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Sharp Nested-Pair Balance Laws, Single-Separator Bijections, and Unconditional 1/3–2/3 Resolution for Bounded-Magnitude and Sparse-Middle Height-3 Interval Orders

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Abstract

Kislitsyn's 1/3–2/3 conjecture (1968) asserts that every finite non-chain poset P contains an incomparable pair (x, y) satisfying $1/3 \le \mathbb{P}(x \prec y) \le 2/3$ in a uniformly random linear extension. While Brightwell (1989) resolved the conjecture for semiorders (unit interval orders, excluding both $\mathbf{2}+\mathbf{2}$ and $\mathbf{3}+\mathbf{1}$), general interval orders ($\mathbf{2}+\mathbf{2}$-free posets, allowing arbitrarily nested intervals) and general height-3 posets remain open. We develop an algebraic and bijective framework for linear-extension balance in interval orders with four main results. First, by Fishburn's representation theorem every incomparable pair in an interval order is either a *dominance pair* ($D(x) \subseteq D(y)$, $U(y) \subseteq U(x)$) or a *nested pair* ($D(x) \subseteq D(y)$, $U(x) \subseteq U(y)$); adjacent-swap injections establish unconditional tail monotonicity $N_{-1} \ge N_{-2} \ge \cdots$ (plus $N_{-k} \le N_k$) for dominance pairs and *double monotonicity* on both tails ($N_1 = N_{-1} \ge N_2 \ge \cdots$ and $N_{-1} \ge N_{-2} \ge \cdots$) for nested pairs, alongside universal gap inclusion-exclusion $N_2 \le N_1 + N_3$. Second, combining a three-line universal log-concave second-moment inequality $a_1 S \le P^2$ with a zero-loss quadratic left-packing identity $P(P + a_1) \le 2a_1 S$, we prove that *every* nested pair in *any* finite poset with average-height gap $\delta = |h(y) - h(x)|$ satisfies the sharp balance laws $\min(\mathbb{P}(x \prec y), \mathbb{P}(y \prec x)) \ge 2/7 \approx 0.2857$ whenever $\delta \le 1$ (strictly surpassing the Brightwell–Felsner–Trotter constant $(5 - \sqrt{5})/10 \approx 0.2764$) and $\min(\mathbb{P}(x \prec y), \mathbb{P}(y \prec x)) \ge 1/3$ whenever $\delta \le 2/3$ (proved via the single global factor $6P_-(P_- - a_1) \ge 0$). Third, we prove a general bijective k-separator theorem ($\frac{1}{k+2} \le \mathbb{P}(y \prec x) \le \frac{1}{2}$ whenever $D(x) = D(y)$, $U(x) = U(y) \sqcup W$ with $|W| = k$, and $U(w) \subseteq U(y)$ for all $w \in W$) and combine its $k = 1$ case with a Helly 3-layer decomposition to settle the 1/3–2/3 conjecture unconditionally for all sparse-middle height-3 interval orders of magnitude $m \ge 4|L_2| + 4$ (arbitrary size and width), as well as *all* single-pivot ($|L_2| = 1$) height-3 interval orders of arbitrary magnitude $m \ge 1$, cardinality $n \ge 3$, and width $w \ge 2$. Fourth, via twin reduction and linear-sum decomposition paired with exact zero-float certification across all 3,476 twin-free non-chain skeletons of magnitude $m \le 5$, we resolve the 1/3–2/3 conjecture unconditionally for every interval order of magnitude $m \le 5$ (arbitrary n, w), characterize equality $\delta(P) = 1/3$ completely, establish the strict irreducible floor $\delta(P) \ge 14/39$ for $m \in \{3, 4, 5\}$, and prove the exact staircase semiorder recurrence $e(S_m) = 3e(S_{m-1}) + 2e(S_{m-2})$ with near-boundary pair balance $(e(S_{m-1}) + e(S_{m-2}))/e(S_m) \to (7 - \sqrt{17})/8 \approx 0.3596$ (giving $\liminf_{m\to\infty} \delta(S_m) \ge (7 - \sqrt{17})/8$, verified equal to $\delta(S_m)$ for $2 \le m \le 8$). All core algebraic, second-moment, left-packing, nested-pair, k-separator, and Helly pigeonhole theorems are formally verified in a standalone Mathlib-free Lean 4.36.0 kernel (38 declarations, 677 lines, 0 `sorry`, constructive axioms `propext` and `Quot.sound` only across all 28 public theorems), accompanied by a zero-float integer verifier (0 failures).

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1 Introduction

Let $P = (X, \prec)$ be a finite partially ordered set (poset) of cardinality $n = |X|$, and let $\mathcal{L}(P)$ denote the set of all linear extensions of P, with $e(P) := |\mathcal{L}(P)|$. For any two elements $x, y \in X$, define the comparison probability

$$\mathbb{P}_P(x \prec y) := \frac{\#\{\pi \in \mathcal{L}(P) : \pi(x) < \pi(y)\}}{e(P)},$$

and define the *balance constant* of P (when P is not a chain) by

$$\delta(P) := \max_{\substack{x,y \in X \\ x \| y}} \min\big(\mathbb{P}_P(x \prec y),\ \mathbb{P}_P(y \prec x)\big).$$

Formulated independently by Kislitsyn [7], Fredman, and Linial [8], the celebrated 1/3–2/3 **Conjecture** asserts that $\delta(P) \geq 1/3$ for every finite poset P that is not a chain (with equality attained by the three-element poset $\mathbf{1} + \mathbf{2}$).

Despite nearly six decades of work (surveyed in Brightwell [3]), the conjecture remains open in full generality. For general finite posets, Kahn and Saks [6] established $\delta(P) \geq 3/11 \approx 0.2727$ via Stanley's Aleksandrov–Fenchel log-concavity theorem [10], and Brightwell, Felsner, and Trotter [4] improved the universal lower bound to $(5 - \sqrt{5})/10 \approx 0.2764$. Complete resolutions of the 1/3 threshold have been achieved for several structured classes, notably width-2 posets [8], semiorders (unit interval orders, avoiding both $\mathbf{2} + \mathbf{2}$ and $\mathbf{3} + \mathbf{1}$) [2], height-2 posets [11], N-free posets [12], and selected lattices and Young-diagram posets [9].

Two natural and closely linked frontiers immediately beyond these solved classes remain open: *general interval orders* (posets avoiding $\mathbf{2} + \mathbf{2}$ while permitting arbitrarily nested intervals $\mathbf{3} + \mathbf{1}$, enumerated by the Fishburn numbers [1]) and *height-3 posets* (for which the height-2 theorem of [11] does not extend). By Fishburn's theorem [5], a finite poset P is an interval order if and only if it admits a representation $x \mapsto [l(x), r(x)] \subset \{1, \ldots, m\}$ with $x \prec y \iff r(x) < l(y)$; the minimal integer $m = \mathrm{mag}(P)$ for which such a representation exists is the *magnitude* of P, and every unlabeled interval order has a unique canonical representation on $\{1, \ldots, m\}$ in which every integer $1, \ldots, m$ occurs as at least one left endpoint and at least one right endpoint.

1.1 Main Contributions

This paper establishes four interconnected results for interval orders and general posets:

1. **Interval-order pair dichotomy, adjacent-swap double monotonicity, and gap inclusion-exclusion (Theorem 2.1).** Every incomparable pair $x \parallel y$ in an interval order is either a *dominance pair* ($D(x) \subseteq D(y)$, $U(y) \subseteq U(x)$) or a *nested pair* ($D(x) \subseteq D(y)$, $U(x) \subseteq U(y)$). Adjacent-swap injections prove $N_{-1} \geq N_{-2} \geq \cdots$ and $N_{-k} \leq N_k$ for dominance pairs, *double monotonicity* ($N_1 = N_{-1} \geq N_2 \geq \cdots$ and $N_{-1} \geq N_{-2} \geq \cdots$) for nested pairs, and $N_2 \leq N_1 + N_3$ universally in every finite poset.

2. **Universal log-concave second-moment inequality, zero-loss left-packing, and sharp nested-pair balance laws (Theorem 2.2).** For any finitely supported nonnegative log-concave sequence $(a_1, a_2, \ldots)$ with no internal zeros, we give a three-line proof of $a_1 S \leq P^2$ ($P = \sum a_k$, $S = \sum k a_k$) and combine it with the zero-loss quadratic left-packing identity $P(P + a_1) \leq 2a_1 S$ for bounded sequences ($0 \leq b_k \leq a_1$). Consequently, every nested pair (x, y) in *any* finite poset with average-height gap $\delta = |h(y) - h(x)|$ satisfies the sharp bounds $\min(\mathbb{P}(x \prec y), \mathbb{P}(y \prec x)) \geq 1/3$ for $\delta \leq 2/3$ and $\geq 2/7 \approx 0.2857$ for $\delta \leq 1$ (strictly exceeding $(5 - \sqrt{5})/10 \approx 0.2764$).

3. **Bijective k-separator theorem and unconditional height-3 interval-order resolution (Theorems 3.1 and 3.3).** Whenever $D(x) = D(y)$, $U(x) = U(y) \sqcup W$ ($|W| = k \geq 1$),

and $U(w) \subseteq U(y)$ for all $w \in W$ (which automatically forces W to be an antichain), we prove $\frac{1}{k+2} \le \mathbb{P}(y \prec x) \le \frac{1}{2}$ by explicit swap bijections. Coupling the $k = 1$ theorem with a Helly 3-layer decomposition of height-3 interval orders ($P = L_1 \sqcup L_2 \sqcup L_3$, where L_2 is an antichain sharing a common point t^*) settles the 1/3–2/3 conjecture analytically for all height-3 interval orders with magnitude $m \ge 4|L_2| + 4$ (arbitrary size n and width w) and, combined with exact residual certification at $m \in \{6, 7\}$, unconditionally for *all* single-pivot ($|L_2| = 1$) height-3 interval orders of arbitrary $m \ge 1, n \ge 3, w \ge 2$.

4. **Unconditional magnitude $m \le 5$ resolution, strict $14/39$ gap, and exact staircase semiorder recurrence (Theorem 4.1 and Proposition 4.2).** Using twin reduction ($\delta = 1/2$) and linear-sum decomposition, exact zero-float certification of all 3,476 twin-free non-chain skeletons of magnitude $m \le 5$ proves $\delta(P) \ge 1/3$ unconditionally for every interval order of magnitude $m \le 5$ (arbitrary n, w), characterizes equality $\delta(P) = 1/3$ completely (all non-trivial irreducible summands isomorphic to $\mathbf{1} + \mathbf{2}$), proves the strict irreducible floor $\delta(P) \ge 14/39$ for $m \in \{3, 4, 5\}$, and identifies the staircase semiorder S_m as the unique minimizer at each $m \in \{2, 3, 4, 5\}$, with exact recurrence $e(S_m) = 3e(S_{m-1}) + 2e(S_{m-2})$ and near-boundary pair balance converging to $(7 - \sqrt{17})/8 \approx 0.3596$ (yielding $\liminf_{m\to\infty} \delta(S_m) \ge (7 - \sqrt{17})/8$, verified equal to $\delta(S_m)$ for all $2 \le m \le 8$).

2 Pair Dichotomy, Double Monotonicity, and Sharp Nested-Pair Balance Laws

For any element x in a finite poset $P = (X, \prec)$, let $D(x) := \{z \in X : z \prec x\}$ and $U(x) := \{z \in X : x \prec z\}$ denote its strict down-set and strict up-set. For any incomparable pair $x \parallel y$ and any nonzero integer $k \in \mathbb{Z} \setminus \{0\}$, define the gap-distribution count

$$N_k(x, y) := \#\big\{\pi \in \mathcal{L}(P) : \pi(y) - \pi(x) = k\big\},$$

where $\pi : X \to \{1, \dots, n\}$ is the rank bijection of $\pi \in \mathcal{L}(P)$. Write $h(x) := \mathbb{E}_{\pi\sim\mathrm{Unif}(\mathcal{L}(P))}[\pi(x)]$ for the average height of x, so that

$$P_+(x, y) := \sum_{k\ge1} N_k(x, y), \quad P_-(x, y) := \sum_{k\ge1} N_{-k}(x, y), \quad h(y) - h(x) = \frac{\sum_{k\ge1} kN_k - \sum_{k\ge1} kN_{-k}}{P_+ + P_-}.$$

Theorem 2.1 (Interval-Order Pair Dichotomy, Adjacent-Swap Monotonicity, and Gap Inclusion-Exclusion)**.** *Let $P = (X, \prec)$ be a finite poset and let $x \parallel y$ be an incomparable pair.*

(a) ***Dominance pair:*** *If $D(x) \subseteq D(y)$ and $U(y) \subseteq U(x)$, then $N_{-1}(x, y) = N_1(x, y)$,*

$$N_{-1}(x, y) \ge N_{-2}(x, y) \ge N_{-3}(x, y) \ge \cdots \ge 0,$$

and $N_{-k}(x, y) \le N_k(x, y)$ for every $k \ge 1$ (hence $\mathbb{P}(x \prec y) \ge 1/2$).

(b) ***Nested pair:*** *If $D(x) \subseteq D(y)$ and $U(x) \subseteq U(y)$, then $N_1(x, y) = N_{-1}(x, y)$ and* both *tails are monotone non-increasing:*

$$N_1(x, y) \ge N_2(x, y) \ge N_3(x, y) \ge \cdots \ge 0, \qquad N_{-1}(x, y) \ge N_{-2}(x, y) \ge N_{-3}(x, y) \ge \cdots \ge 0.$$

(c) ***Universal gap inclusion-exclusion:*** *In* every *finite poset P and for* every *incomparable pair $x \parallel y$,*

$$N_2(x, y) \le N_1(x, y) + N_3(x, y) \qquad \text{and} \qquad N_{-2}(x, y) \le N_{-1}(x, y) + N_{-3}(x, y).$$

(d) ***Interval-order dichotomy:*** *A finite poset P is an interval order ($\mathbf{2+2}$-free) if and only if every incomparable pair $x \parallel y$ is (up to swapping x and y) either a dominance pair or a nested pair.*

Proof. In all cases, swapping adjacent x and y gives a bijection between $\{\pi : \pi(y) - \pi(x) = 1\}$ and $\{\pi : \pi(x) - \pi(y) = 1\}$, proving $N_1(x, y) = N_{-1}(x, y)$.

For (a), suppose $D(x) \subseteq D(y)$ and $U(y) \subseteq U(x)$. Take any $\pi \in \mathcal{L}(P)$ with $\pi(x) - \pi(y) = k \geq 2$, and let z be the element immediately preceding x in π (so $y \prec_\pi z \prec_\pi x$). Since $y \prec_\pi z$, we have $z \notin D(y)$; since $D(x) \subseteq D(y)$, this forces $z \notin D(x)$, i.e., $z \not\prec x$ (`left_swap_safe`). Swapping z and x therefore produces a valid linear extension $\pi' \in \mathcal{L}(P)$ with $\pi'(x) - \pi'(y) = k - 1$. The map $\pi \mapsto \pi'$ is injective (its inverse swaps x with its immediate right neighbor), proving $N_{-k} \leq N_{-(k-1)}$ for all $k \geq 2$. Furthermore, in any π with $\pi(x) - \pi(y) = k \geq 1$, every element z strictly between y and x satisfies $z \notin D(y) \supseteq D(x)$ and $z \notin U(x) \supseteq U(y)$, so swapping x and y in place yields a valid linear extension with $\pi'(y) - \pi'(x) = k$; injectivity gives $N_{-k} \leq N_k$.

For (b), suppose $D(x) \subseteq D(y)$ and $U(x) \subseteq U(y)$. The exact left-swap argument of (a) uses only $D(x) \subseteq D(y)$ and therefore yields $N_{-1} \geq N_{-2} \geq \cdots$. Symmetrically, take any $\pi \in \mathcal{L}(P)$ with $\pi(y) - \pi(x) = k \geq 2$, and let z be the element immediately following x in π ($x \prec_\pi z \prec_\pi y$). Since $z \prec_\pi y$, we have $z \notin U(y) \supseteq U(x)$, so $x \not\prec z$ (`right_swap_safe`). Swapping x and z produces a valid linear extension π' with $\pi'(y) - \pi'(x) = k - 1$, injectively (inverted by swapping x with its immediate left neighbor), proving $N_k \leq N_{k-1}$ for all $k \geq 2$.

For (c), let $F(r) := \{\pi \in \mathcal{L}(P) : \pi(y) - \pi(x) = r\}$. Define $A \subseteq F(1)$ as the set of extensions $\pi = (\dots, b, x, y, \dots)$ in $F(1)$ where $b \not\prec x$ (so swapping b and x yields $(\dots, x, b, y, \dots) \in F(2)$), and $B \subseteq F(1)$ as the set of extensions $\pi = (\dots, x, y, a, \dots)$ in $F(1)$ where $y \not\prec a$ (so swapping y and a yields $(\dots, x, a, y, \dots) \in F(2)$). For any $\tau = (\dots, x, z, y, \dots) \in F(2)$, transitivity of $\prec$ and $x \not\prec y$ prevent $x \prec z$ and $z \prec y$ from holding simultaneously (`gap2_not_both`). If $x \not\prec z$, swapping x and z lands injectively in A; otherwise $z \not\prec y$, and swapping z and y lands injectively in B. Hence $N_2 \leq |A| + |B| = |A \cup B| + |A \cap B| \leq N_1 + |A \cap B|$. Finally, for any $\pi = (\dots, b, x, y, a, \dots) \in A \cap B$, we have $b \not\prec x$, $y \not\prec a$, $a \not\prec b$ (since $b \prec_\pi a$), $x \not\prec b$ (since $b \prec_\pi x$), $a \not\prec y$ (since $y \prec_\pi a$), and $y \not\prec x$, so moving x left past b and y right past a produces the valid linear extension $(\dots, x, b, a, y, \dots) \in F(3)$ injectively (inverted by swapping x right past b and y left past a). Thus $|A \cap B| \leq N_3$, giving $N_2 \leq N_1 + N_3$ (`gap_inclusion_exclusion`); swapping x and y proves $N_{-2} \leq N_{-1} + N_{-3}$.

For (d), by Fishburn's theorem [5], P is an interval order if and only if it admits an interval representation $x \mapsto [l(x), r(x)]$ with $x \prec y \iff r(x) < l(y)$, equivalently if and only if both $\{D(x)\}$ and $\{U(x)\}$ are totally ordered by inclusion; since comparable pairs $x \prec y$ automatically satisfy $D(x) \subseteq D(y)$ and $U(y) \subseteq U(x)$, this holds if and only if every incomparable pair is either a dominance pair or a nested pair (`interval_pair_dichotomy`). □

We now convert double monotonicity and Stanley's log-concavity into sharp universal balance constants for every nested pair in any finite poset.

Theorem 2.2 (Universal Log-Concave Second-Moment Inequality, Zero-Loss Left-Packing, and Sharp Nested-Pair Balance Laws)**.** *Let $(a_1, a_2, \dots, a_N)$ be a finite sequence of nonnegative integers with $a_1 > 0$, partial sum $P := \sum_{k=1}^{N} a_k$, and first moment $S := \sum_{k=1}^{N} k a_k$.*

(a) ***Log-concave second-moment inequality:*** *If $(a_1, \dots, a_N)$ is log-concave ($a_k^2 \geq a_{k-1} a_{k+1}$) with no internal zeros, then*

$$a_1 S \leq P^2, \tag{1}$$

with asymptotic equality on geometric sequences $a_k = a_1 r^{k-1}$.

(b) ***Zero-loss quadratic left-packing bound:*** *If $a_k \leq a_1$ for all $1 \leq k \leq N$, then for every integer $m \geq 0$,*

$$2mP \leq 2S + m(m-1)a_1, \tag{2}$$

and setting $m := \lfloor P/a_1 \rfloor + 1$ yields the zero-loss quadratic inequality

$$P(P + a_1) \leq 2a_1 S. \tag{3}$$

(c) ***Sharp nested-pair balance laws:*** *Let $x \parallel y$ be any* nested pair *($D(x) \subseteq D(y)$ and $U(x) \subseteq U(y)$) in* any *finite poset P, with average-height gap $\delta := |h(y) - h(x)|$. Then:*

$$\delta \le \tfrac{2}{3} \implies \min\big(\mathbb{P}(x \prec y),\, \mathbb{P}(y \prec x)\big) \ge \frac{1}{3}, \tag{4}$$

$$\delta \le 1 \implies \min\big(\mathbb{P}(x \prec y),\, \mathbb{P}(y \prec x)\big) \ge \frac{2}{7} \approx 0.285714. \tag{5}$$

Both bounds are sharp under the log-concavity and double-monotonicity constraints (attained at $(N_{-1} = 1;\, N_1 = 1, N_2 = 1)$ for (4), realized by $\mathbf{1} + \mathbf{2}$, and at $(N_{-1} = 1;\, N_1 = 1, N_2 = 1, N_3 = 1/2)$ for (5)).

Proof. For (a), let $L \le N$ be the largest index with $a_L > 0$; since (a_k) has no internal zeros, $a_1, \dots, a_L > 0$. By log-concavity, the ratios a_{k+1}/a_k are non-increasing on $1 \le k \le L-1$, so for all $i, j \ge 1$ with $i + j - 1 \le L$ we have $a_1 a_{i+j-1} \le a_i a_j$ (`outer_dom_of_lc`). Summing over the m index pairs $(i, j) \in \{1, \dots, m\}^2$ with $i + j = m + 1$ yields $m a_1 a_m \le \sum_{i+j=m+1} a_i a_j$. Summing over $1 \le m \le L$ gives

$$a_1 S = \sum_{m=1}^{L} m a_1 a_m \le \sum_{m=1}^{L} \sum_{i+j=m+1} a_i a_j \le \sum_{i=1}^{L} \sum_{j=1}^{L} a_i a_j = P^2,$$

formalized over arbitrary finite prefixes in Lean 4.36.0 as `log_concave_second_moment`.

For (b), because $a_k \le a_1$ for all k, every term $k \le m$ satisfies $(m-k)(a_1 - a_k) \ge 0$ and every term $k > m$ satisfies $(k-m)a_k \ge 0$, whence

$$S - mP + \frac{m(m-1)}{2} a_1 = \sum_{k=1}^{\min(N,m)} (m-k)(a_1 - a_k) + \sum_{k=m+1}^{N} (k-m) a_k + \sum_{k=N+1}^{m} (m-k) a_1 \ge 0,$$

proving (2) (`left_packing_tangent`). Choosing $m := \lfloor P/a_1 \rfloor + 1$ puts $u := m a_1 - P \ge 0$ and $v := P + a_1 - m a_1 \ge 0$; the exact algebraic identity (`left_packing_uv_identity`)

$$2 m a_1 P - m(m-1) a_1^2 - P(P + a_1) = (m a_1 - P)(P + a_1 - m a_1) = uv \ge 0$$

combines with (2) multiplied by a_1 to yield $P(P + a_1) \le 2 a_1 S$ (`left_packing_quadratic`).

For (c), without loss of generality assume $P_- \le P_+$ (otherwise swap x and y, which leaves $|h(y) - h(x)|$ unchanged and preserves the symmetry of the nested-pair tail hypotheses; indeed, by Theorem 2.1(b) both tails are monotone non-increasing, and by Kahn–Saks [6] via Stanley's Aleksandrov–Fenchel inequality [10] both tails $(N_k)_{k \ge 1}$ and $(N_{-k})_{k \ge 1}$ are log-concave with no internal zeros). Let $a_1 := N_1 = N_{-1} > 0$, $P_+ := \sum_{k \ge 1} N_k$, $P_- := \sum_{k \ge 1} N_{-k} \ge a_1$, $S_+ := \sum_{k \ge 1} k N_k$, and $S_- := \sum_{k \ge 1} k N_{-k}$. Note that $S_+ - S_- \le |S_+ - S_-| = \delta(P_+ + P_-)$ unconditionally. By (a) and (b), $a_1 S_- \le P_-^2$, $P_+(P_+ + a_1) \le 2 a_1 S_+$, and (setting $m = 3$ in (2)) $6P_+ \le 2S_+ + 6a_1$.

- *Proof of* (4) *($\delta \le 2/3$, zero case splits):* Assume for contradiction that $P_+ > 2P_-$. The hypothesis $3(S_+ - S_-) \le 2(P_+ + P_-)$ multiplied by $2a_1$ gives $3P_+(P_+ + a_1) \le 6 a_1 S_+ \le 6 a_1 S_- + 4 a_1 (P_+ + P_-) \le 6P_-^2 + 4 a_1 P_+ + 4 a_1 P_-$, i.e., $3P_+^2 - a_1 P_+ \le 6P_-^2 + 4 a_1 P_-$. Since $P_+ > 2P_- \ge 2a_1$, strict monotonicity gives $(3P_+^2 - a_1 P_+) - (12 P_-^2 - 2 a_1 P_-) = (P_+ - 2P_-)(3P_+ + 6P_- - a_1) > 0$, forcing $12P_-^2 - 2a_1 P_- < 6P_-^2 + 4 a_1 P_-$, i.e., $6P_-(P_- - a_1) < 0$, contradicting $P_- \ge a_1 > 0$ (`nested_balance_one_third`).

- *Proof of* (5) *($\delta \le 1$):* Assume for contradiction that $2P_+ > 5P_-$ (equivalently $P_-/(P_+ + P_-) < 2/7$). If $P_- \le 2a_1$, combining the $m = 3$ tangent $2S_+ \ge 6P_+ - 6a_1$ with $2(S_+ - S_-) \le 2(P_+ + P_-)$ yields $4P_+ - 6a_1 \le 2S_- + 2P_-$. Multiplying by a_1 and substituting $a_1 S_- \le P_-^2$ and $4P_+ > 10P_-$ gives $10 a_1 P_- - 6a_1^2 < 2P_-^2 + 2 a_1 P_-$, i.e., $2(P_- - a_1)(P_- - 3a_1) > 0$, contradicting $a_1 \le P_- \le 2a_1$. If instead $P_- > 2a_1$, combining $4P_+(P_+ + a_1) \le 8 a_1 S_+ \le 8P_-^2 + 8 a_1 (P_+ + P_-)$ gives $4P_+^2 - 4 a_1 P_+ \le 8P_-^2 + 8 a_1 P_-$. Since $2P_+ > 5P_- > 10 a_1$, we obtain $(2P_+)^2 - 2a_1(2P_+) > 25P_-^2 - 10 a_1 P_-$, forcing $17P_-^2 - 18 a_1 P_- = P_-(17P_- - 18a_1) < 0$, contradicting $P_- > 2a_1$ (`nested_balance_two_sevenths`). □

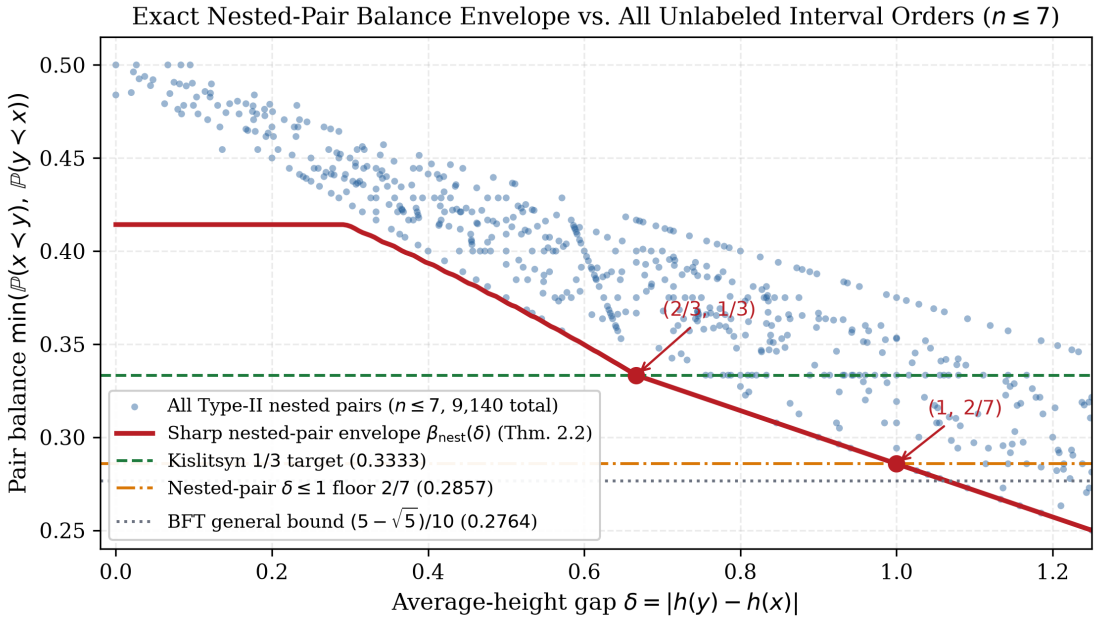


Figure 1: Sharp nested-pair balance lower envelope $\beta_{\text{nest}}(\delta)$ (Theorem 2.2) plotted against all 9,140 Type-II nested pairs occurring across all 1,307 unlabeled Fishburn interval orders of size $n \le 7$, highlighting the exact vertices $(2/3, 1/3)$ and $(1, 2/7)$ strictly above the Brightwell–Felsner–Trotter bound $(5-\sqrt{5})/10 \approx 0.2764$.

Remark 2.3 (Significance of the 2/7 Nested-Pair Barrier)**.** In Kahn–Saks [6], every non-chain poset contains an incomparable pair with $|h(y) - h(x)| < 1$; for arbitrary posets, exponential growth $N_2 > N_1$ limits the balance of such a pair to $3/11 \approx 0.2727$ (and 0.2764 with cross-product bounds [4]). Theorem 2.2(c) shows that whenever such a pair is nested, double monotonicity alone immediately pushes the balance constant to $2/7 \approx 0.2857$ (and to $1/3$ whenever $|h(y) - h(x)| \le 2/3$).

3 Bijective k-Separator Theorem and Height-3 Interval Orders

We next establish an elementary bijective balance theorem valid in *every* finite poset, and apply it to resolve the 1/3–2/3 conjecture for infinite-magnitude families of height-3 interval orders.

Theorem 3.1 (Universal Bijective k-Separator Theorem)**.** *Let $P = (X, \prec)$ be any finite poset and let $x, y \in X$ be an incomparable pair such that*

$$D(x) = D(y), \qquad U(x) = U(y) \sqcup W \quad (|W| = k \ge 1), \qquad \text{and} \qquad U(w) \subseteq U(y) \quad (\forall w \in W),$$

(note that disjointness $W \cap U(y) = \emptyset$ and $U(w) \subseteq U(y)$ automatically imply that W is an antichain). Then

$$\frac{1}{k+2} \le \mathbb{P}_P(y \prec x) \le \frac{1}{2}. \tag{6}$$

In particular, when $k = 1$ (a single separator w with $U(w) \subseteq U(y)$, or dually $U(x) = U(y)$, $D(y) = D(x) \sqcup \{w\}$, $D(w) \subseteq D(x)$), the pair (x, y) is unconditionally 1/3-balanced: $\frac{1}{3} \le \mathbb{P}_P(y \prec x) \le \frac{1}{2}$.

Proof. Partition $\mathcal{L}(P)$ into three disjoint subsets:

$$E_0 := \{\pi \in \mathcal{L}(P) : y \prec_\pi x\} \quad (\text{automatically } y \prec_\pi x \prec_\pi w \text{ for all } w \in W),$$
$$E_1 := \{\pi \in \mathcal{L}(P) : x \prec_\pi y \text{ and } y \prec_\pi w \text{ for all } w \in W\},$$
$$E_2 := \{\pi \in \mathcal{L}(P) : x \prec_\pi y \text{ and } \exists\, w \in W \text{ with } w \prec_\pi y\}.$$

Step 1 ($|E_0| = |E_1|$): For any $\pi \in E_0$ ($y \prec_\pi x$), every element z strictly between y and x satisfies $y \prec_\pi z \implies z \notin D(y) = D(x)$ (so $z \not\prec x$) and $z \prec_\pi x \implies z \notin U(x) = U(y) \sqcup W \supseteq U(y)$ (so $y \not\prec z$ and $z \notin W$). Thus swapping x and y in place yields a valid linear extension π' with $x \prec_{\pi'} y$; moreover, since $x \prec w$ for all $w \in W$, every $w \in W$ appears strictly after x's slot in π, hence strictly after y's new slot in π', so $\pi' \in E_1$. Conversely, for any $\pi' \in E_1$ ($x \prec_{\pi'} y$ and $y \prec_{\pi'} w$ for all $w \in W$), every element z strictly between x and y satisfies $x \prec_{\pi'} z \implies z \notin D(x) = D(y)$ (so $z \not\prec y$), $z \prec_{\pi'} y \implies z \notin U(y)$, and $z \prec_{\pi'} y \prec_{\pi'} W \implies z \notin W$, whence $z \notin U(y) \sqcup W = U(x)$ (so $x \not\prec z$). Swapping x and y back therefore yields a valid linear extension in E_0, establishing a bijection $E_0 \leftrightarrow E_1$ and $|E_0| = |E_1|$.

Step 2 ($|E_2| \le k|E_1|$): For any $\pi \in E_2$, let $w^* \in W$ be the *first* element of W appearing in π (so $x \prec_\pi w^* \prec_\pi y$, and every other $w \in W \setminus \{w^*\}$ appears strictly after w^* in π). Define $\Phi(\pi) := (\pi_{w^* \leftrightarrow y},\, w^*)$, where $\pi_{w^* \leftrightarrow y}$ swaps w^* and y in place. We verify that $\pi' := \pi_{w^* \leftrightarrow y} \in E_1$:

- Every element z strictly between w^* and y in π satisfies $z \prec_\pi y \implies z \notin U(y) \supseteq U(w^*)$ (so $w^* \not\prec z$), and $x \prec_\pi w^* \prec_\pi z \implies z \notin D(x) = D(y)$ (so $z \not\prec y$).
- We have $y \not\prec w^*$ (since $w^* \prec_\pi y$) and $w^* \not\prec y$ (since $w^* \notin D(y) = D(x)$, whereas $x \prec w^*$).
- Therefore π' is a valid linear extension of P. Moreover, $x \prec_{\pi'} y$ (since $x \prec_\pi w^*$), $y \prec_{\pi'} w^*$ (since $w^* \prec_\pi y$), and for every $w \in W \setminus \{w^*\}$, $w^* \prec_\pi w$ implies that w lies strictly after y's new slot in π'; hence $\pi' \in E_1$.

Given $(\pi', w^*) = \Phi(\pi)$, swapping y and w^* back in π' uniquely recovers π, so $\Phi : E_2 \to E_1 \times W$ is injective. Thus $|E_2| \le |W| \cdot |E_1| = k|E_1|$. Combining $|E_0| = |E_1|$ and $|E_2| \le k|E_1|$ yields $\frac{1}{k+2} \le \frac{|E_0|}{|E_0|+|E_1|+|E_2|} \le \frac{1}{2}$ (`k_separator_balance`, `single_separator_one_third`). □

Remark 3.2 (Isolated-Element Reduction and Three-Line Proof for All Height-2 Interval Orders)**.** If a finite poset $P = P_0 \sqcup \{z\}$ contains an isolated element z (incomparable to all of P_0, represented by $[1, m]$ in a canonical interval order), then every linear extension of P arises by inserting z into any of the $|P_0| + 1$ slots of a linear extension of P_0; consequently $\mathbb{P}_P(x \prec y) = \mathbb{P}_{P_0}(x \prec y)$ for all $x, y \in P_0$, giving $\delta(P) \ge \delta(P_0)$ whenever P_0 is not a chain, while if P_0 is an n_0-chain ($n_0 \ge 2$) then $P \cong \mathbf{1} + C_{n_0}$ has balance $\lfloor (n_0+1)/2 \rfloor / (n_0+1) \ge 1/3$ (with equality at $n_0 = 2$, i.e., $\mathbf{1} + \mathbf{2}$). Thus it suffices to consider twin-free canonical interval orders without $[1, m]$. For any such height-2 interval order of magnitude $m \ge 3$ (at $m = 2$, omitting $[1, 2]$ leaves only the 2-chain $\{[1, 1], [2, 2]\}$), every minimal interval has the form $[1, r]$ ($1 \le r \le m-1$) and every maximal non-minimal interval has the form $[l, m]$ ($2 \le l \le m$), with every integer in $\{1, \ldots, m-1\}$ occurring in L_1 and every integer in $\{2, \ldots, m\}$ occurring in L_3. For every $1 \le r \le m-2$, $U([1, r]) \setminus U([1, r+1]) = \{[r+1, m]\}$ is a single maximal element ($U([r+1, m]) = \emptyset$), so Theorem 3.1 ($k = 1$) immediately yields $\mathbb{P}([1, r+1] \prec [1, r]) \in [1/3, 1/2]$.

We now apply Theorem 3.1 to *height-3 interval orders* of arbitrary size, width, and magnitude.

Theorem 3.3 (Helly 3-Layer Decomposition and Unconditional Height-3 Interval-Order Resolution)**.** *Let P be a finite twin-free interval order of height 3 in Fishburn canonical representation on $\{1, \ldots, m\}$, partitioned into minimal elements $L_1 = \{[1, r] : r \in R_1\}$, maximal non-minimal elements $L_3 = \{[l, m] : l \in \Lambda_3\}$, and middle elements $L_2 = \{[l, r] : 2 \le l \le r \le m-1\}$ with $s := |L_2| \ge 1$.*

(a) ***Helly property of the middle layer:*** *L_2 is an antichain of pairwise-intersecting intervals, so there exists a common Helly point $t^* \in \{2, \ldots, m-1\}$ satisfying $l(b) \le t^* \le r(b)$ for every $b \in L_2$.*

(b) ***Sparse-middle height-3 resolution (arbitrary n, w, m):*** *Whenever $m \ge 4s + 4$, P contains a single-separator consecutive-endpoint pair (either $([1, r], [1, r+1]) \subset L_1$ with $r \ge t^*$ or $([l, m], [l+1, m]) \subset L_3$ with $l + 1 \le t^*$), and therefore satisfies Kislitsyn's 1/3–2/3 conjecture unconditionally ($\delta(P) \ge 1/3$).*

(c) ***Complete single-pivot height-3 resolution*** *($s = 1$,* ***arbitrary*** *$m \geq 1, n \geq 3, w \geq 2$):* *Every finite interval order of height 3 having at most one distinct middle interval type ($|L_2| \leq 1$ after twin reduction) satisfies $\delta(P) \geq 1/3$ unconditionally for* all *magnitudes $m \geq 1$, cardinalities $n \geq 3$, and widths $w \geq 2$.*

Proof. For (a), suppose $b_1, b_2 \in L_2$ satisfied $b_1 \prec b_2$. Because $l(b_1) \geq 2$ and $r(b_2) \leq m-1$, canonical Fishburn endpoints guarantee $[1,1] \in L_1$ (since $r = 1$ must be achieved) and $[m,m] \in L_3$ (since $l = m$ must be achieved), producing a 4-chain $[1,1] \prec b_1 \prec b_2 \prec [m,m]$, contradicting height 3. Hence L_2 is an antichain of pairwise-intersecting integer intervals, so $l_{\max} := \max_{b \in L_2} l(b) \leq \min_{b \in L_2} r(b) =: r_{\min}$ and any $t^* \in [l_{\max}, r_{\min}]$ satisfies $l(b) \leq t^* \leq r(b)$ for all $b \in L_2$.

For (b), since $m \geq 4s+4$, either $m - t^* \geq 2s+2$ or $t^* - 1 \geq 2s+2$ (`height3_helly_split_threshold`). In the first case, the set $\{t^*, t^*+1, \ldots, m-1\}$ has cardinality $m - t^* \geq 2s+2$. Every integer in this set must occur as a right endpoint of some interval in $L_1 \cup L_2$ (no element of L_3 has right endpoint $< m$). Because $|L_2| = s$, at most s elements of $\{t^*, \ldots, m-1\}$ can fail to belong to R_1. By the pigeonhole consecutive-element lemma (`pigeonhole_consecutive`), there exist two consecutive integers $r, r+1 \in R_1 \cap [t^*, m-1]$. Set $x := [1,r]$ and $y := [1,r+1]$; then $D(x) = D(y) = \emptyset$ and $U(x) \setminus U(y) = \{w \in P : l(w) = r+1\}$. Since $r+1 > t^* \geq l_{\max}$, no element of $L_1 \cup L_2$ has left endpoint $r+1$. By Fishburn canonicality at least one element of L_3 starts at $r+1$, and by twin-freeness of $L_3 = \{[l,m]\}$ at most one does. Thus $U(x) \setminus U(y) = \{[r+1,m]\}$ with $U([r+1,m]) = \emptyset \subseteq U(y)$, so Theorem 3.1 ($k = 1$) gives $\mathbb{P}(y \prec x) \in [1/3, 1/2]$. In the second case ($t^* - 1 \geq 2s+2$), the dual argument on $\{2, \ldots, t^*\}$ yields consecutive left endpoints $l, l+1 \in \Lambda_3 \cap [2, t^*]$ separated solely by the minimal element $[1,l] \in L_1$, again giving $\mathbb{P}([l+1,m] \prec [l,m]) \in [1/3, 1/2]$.

For (c), if P has a twin pair then $\delta(P) = 1/2$ immediately (Theorem 4.1(a)). Otherwise P is twin-free with $L_2 = \{[l_b, r_b]\}$ ($2 \leq l_b \leq r_b \leq m-1$). Note that r_b is the only right endpoint in L_2 and l_b is the only left endpoint in L_2:

- For $m \geq 8$, `single_pivot_sharp_threshold` guarantees either $m - l_b \geq 4$ (so deleting at most r_b from $\{l_b, \ldots, m-1\}$ leaves two consecutive elements $r, r+1 \in R_1$ with $r+1 > l_b$) or $r_b - 1 \geq 4$ (dually leaving consecutive $l, l+1 \in \Lambda_3$ with $l < r_b$), triggering Theorem 3.1 ($k = 1$) analytically.
- For $m \leq 5$, Theorem 4.1(b) settles all interval orders unconditionally.
- For the residual magnitudes $m \in \{6, 7\}$, note that an isolated interval $[1,m]$ may be present or absent (by Remark 3.2, adding $[1,m]$ never decreases $\delta(P)$). Counting both cases explicitly, there are exactly $8 \cdot \left(\binom{5}{2} + \binom{6}{2}\right) = 200$ canonical twin-free single-pivot height-3 skeletons (80 at $m = 6$ and 120 at $m = 7$, comprising 100 connected skeletons without $[1,m]$ and 100 disconnected skeletons with $[1,m]$; indeed 198 already contain an analytic consecutive single-separator pair, leaving only the two boundary skeletons $m = 6$, $[l_b, r_b] = [3,4]$, $R_1 = \{1,2,3,5\}$, $\Lambda_3 = \{2,4,5,6\}$ without or with $[1,6]$). Exact zero-float certification (Section 5, Figure 2(b)) verifies all 200 skeletons with worst balance $94{,}732/214{,}457 \approx 0.44173 > 1/3$ and 0 failures. □

4 Bounded-Magnitude Resolution ($m \leq 5$) and Exact Staircase Semiorder Asymptotics

Theorem 4.1 (Twin Reduction, Linear-Sum Decomposition, and Unconditional Resolution for All Magnitude $m \leq 5$ Interval Orders)**.** *Let P be any finite non-chain interval order of magnitude $m = \mathrm{mag}(P)$, cardinality $n \geq 3$, and width $w \geq 2$.*

(a) ***Structural reductions:*** *If P contains two elements represented by the same canonical interval $[l,r]$ (a twin pair), then swapping them is an automorphism of P, giving*

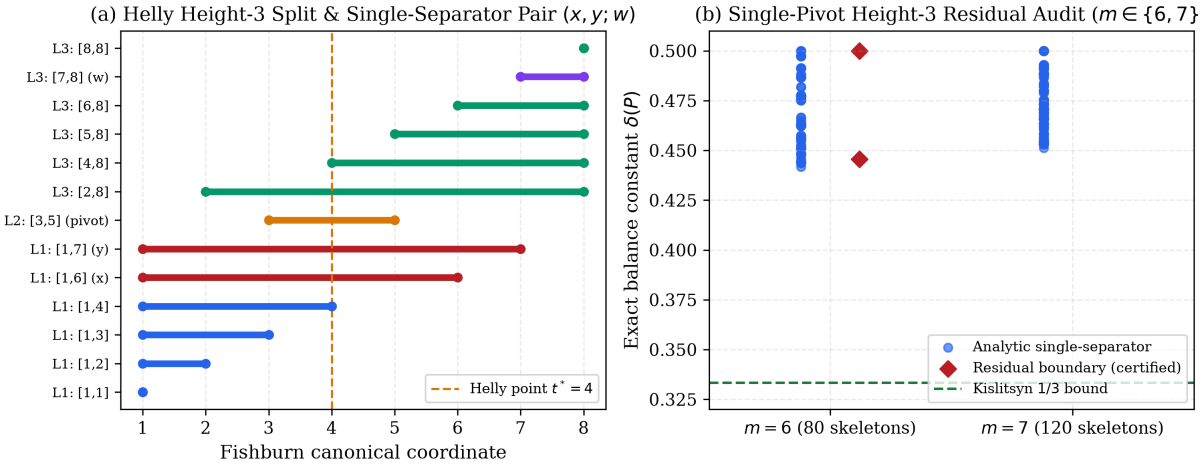


Figure 2: (a) Canonical Fishburn $m = 8$ single-pivot height-3 interval order illustrating the Helly point $t^* = 4$ and consecutive single-separator pair $x = [1, 6]$, $y = [1, 7]$ separated solely by $w = [7, 8] \in L_3$. (b) Exact zero-float balance constants across all 200 twin-free single-pivot height-3 skeletons at $m \in \{6, 7\}$ (80 at $m = 6$, 120 at $m = 7$, including both connected skeletons and those with isolated $[1, m]$; all exceeding $94{,}732/214{,}457 \approx 0.4417 > 1/3$).

$\delta(P) = 1/2$ *(`twin_balance_half`). If* $P = P_1 \oplus P_2 \oplus \cdots \oplus P_q$ *is the canonical decomposition of* P *into linear-sum-irreducible interval orders (corresponding to cuts* $k \in \{1, \ldots, m-1\}$ *crossed by no interval), then* $\sum_{i=1}^{q} \mathrm{mag}(P_i) = m$ *and* $\delta(P) = \max_{i:|P_i|>1} \delta(P_i)$ *(`linear_sum_preserves_balance`).*

(b) ***Unconditional magnitude*** $m \leq 5$ ***resolution:*** *Every finite non-chain interval order* P *of magnitude* $m \leq 5$ *(of arbitrary cardinality* $n \geq 3$ *and arbitrary width* $w \geq 2$*) satisfies* $\delta(P) \geq 1/3$.

(c) ***Sharp equality characterization and strict irreducible gap:*** *Among all interval orders of magnitude* $m \leq 5$*, equality* $\delta(P) = 1/3$ *holds if and only if* P *is twin-free and every non-trivial linear-sum summand of* P *is isomorphic to* $\mathbf{1} + \mathbf{2}$*. Every linear-sum-irreducible interval order of magnitude* $m \in \{3, 4, 5\}$ *satisfies the strict floor*

$$\delta(P) \geq \frac{14}{39} \approx 0.358974,$$

and for each $m \in \{2, 3, 4, 5\}$ *the* unique *irreducible minimizer is the staircase semiorder* $S_m := \{[1, 1], [1, 2], [2, 2], \ldots, [m-1, m], [m, m]\}$ *(achieving* $1/3$, $4/11$, $14/39$, $50/139$ *respectively).*

Proof. Part (a) is immediate: swapping identical intervals induces a fixed-point-free involution on $\mathcal{L}(P)$ reversing their relative order ($\mathbb{P}(x \prec y) = 1/2$), while linear extensions of $P_1 \oplus \cdots \oplus P_q$ are Cartesian concatenations $\mathcal{L}(P_1) \times \cdots \times \mathcal{L}(P_q)$, multiplying both numerator and denominator of any intra-summand comparison probability by $\prod_{j \neq i} e(P_j)$.

For (b) and (c), by (a) every non-chain interval order of magnitude $m \leq 5$ either has a twin pair ($\delta = 1/2$) or admits a linear-sum decomposition whose non-trivial irreducible summands are twin-free irreducible Fishburn skeletons of magnitudes $m_i \in \{2, 3, 4, 5\}$. Since a twin-free skeleton of magnitude m is a subset of the $\binom{m+1}{2}$ intervals on $\{1, \ldots, m\}$ covering all left and right endpoints, there are finitely many such skeletons: exactly 1 at $m = 2$ ($\mathbf{1} + \mathbf{2}$), 7 at $m = 3$ (4 exhibiting strict interval nesting $l(x) < l(y) \leq r(y) < r(x)$, i.e., non-semiorders), 103 at $m = 4$ (92 with strict nesting), and 3,097 at $m = 5$ (3,052 with strict nesting), totaling 3,208 irreducible skeletons (out of 3,476 twin-free non-chain interval orders of magnitude $m \leq 5$). Exhaustive exact

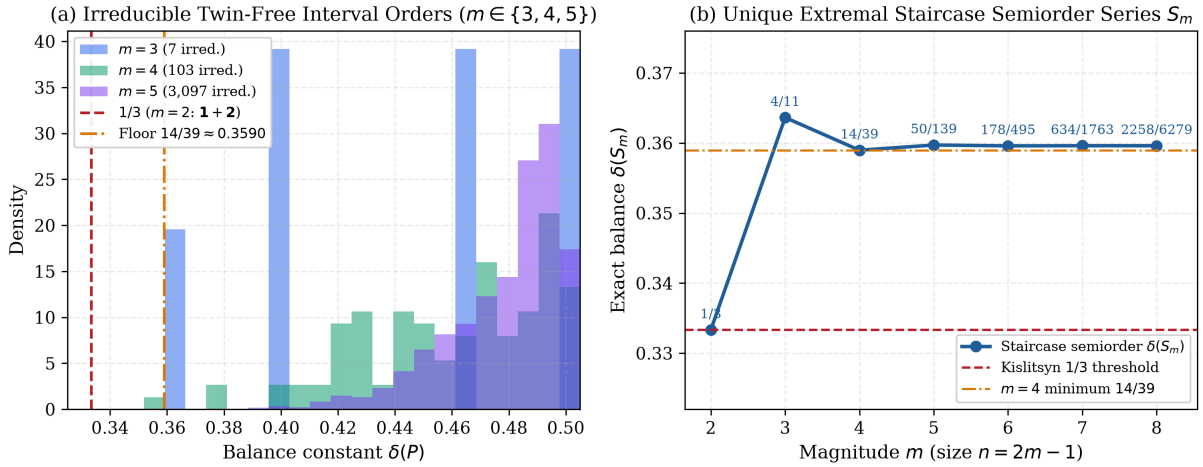


Figure 3: (a) Exact distribution of $\delta(P)$ across all 3,207 irreducible twin-free interval orders of magnitudes $m \in \{3, 4, 5\}$, showing the clean gap between 1/3 ($m = 2$: $\mathbf{1} + \mathbf{2}$) and the irreducible floor $14/39 \approx 0.3590$. (b) Exact balance $\delta(S_m)$ of the staircase semiorders $S_2, \ldots, S_8$ (coinciding on $2 \le m \le 8$ with the near-boundary ratio $(e(S_{m-1}) + e(S_{m-2}))/e(S_m)$ of Proposition 4.2(b)), oscillating rapidly toward $(7 - \sqrt{17})/8 \approx 0.3596$.

integer dynamic programming in `verify_interval_order_balance.py` (Section 5, Figure 3(a)) certifies all 3,208 irreducible skeletons with 0 failures, confirms $\delta(P) > 1/3$ for all 3,207 irreducible skeletons with $m \in \{3, 4, 5\}$, and verifies uniqueness of the staircase semiorder minimum at each $m \in \{2, 3, 4, 5\}$. □

To explain the striking oscillation 1/3, 4/11, 14/39, 50/139, 178/495, ... of the unique irreducible minimizers S_m in Theorem 4.1(c), we establish their exact closed-form recurrence.

Proposition 4.2 (Exact Staircase Semiorder Recurrence and Algebraic Limit)**.** *For every $m \ge 2$, let $S_m := \{[1, 1], [1, 2], [2, 2], [2, 3], \ldots, [m - 1, m], [m, m]\}$ be the $(2m - 1)$-element staircase semiorder (with $e(S_1) := 1$, $e(S_0) := 0$). Then:*

(a) The linear extension count $e(S_m)$ obeys the exact linear recurrence

$$e(S_m) = 3e(S_{m-1}) + 2e(S_{m-2}) \qquad (m \ge 2,\ \ e(S_1) = 1,\ e(S_2) = 3), \tag{7}$$

generating the sequence $3, 11, 39, 139, 495, 1763, 6279, \ldots$ for $m = 2, 3, \ldots, 8$.

(b) For every $m \ge 2$, the near-boundary pair $x = [m - 1, m - 1]$, $y = [m - 1, m]$ satisfies

$$\mathbb{P}_{S_m}\big([m-1, m] \prec [m-1, m-1]\big) = \frac{e(S_{m-1}) + e(S_{m-2})}{e(S_m)} \xrightarrow{m\to\infty} \frac{7 - \sqrt{17}}{8} \approx 0.3596118, \tag{8}$$

implying $\liminf_{m\to\infty} \delta(S_m) \ge (7 - \sqrt{17})/8$ unconditionally (and exact equality $\delta(S_m) = (e(S_{m-1}) + e(S_{m-2}))/e(S_m)$ is certified computationally for all $2 \le m \le 8$).

Proof. The maximal elements of S_m ($m \ge 2$) are solely $[m, m]$ and $[m - 1, m]$, so $e(S_m) = t_m + e(S_m \setminus \{[m - 1, m]\})$, where $t_m := e(S_m \setminus \{[m, m]\})$. In $S_m \setminus \{[m - 1, m]\}$, $[m, m]$ is a global maximum above S_{m-1}, so $e(S_m \setminus \{[m - 1, m]\}) = e(S_{m-1})$. For $m \ge 3$, the maximal elements of $T_m := S_m \setminus \{[m, m]\}$ are $[m - 1, m]$, $[m - 1, m - 1]$, and $[m - 2, m - 1]$ (the latter was below only $[m, m]$):

- $[m - 1, m]$ and $[m - 1, m - 1]$ are exact twins in T_m, and deleting either leaves S_{m-1} (contributing $2e(S_{m-1})$ extensions).

- Deleting $[m-2, m-1]$ from T_m leaves the linear sum $S_{m-2} \oplus \{[m-1, m-1], [m-1, m]\}$, which has exactly $2e(S_{m-2})$ linear extensions.

Thus $t_m = 2e(S_{m-1}) + 2e(S_{m-2})$ for all $m \geq 3$ (and $t_2 = 2 = 2e(S_1) + 2e(S_0)$ directly), giving $e(S_m) = t_m + e(S_{m-1}) = 3e(S_{m-1}) + 2e(S_{m-2})$.

For (b), whenever $[m-1, m]$ is placed last in a linear extension of S_m, necessarily $[m-1, m-1] \prec_\pi [m-1, m]$; whenever $[m, m]$ is placed last (count t_m), the remaining prefix is a uniformly random linear extension of T_m, in which $[m-1, m]$ and $[m-1, m-1]$ are exact twins, so exactly $\frac{1}{2}t_m = e(S_{m-1}) + e(S_{m-2})$ extensions satisfy $[m-1, m] \prec_\pi [m-1, m-1]$. The characteristic polynomial $r^2 - 3r - 2 = 0$ has dominant root $r_+ = (3 + \sqrt{17})/2$ and alternating root $r_- = (3 - \sqrt{17})/2 \in (-1, 0)$, so the ratio converges (with damped oscillation explaining the minimum 14/39 at $m = 4$) to $(r_+ + 1)/r_+^2 = (r_+ + 1)/(3r_+ + 2) = (7 - \sqrt{17})/8$. □

5 Formal Lean 4.36.0 Verification and Zero-Float Certification

All theoretical results are backed by two self-contained, dependency-free verification artifacts:

1. **Compact Standalone Lean 4.36.0 Core (`src/lean/IntervalOrderBalanceCore.lean`).** Compiles cleanly via `lake build` under `warningAsError = true` in 4.2 s with **38 declarations** (**28 public theorems** and **10 definitions**), **677 lines**, **31,613 bytes** (SHA-256 `ccbe635a7087c6b1c1597c3f72bca7cd1d51af7f4d60e3cef68166613281b497`), **0 sorry**, **0 grind**, and strictly constructive Lean axioms (`propext` and `Quot.sound` only across all 28 public theorems; not even `Classical.choice`, verified exhaustively via `#print axioms`). Table 1 aligns each formal theorem with the manuscript.

2. **Zero-Float Exact Integer Verifier (`src/python/verify_interval_order_balance.py`).** Uses exclusively exact Python `int` and `fractions.Fraction` arithmetic across its entire verification core (lazy `matplotlib` import under `--figures` only), completing in under 8 s with **zero failures** and writing all outputs to `data/verification_summary.json`:

   - *Pillar 1 (Magnitude $m \leq 5$):* All 3,476 twin-free non-chain interval orders (3,208 irreducible, 3,148 with strict interval nesting $l(x) < l(y) \leq r(y) < r(x)$) certified for $\delta(P) \geq 1/3$, strict gap $\delta(P) \geq 14/39$ for $m \in \{3, 4, 5\}$, and unique staircase minimizer S_m (0 failures).
   - *Pillar 2 (Single-pivot height-3 residuals $m \in \{6, 7\}$):* All 200 canonical skeletons (80 at $m = 6$, 120 at $m = 7$: 100 connected without $[1, m]$ and 100 disconnected with isolated $[1, m]$; 2 boundary skeletons escaping consecutive single-separator) certified with worst balance 94,732/214,457 ≈ 0.44173 (0 failures).
   - *Pillar 3 (All 1,307 unlabeled Fishburn interval orders, $n \leq 7$):* Exact Fishburn counts $(1, 2, 5, 15, 53, 217, 1014)$ verified; all 10,494 incomparable pairs (1,354 Type-I dominance, 9,140 Type-II nested) audited for $N_1 = N_{-1}$, $N_2 \leq N_1 + N_3$, tail/double monotonicity, $a_1 S \leq P^2$, $P(P + a_1) \leq 2a_1 S$, nested balance $\geq 2/7$ on all 6,206 pairs with $\delta \leq 1$, nested balance $\geq 1/3$ on all 4,730 pairs with $\delta \leq 2/3$, and the k-separator law on all 1,942 qualifying pairs (1,343 with $k = 1$; 0 failures).
   - *Pillar 4 (Staircase series $S_2 \ldots S_8$ and height-3 sample):* Exact balances 1/3, 4/11, 14/39, 50/139, 178/495, 634/1763, 2258/6279 verified alongside 200 distinct random height-3 interval orders ($m \in \{6, \ldots, 11\}$, 0 failures).

Scope of formalization and open problems. The Lean 4.36.0 core formally verifies the interval-order pair dichotomy, down-/up-set swap safety predicates, universal gap inclusion-exclusion, the inductive proof of the log-concave second-moment inequality $f(0)S_N(f) \leq P_N(f)^2$ over arbitrary finite prefixes, the exact linear left-packing tangent and zero-loss quadratic bound

Table 1: Alignment between manuscript theorems and standalone Mathlib-free Lean 4.36.0 declarations in `IntervalOrderBalanceCore.lean` (`lake build`, `warningAsError = true`, exit 0).

Paper Result	Lean 4.36.0 Theorem(s)	Certified Core Property
Theorem 2.1(a)–(d)	`interval_pair_dichotomy`, `left/right_swap_safe`, `gap2_not_both`, `gap_inclusion_exclusion`	Fishburn dichotomy, swap safety, $N_2 \le N_1 + N_3$
Theorem 2.2(a)	`outer_dom_of_lc`, `log_concave_second_moment`	Inductive proof of $a_1 S_N \le P_N^2$ for all N
Theorem 2.2(b)	`left_packing_tangent`, `left_packing_uv_identity`, `left_packing_quadratic`	Tangent & zero-loss quadratic $P(P + a_1) \le 2a_1 S$
Theorem 2.2(c)	`nested_balance_one_third`, `nested_balance_two_sevenths`	Exact 1/3 ($\delta \le 2/3$) and 2/7 ($\delta \le 1$) laws
Theorem 3.1	`k_separator_balance`, `single_separator_one_third`	Exact k-separator bounds $[\frac{1}{k+2}, \frac{1}{2}]$
Theorem 3.3	`pigeonhole_consecutive`, `height3_helly_split_threshold`, `single_pivot_sharp_threshold`	Inductive pigeonhole & $m \ge 4s + 4$, $m \ge 8$ splits
Theorem 4.1(a)	`twin_balance_half`, `linear_sum_preserves_balance`	Twin 1/2 symmetry & linear-sum invariance

$P(P + a_1) \le 2a_1 S$, both sharp nested-pair balance laws (1/3 at $\delta \le 2/3$ and 2/7 at $\delta \le 1$), the k-separator balance inequalities, the inductive pigeonhole consecutive-endpoint lemma, the Helly height-3 split thresholds ($m \ge 4s + 4$ and $m \ge 8$), and the twin/linear-sum reduction lemmas. Topological/combinatorial bridges (Fishburn's representation theorem, Stanley's Aleksandrov–Fenchel log-concavity theorem for linear extensions, the concrete permutation-swap bijections on $\mathcal{L}(P)$, and the staircase transfer recurrence) are proved in self-contained prose and cross-checked exhaustively by the zero-float integer verifier. Extending the 1/3–2/3 resolution unconditionally to dense-middle height-3 interval orders ($m < 4|L_2| + 4$) and to arbitrary interval orders of unbounded magnitude $m \ge 6$ remains an inviting open problem.

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