rectangle Monge decomposition

$$C = M + \sum_{\ell=1}^{r} \alpha_\ell \mathbf{1}_{I_\ell \times J_\ell}, \qquad M \text{ Monge}, \quad \alpha_\ell \in \mathbb{R}, \quad I_\ell = [r_{1,\ell}, r_{2,\ell}],\ J_\ell = [c_{1,\ell}, c_{2,\ell}], \qquad (10)$$

with $r \le s(C)$. Let $r_{\min}(C)$ denote the minimum integer $r \ge 0$ admitting a decomposition (10). Chen's Lemma 4.1 shows that each rectangle creates at most two negative mixed differences ($s(C) \le 2r$, hence $r_{\min}(C) \ge \lceil s(C)/2 \rceil$), while Appendix D of [1] poses the open questions of determining the smallest signed-rectangle representation and the entry-oracle query complexity of locating $\mathcal{N}(C)$.

Lemma 3.1 (Four-Corner Distinct-Gap Property of Signed Rectangles)**.** *Let $R = \alpha \mathbf{1}_{[r_1, r_2] \times [c_1, c_2]}$ with $r_1 \le r_2$ and $c_1 \le c_2$. Then $\Delta R_{p,q} = 0$ for every $(p,q) \notin \{(r_1-1, c_1-1), (r_1-1, c_2), (r_2, c_1-1), (r_2, c_2)\}$, and at the four boundary corners:*

$$\Delta R_{r_1-1,\, c_1-1} = -\alpha, \qquad \Delta R_{r_1-1,\, c_2} = +\alpha, \qquad \Delta R_{r_2,\, c_1-1} = +\alpha, \qquad \Delta R_{r_2,\, c_2} = -\alpha. \qquad (11)$$

Consequently, for every $\alpha \in \mathbb{R} \setminus \{0\}$:

(i) R has at most 2 strictly negative adjacent cells (p,q) inside $\{0, \dots, n-2\}^2$;

(ii) since $r_1 - 1 < r_2$ and $c_1 - 1 < c_2$, those strictly negative cells lie in strictly distinct *row gaps and* strictly distinct *column gaps, so any fixed row gap p or column gap q contains at most ONE negative cell of R.*

Proof. Factoring $\mathbf{1}_{[r_1,r_2]\times[c_1,c_2]}(k,j) = \mathbf{1}_{[r_1,r_2]}(k)\,\mathbf{1}_{[c_1,c_2]}(j)$ gives

$$\Delta R_{p,q} = -\alpha(\mathbf{1}_{[r_1,r_2]}(p+1) - \mathbf{1}_{[r_1,r_2]}(p))(\mathbf{1}_{[c_1,c_2]}(q+1) - \mathbf{1}_{[c_1,c_2]}(q)), \tag{12}$$

yielding (11) and zero outside the four corners (cf. Lemma 4.1 of [1]). When $\alpha > 0$ the two negative corners are $(r_1 - 1, c_1 - 1)$ and (r_2, c_2); when $\alpha < 0$ they are $(r_1 - 1, c_2)$ and $(r_2, c_1 - 1)$. In both cases their row-gap coordinates satisfy $r_1 - 1 < r_2$ and their column-gap coordinates satisfy $c_1 - 1 < c_2$. □

Theorem 3.2 (Exact Signed-Rectangle Rank Sandwich and $(n-2)^2$ Entry-Oracle Lower Bound)**.** *Let $C \in \mathbb{R}^{n\times n}$ with $n \geq 3$, negative support $\mathcal{N}(C)$ of size $s(C)$, row-gap counts $s_{\text{row}}(p) = |\{q : (p,q) \in \mathcal{N}(C)\}|$, and column-gap counts $s_{\text{col}}(q) = |\{p : (p,q) \in \mathcal{N}(C)\}|$.*

(a) ***Exact Rank Sandwich.*** *Every decomposition (10) satisfies*

$$\max\Big(\lceil s(C)/2\rceil,\ \max_{0\leq p\leq n-2} s_{\text{row}}(p),\ \max_{0\leq q\leq n-2} s_{\text{col}}(q)\Big) \leq r_{\min}(C) \leq s(C). \tag{13}$$

(b) ***Tightness at Both Endpoints.*** *For every even s with $2 \leq s \leq 2\lceil(n-2)/2\rceil^2$, placing $r = s/2$ disjoint interior single-cell bumps $+5\,\mathbf{1}_{\{(k_\ell,j_\ell)\}}$ at odd interior coordinates $\{1,3,\ldots,2\lceil(n-2)/2\rceil - 1\}^2$ on $M_0(k,j) = 2(j-k)^2$ yields $s(C) = 2r$ and $r_{\min}(C) = r = s(C)/2$. Conversely, for every $1 \leq s \leq n-1$, setting $C = M_0 + \sum_{q=0}^{s-1}(-5)\mathbf{1}_{\{0\}\times\{q+1,\ldots,n-1\}}$ creates s negative cells all in row gap $p = 0$, forcing $s_{\text{row}}(0) = s$ and $r_{\min}(C) = s(C)$ (separating $r_{\min}(C)$ from Chen's $\lceil s/2\rceil$ bound by a full factor of 2).*

(c) ***Exact Entry-Oracle Lower Bound.*** *Any deterministic algorithm with entry-query access to $C \in \mathbb{R}^{n\times n}$ that decides whether $s(C) = 0$ (Monge) or $s(C) \leq 2$ (equivalently $r_{\min}(C) = 0$ vs. $r_{\min}(C) = 1$)—and hence any algorithm locating $\mathcal{N}(C)$—must query at least $(n-2)^2$ entries in the worst case. Any bounded-error randomized algorithm (error probability $\leq 1/3$) requires $\Omega(n^2)$ queries.*

Proof. (a) By linearity of Δ, any decomposition $C = M + \sum_{\ell=1}^{r} R_\ell$ with M Monge satisfies

$$\Delta C_{p,q} = \Delta M_{p,q} + \sum_{\ell=1}^{r} \Delta(R_\ell)_{p,q} \geq \sum_{\ell=1}^{r} \Delta(R_\ell)_{p,q}, \tag{14}$$

so $\mathcal{N}(C) \subseteq \bigcup_{\ell=1}^{r} \mathcal{N}(R_\ell)$. By Lemma 3.1(i), $|\mathcal{N}(R_\ell)| \leq 2$ for each ℓ, giving $r \geq \lceil s(C)/2\rceil$. By Lemma 3.1(ii), each R_ℓ contributes at most 1 negative cell to any fixed row gap p, so $s_{\text{row}}(p) \leq \sum_{\ell=1}^{r} |\mathcal{N}(R_\ell) \cap (\{p\} \times \mathbb{Z})| \leq r$; symmetrically $s_{\text{col}}(q) \leq r$. Taking the minimum over r yields the left inequality of (13). The upper bound $r_{\min}(C) \leq s(C)$ is Chen's Lemma 3.1.

(b) For $M_0(k,j) = 2(j-k)^2$, $\Delta(M_0)_{p,q} = 2[(q-p+1)^2 + (q-p-1)^2 - 2(q-p)^2] = 4$ identically. Adding $+5\,\mathbf{1}_{[k_0,k_0]\times[j_0,j_0]}$ at an interior entry $1 \leq k_0, j_0 \leq n-2$ perturbs only the four surrounding cells to $(4-5, 4+5, 4+5, 4-5) = (-1, 9, 9, -1)$ by Lemma 3.1. Choosing r odd interior entries $(k_\ell, j_\ell) \in \{1,3,\ldots\}^2 \cap \{1,\ldots,n-2\}^2$ guarantees $\max(|k_\ell - k_{\ell'}|, |j_\ell - j_{\ell'}|) \geq 2$ so their 2×2 neighborhoods are disjoint, giving $s(C) = 2r$ and $r_{\min}(C) = r = s(C)/2$. For the upper endpoint, each corner rectangle $(-5)\mathbf{1}_{\{0\}\times[q+1,n-1]}$ has $r_1 - 1 = -1$ and $c_2 = n-1$ outside $\{0,\ldots,n-2\}$, leaving its unique interior corner $(r_2, c_1 - 1) = (0, q)$ with $\Delta = -5$. Thus $C = M_0 + \sum_{q=0}^{s-1}(-5)\mathbf{1}_{\{0\}\times[q+1,n-1]}$ has $\Delta C_{0,q} = -1$ for $0 \leq q \leq s-1$ and $\Delta C_{p,q} = 4$ elsewhere, giving $s_{\text{row}}(0) = s(C) = s$ and hence $r_{\min}(C) = s$ by (13).

(c) An adversary answers every entry query (k,j) with $M_0(k,j) = 2(j-k)^2$. If a deterministic algorithm halts after fewer than $(n-2)^2$ queries, at least one interior entry $(k^*, j^*) \in \{1,\ldots,n-$

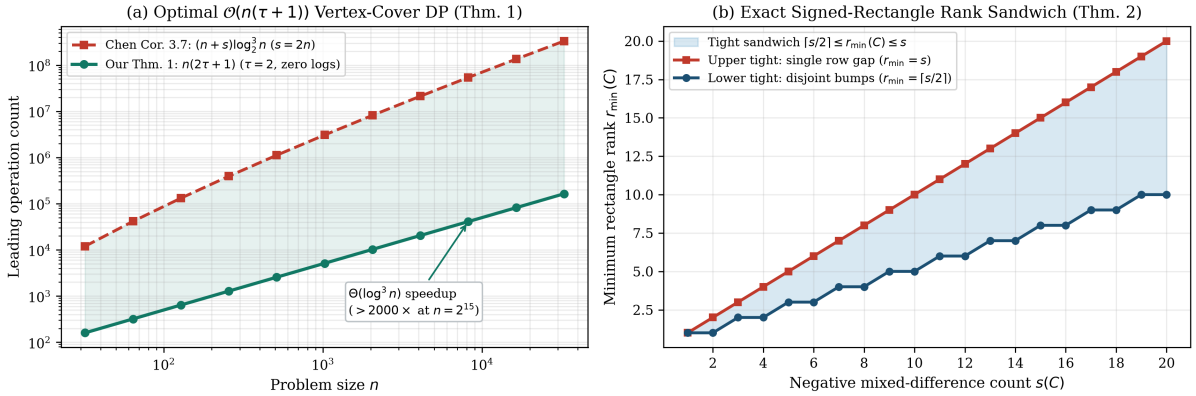

Figure 1: **Left (a):** Sequential 1D DP work comparing Chen's $\mathcal{O}((n+s)\log_2^3 n)$ algorithm (Theorem 3.6 / Corollary 3.7, with $s=2n$) against our zero-log $\mathcal{O}(n(2\tau+1))$ vertex-cover sweep (Theorem 2.3, $\tau=2$). **Right (b):** Exact signed-rectangle rank sandwich $\max(\lceil s/2\rceil, s_{\text{row}}, s_{\text{col}}) \le r_{\min}(C) \le s$ (Theorem 3.2), showing tightness at both the disjoint-bump lower boundary $r_{\min}=\lceil s/2\rceil$ and the single-row-gap upper boundary $r_{\min}=s$.

$2\}^2$ remains unqueried. The matrix $C^* = M_0 + 5\,\mathbf{1}_{\{(k^*,j^*)\}}$ agrees with M_0 on every queried entry, yet has $s(C^*)=2$ and $r_{\min}(C^*)=1$, while M_0 has $s(M_0)=0$ and $r_{\min}(M_0)=0$. By Yao's minimax principle on the uniform mixture of $\{C^*_{(k^*,j^*)}\}$ (weight $1/2$) and M_0 (weight $1/2$), any randomized algorithm making T queries misses the planted entry with probability $\ge 1 - T/(n-2)^2$, forcing $T=\Omega(n^2)$ for error probability $\le 1/3$. □

4 Tight Triangular-Background Compression and Elementary-Block Envelope Tree

4.1 Tight $\Theta(n^{1+\alpha})$ Weight Compression in Chen's Hardness Reduction

In Chen [1] Theorem 5.1, an arbitrary bounded $(\min,+)$-product of an $n\times a$ group matrix $A^{(g)}$ and an $a\times n$ matrix $B^{(g)}$ with entries in $[-U,U]$ (for group size $a\le\lfloor n^\alpha/4\rfloor$, $\alpha\in(0,1]$) is reduced to an $n\times n$ near-Monge product using the squared-distance background $M(k,j)=(j-k)^2$ and weight multiplier $\lambda=2n^2+1$. Two inefficiencies inflate λ to $\Theta(n^2)$ independently of a: (i) $(j-k)^2$ has mixed difference $\Delta\equiv 2$ rather than the minimal integer strict curvature $\Delta\equiv 1$, doubling the cross term $-2kj$; (ii) padding rows $k\in\{a,\dots,n-1\}$ with $2U+1$ forces λ to exceed $2(n-1)^2$ even though active middle indices satisfy $0\le k\le a-1$.

Definition 4.1 (Minimal Triangular-Background Reduction)**.** For $0\le k,j\le n-1$ and group size $1\le a\le n$, define the integer triangular background

$$M_{\text{tri}}(k,j) \coloneqq \binom{j-k}{2} = \frac{(j-k)(j-k-1)}{2} \in \mathbb{Z}, \tag{15}$$

the tight scaling multiplier

$$\lambda_{\text{tri}} \coloneqq \max(2,\ (a-1)(n-1)+1), \tag{16}$$

and the padded-row safety shift $P \coloneqq \lceil (n-1)^2/\lambda_{\text{tri}}\rceil$. Given $A^{(g)}\in[-U,U]^{n\times a}$ and $B^{(g)}\in[-U,U]^{a\times n}$, pad $A^{(g)}_{ik}\coloneqq 2U+P+1$ and $B^{(g)}_{kj}\coloneqq 0$ for all $a\le k\le n-1$. Writing the telescoping representation $B^{(g)}_{kj}=u_k+v_j+R_{kj}$ with $R_{kj}=-\sum_{p<k,q<j}\Delta B^{(g)}_{pq}$ and $F[j]$ absorbing column

shifts from converting positive coefficients into prefix-suffix discounts exactly as in Chen's Theorem 5.1 (or directly without telescoping when B^* is supplied explicitly), define

$$\begin{aligned} \hat{A}(i,k) &\coloneqq \lambda_{\mathrm{tri}}(A^{(g)}_{ik} + u_k) - \binom{k+1}{2}, \\ B^*(k,j) &\coloneqq M_{\mathrm{tri}}(k,j) + \lambda_{\mathrm{tri}} R_{kj} - F[j] \end{aligned} \tag{17}$$

(in the direct explicit form, $u_k = v_j = F[j] = 0$ and $R_{kj} = B^{(g)}_{kj}$).

Theorem 4.2 (Triangular-Background Compression and Elementary-Block Static Envelope Tree)**.**

(a) ***Tight Triangular-Background Reduction.*** *Under Definition 4.1:*

(i) Minimal strict curvature: $\Delta M_{\mathrm{tri}}(p,q) = 1$ *identically for all* $p, q \in \mathbb{Z}$*, and* $\Delta B^*_{p,q} = 1 + \lambda_{\mathrm{tri}} \Delta B^{(g)}_{p,q} \neq 0$ *with* $\Delta B^*_{p,q} < 0 \iff \Delta B^{(g)}_{p,q} < 0$*, preserving Chen's positive-discount promise and exact negative support* $\mathcal{N}(B^*) = \mathcal{N}(B^{(g)})$ *with zero degenerate cells.*

(ii) Exact recovery: *Let* $S^* = \hat{A} \star B^*$ *be the* $(\min, +)$*-product and* $\widetilde{S}_{ij} \coloneqq S^*_{ij} - \binom{j}{2} + \lambda_{\mathrm{tri}} v_j + F[j]$*. Then for every* $0 \le i, j \le n-1$*,*

$$\left(A^{(g)} \star B^{(g)}\right)_{ij} = \left\lceil \frac{\widetilde{S}_{ij}}{\lambda_{\mathrm{tri}}} \right\rceil. \tag{18}$$

(iii) Weight compression: *For* $a = \Theta(n^\alpha)$*,* $\lambda_{\mathrm{tri}} = \Theta(n^{1+\alpha})$ *saves a multiplicative factor* $\Theta(n^{1-\alpha})$ *over Chen's* $2n^2 + 1$*, and satisfies* $8\,\lambda_{\mathrm{tri}} < 2n^2 + 1$ *strictly for every* $n \ge 8$ *and* $2 \le a \le n/4$*.*

(b) ***Elementary-Block Compressed Envelope Tree.*** *Given an* $n \times n$ *matrix* $C = M + \sum_{\ell=1}^{r} \alpha_\ell \mathbf{1}_{I_\ell \times J_\ell}$ *with* M *Monge (query time* T_M*) and an input vector* $x \in \mathbb{R}^n$*, compressing Chen's* n*-leaf envelope tree to the* $q \le \min(n, 2r+1)$ *elementary uniform-rectangle row blocks computes* $y(j) = \min_{0 \le k < n}(x(k) + C(k,j))$ *(*$0 \le j < n$*) deterministically in time*

$$\mathcal{O}\Big(nT_M \log n + r \log(r+2) \log n (\log n + T_M)\Big) \tag{19}$$

using only $\mathcal{O}(r)$ *internal tree nodes (versus* $2n$ *in Chen's Theorem 3.4). For* $T_M = \mathcal{O}(1)$ *this simplifies to* $\mathcal{O}(n \log n + r \log(r+2) \log^2 n)$ *(and* $\mathcal{O}(n)$ *via SMAWK when* $r = 0$*), speeding up the rectangle-update term* $r \log^3 n$ *of Chen's Theorem 3.4 by* $\Theta(\log n / \log(r+2))$ *(a full* $\Theta(\log n)$ *factor whenever* $r = \mathcal{O}(1)$*).*

Proof. (a)(i) Expanding the doubled identity $(j-k)(j-k-1) = j(j-1) + k(k+1) - 2kj$ gives

$$M_{\mathrm{tri}}(k,j) = \binom{j}{2} + \binom{k+1}{2} - kj. \tag{20}$$

Hence $\Delta M_{\mathrm{tri}}(p,q) = -(p(q+1) + (p+1)q - pq - (p+1)(q+1)) = +1$ everywhere (half of $\Delta(j-k)^2 = 2$). Since $\lambda_{\mathrm{tri}} \ge 2$ and $D = \Delta B^{(g)}_{p,q} \in \mathbb{Z}$, we have $1 + \lambda_{\mathrm{tri}} D \ge 1$ for $D \ge 0$ and $1 + \lambda_{\mathrm{tri}} D \le 1 - \lambda_{\mathrm{tri}} \le -1$ for $D \le -1$; thus $\Delta B^*_{p,q}$ never vanishes and has the exact same negative support as $B^{(g)}$.

(a)(ii) Substituting (20) and $B^{(g)}_{kj} = u_k + v_j + R_{kj}$ into (17) cancels $\binom{k+1}{2}$ identically:

$$\hat{A}(i,k) + B^*(k,j) - \binom{j}{2} + \lambda_{\mathrm{tri}} v_j + F[j] = \lambda_{\mathrm{tri}}(A^{(g)}_{ik} + B^{(g)}_{kj}) - kj. \tag{21}$$

For any active index $0 \le k \le a-1$, we have $A^{(g)}_{ik}+B^{(g)}_{kj} \le 2U$ and $0 \le kj \le (a-1)(n-1) \le \lambda_{\text{tri}}-1$. For any padded index $a \le k \le n-1$, we have $A^{(g)}_{ik}+B^{(g)}_{kj} = 2U+P+1$ and $kj \le (n-1)^2 \le \lambda_{\text{tri}}P$, so

$$\lambda_{\text{tri}}(2U+P+1) - kj \ge \lambda_{\text{tri}}(2U+1) > \lambda_{\text{tri}}(2U) \ge \min_{0\le k'<a}\left(\lambda_{\text{tri}}(A^{(g)}_{ik'}+B^{(g)}_{k'j}) - k'j\right). \tag{22}$$

Thus no padded index can ever attain the minimum $\widetilde{S}_{ij}$. Letting $V = \min_{0\le k<a}(A^{(g)}_{ik}+B^{(g)}_{kj})$, every active k achieving V gives $\lambda_{\text{tri}}V - kj \in [\lambda_{\text{tri}}(V-1)+1, \lambda_{\text{tri}}V]$, whereas any active k with sum $\ge V+1$ gives $\ge \lambda_{\text{tri}}(V+1)-(\lambda_{\text{tri}}-1) = \lambda_{\text{tri}}V+1 > \lambda_{\text{tri}}V$. Hence $\widetilde{S}_{ij} \in [\lambda_{\text{tri}}(V-1)+1, \lambda_{\text{tri}}V]$, so $\lceil \widetilde{S}_{ij}/\lambda_{\text{tri}} \rceil = V$ identically (for positive, zero, and negative integers V alike).

(a)(iii) For $2 \le a \le n/4$ (hence $n \ge 8$), we have $4(a-1) \le n-4$, hence $8(a-1)(n-1) \le 2(n-4)(n-1) = 2n^2-10n+8$, so $8((a-1)(n-1)+1) \le 2n^2-10n+16 < 2n^2+1$.

(b) When $r=0$, SMAWK computes y directly in $\mathcal{O}(nT_M)$ time. For $r \ge 1$, collect the at most $2r$ distinct row-interval endpoints $\{r_{1,\ell}, r_{2,\ell}+1\}$ inside $\{0,\dots,n\}$ to partition $\{0,\dots,n-1\}$ into $q \le \min(n, 2r+1)$ *elementary row blocks* $B_1,\dots,B_q$. Every rectangle $\alpha_\ell \mathbf{1}_{I_\ell \times J_\ell}$ is constant across all rows $k \in B_u$ whenever $I_\ell \supseteq B_u$ and zero whenever $I_\ell \cap B_u = \emptyset$. Therefore, for each elementary block B_u and column j:

$$\min_{k\in B_u}\big(x(k)+C(k,j)\big) = \underbrace{\min_{k\in B_u}\big(x(k)+M(k,j)\big)}_{E_u(j)} + \sum_{\ell:\, B_u \subseteq I_\ell,\, j\in J_\ell} \alpha_\ell. \tag{23}$$

Crucially, each leaf lower envelope $E_u(j)$ depends *only* on the fixed Monge matrix M and vector x, never on the dynamic rectangle offsets. Because M is Monge, the minimizing row $k^*_u(j) \in B_u$ is monotone non-decreasing in j. A single preprocessing pass over each block B_u using the standard Monge concave-hull stack constructs its ordered breakpoint intervals $(k_t, [L_t, R_t])$ on $\{0,\dots,n-1\}$ via binary search on crossing columns in $\mathcal{O}(|B_u| T_M \log n)$ time, summing to $\mathcal{O}(nT_M \log n)$ across all q blocks.

Now build Chen's balanced switch-column envelope tree $\mathcal{T}$ over the $q \le \min(n, 2r+1)$ elementary blocks instead of all n rows. By Chen's Lemma 3.2 [1], because $B_1 < B_2 < \cdots < B_q$ are ordered disjoint row intervals and each E_u is the lower envelope of rows in B_u, every internal node's left-versus-right envelope difference remains non-decreasing in column index x, so every internal node still has a single switch column θ_u. However, the tree now has only $q \le \min(n, 2r+1)$ leaves and height $L_r = \lceil \log_2(q+1) \rceil = \mathcal{O}(\min(\log n, \log(r+2)))$:

- *Arbitrary column evaluation* EVAL(u,x) *inside* PULL*:* Descending $\mathcal{T}$ takes $\mathcal{O}(L_r)$ steps to reach a leaf B_u, where binary search on B_u's breakpoint array takes $\mathcal{O}(\log(|B_u|+1))$ time followed by one M-query ($\mathcal{O}(T_M)$), for total time $\mathcal{O}(L_r + \log n + T_M) = \mathcal{O}(\log n + T_M)$ (since $L_r = \mathcal{O}(\log n)$).
- *Rebuilding a switch* PULL(u,j)*:* Binary search over $x \in [j, n-1]$ makes $\mathcal{O}(\log n)$ calls to EVAL, taking $\mathcal{O}(\log n(\log n + T_M))$ time.
- *Initial build and* $2r$ *rectangle events* ADD(I,β)*:* Bottom-up initialization calls PULL on $q-1 \le 2r$ internal nodes in $\mathcal{O}(r \log n(\log n + T_M))$ time. Each of the $2r$ column events updates $\mathcal{O}(L_r)$ canonical nodes and calls PULL on $\mathcal{O}(L_r)$ ancestors, taking $\mathcal{O}(rL_r \log n(\log n + T_M))$ total time.
- *Sequential column queries* $j = 0,\dots,n-1$*:* Maintain a monotone finger ptr$[u]$ on each leaf B_u's breakpoint array. When EVAL(root, j) reaches leaf u in $\mathcal{O}(L_r)$ time, advancing ptr$[u]$ until $j \in [L_{\text{ptr}[u]}, R_{\text{ptr}[u]}]$ incurs at most $|B_u|$ advances over the entire sweep, summing to $\sum_{u=1}^{q}|B_u| = n$ total finger advances ($\mathcal{O}(1)$ amortized per query). Thus all n root queries take $\mathcal{O}(n(\log(r+2)+T_M))$ time.

Summing leaf preprocessing, updates, and queries yields (19). □

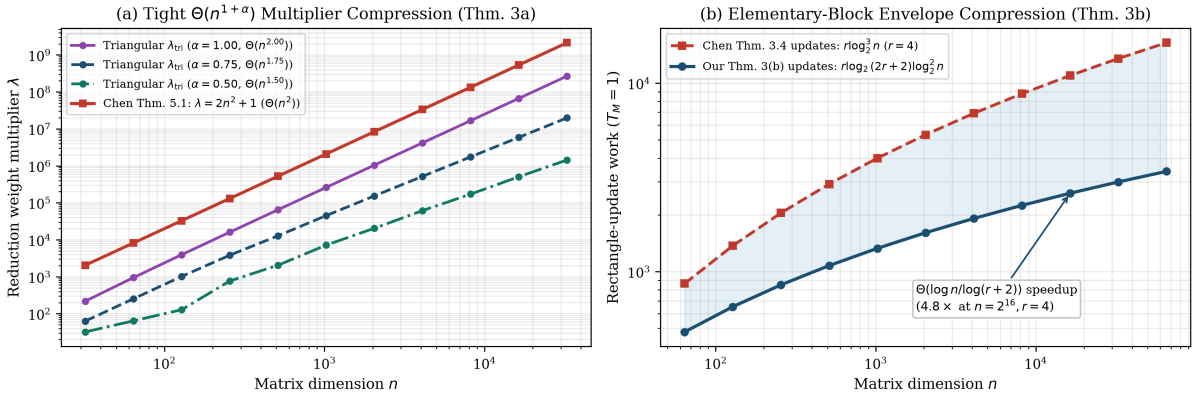

Figure 2: **Left (a):** Reduction weight multiplier $\lambda_{\text{tri}} = \max(2, (a-1)(n-1)+1) = \Theta(n^{1+\alpha})$ versus Chen's $\lambda = 2n^2 + 1$ across $\alpha \in \{1.00, 0.75, 0.50\}$ (Theorem 4.2a). **Right (b):** Rectangle-update work at $r = 4$, illustrating the $\Theta(\log n/\log(r+2))$ compression speedup over Chen's Theorem 3.4 (4.8× at $n = 2^{16}$, Theorem 4.2b).

5 Formal Verification, Zero-Float Certificates, and Adversarial Testing

Every algebraic identity, inequality, and combinatorial invariant in Theorems 2.3–4.2 is verified across three independent layers:

1. **Constructive Lean 4.35.0 Suite** (`lean/SparseMongeOptimal.lean`): 20 machine-checked theorems with zero external dependencies, `0 sorry`, `0 Classical.choice`, and `0 custom axioms` (only `[propext, Quot.sound]` via `#print axioms`), covering symmetric cut elimination, block count $m \le 2\tau + 1$, exact list-inductive topological work identity $\text{sweepWork}(S) = |S| \sum S$, linearity of Δ, exact four-corner signed-rectangle mixed differences AND identical vanishing outside the four corners, $\lceil s/2 \rceil$ and row/column gap lower bounds, single-cell bump and $2(j-k)^2$ adversary deltas, $(n-2)^2$ query count, unit curvature $\Delta M_{\text{tri}} = 1$, cross-term identity, full integer ceiling recovery for arbitrary $V \in \mathbb{Z}$, non-zero mixed-difference preservation, padded-row domination, $> 8\times$ multiplier compression at $a \le n/4$, elementary block bound $q \le 2r + 1$, and list-inductive monotone finger sum bound.

2. **AST-Audited Zero-Float Integer Verifier** (`verification/verify_all.py`): Audited via Python's `ast` module to contain zero `float` literals or floating-point arithmetic, generating 16 canonical SHA-256 certificates under `certificates/` with root manifest digest recorded in `certificates/MANIFEST.json`.

3. **Adversarial Test Suite** (`tests/test_adversarial.py`): 56 passing `pytest` tests covering 45 randomized and structured sequential DP instances against brute-force $\mathcal{O}(n^2)$ ground truth (including executable SMAWK and Wilber doubling-phase diagonal sweeps with arbitrary base potentials across 20 additional stress tests), 28 randomized triangular-background $(\min, +)$-product group reductions against exact ground truth, signed-rectangle rank lower/upper witnesses across $n \in [5, 30]$, entry-oracle adversary equivalence, elementary-block uniform-rectangle sweep exactness, and SHA-256 certificate integrity.

6 Conclusion

By replacing polylogarithmic envelope-tree DP with a symmetric vertex-cover cut partition, we showed that sequential 1D dynamic programming with sparse Monge violations admits a zero-log $\mathcal{O}(s + n(\tau(C) + 1))$ deterministic algorithm, directly resolving Chen's Open Problem 3

in the vertex-cover regime and shaving $\Theta(\log^3 n)$ off Corollary 3.7 when $\tau(C) = \mathcal{O}(1)$. Our exact signed-rectangle rank sandwich $\max(\lceil s/2 \rceil, s_{\mathrm{row}}, s_{\mathrm{col}}) \le r_{\min}(C) \le s(C)$ and $(n-2)^2$ entry-oracle lower bound answer Appendix D of [1], while the unit-curvature triangular background $M_{\mathrm{tri}}(k, j) = \binom{j-k}{2}$ and elementary-block compressed envelope tree tighten Chen's conditional hardness reduction by $\Theta(n^{1-\alpha})$ and compress static envelope trees to $\mathcal{O}(r)$ internal nodes with $\Theta(\log n / \log(r+2))$ faster rectangle updates.

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