

Characteristic-Free Bounds and Sharp Gauss-Fiber Examples for Point-Span Tangent Absorption

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Abstract

Let $X^d \subset \mathbf{P}(V)$ be a smooth projective variety over an algebraically closed field, let $H = \mathcal{O}_X(1)$, and use the complete embedding defined by H^m , $m \geq 3$. A finite reduced set $Z \subset X$ is point-span tangent-absorbing if the span S_Z of its evaluation functionals contains the affine embedded tangent space at every support. We prove, in arbitrary characteristic,

$$\dim S_Z, |Z| \geq \max \left\{ J(d, m), \min \left((d+1)(d+2), \binom{d+m}{d} \right) \right\},$$

where

$$J(d, m) = \left\lfloor \frac{m+1}{2} \right\rfloor (d+1) + \begin{cases} 1, & m \text{ even,} \\ 0, & m \text{ odd.} \end{cases}$$

The proof retains the escape-or-Gauss-fiber alternative. The characteristic-free step is a minimal-degree argument: in characteristic p , a polynomial with all first partials zero is a p th power, and radicality replaces the characteristic-zero derivative contradiction.

We give two complementary geometric results. If the original embedding is the complete H -embedding and $H = A \otimes B$ with both factors separating distinct points, its Gauss map is injective and only the strong branch can occur. Conversely, over $\mathbf{C}$, for every $d \geq 2$ and $m \geq 3$ we construct a smooth hypersurface whose exceptional Gauss branch contains an absorbing set with

$$\dim S_Z = |Z| = \binom{d+m}{d}.$$

Thus the binomial estimate is exact within the exceptional branch in all these dimensions and degrees. The construction is existential and uses finite jet interpolation and Bertini; no classification or broad priority claim is made. A companion paper proves the sharper exact global binomial floor in characteristic zero; the present paper's distinct contribution is the characteristic-free alternative and its accompanying Gauss-map geometry.

1 Introduction and statements

Let $\mathbf{k}$ be an algebraically closed field, let $X^d \subset \mathbf{P}(V)$ be smooth and integral, and put $H = \mathcal{O}_X(1)$. Write

$$\varphi_m = \varphi|_{H^m} : X \longrightarrow \mathbf{P}(H^0(X, H^m)^*)$$

for the complete embedding. For a nonempty finite reduced set $Z \subset X$, set

$$S_Z := \text{Span}_{\mathbf{k}} \{ \text{ev}_p : p \in Z \} \subset H^0(X, H^m)^*.$$

Definition 1.1. *The set Z is point-span tangent-absorbing with respect to H^m if*

$$\hat{T}_{\varphi_m(p)} \varphi_m(X) \subseteq S_Z \quad (p \in Z),$$

where the left side is the affine tangent space to the cone over the complete H^m -embedding.

Let

$$\gamma_H : X \longrightarrow \mathrm{Gr}(d+1, V), \quad p \longmapsto \widehat{T}_p X$$

be the Gauss map of the original H -embedding. The main result is the following.

Theorem 1.2 (Characteristic-free rank alternative). *Let $\mathbf{k}$ be algebraically closed of arbitrary characteristic, let $X^d \subset \mathbf{P}(V)$ be smooth and integral with $d \geq 1$, and let $m \geq 3$. If $Z \subset X$ is a nonempty finite reduced point-span tangent-absorbing set for the complete H^m -embedding, then at least one of the following nonexclusive alternatives holds:*

- (a) *Z contains $d+2$ supports whose first infinitesimal neighborhoods impose independent conditions on H^m , and hence*

$$\dim S_Z, |Z| \geq (d+1)(d+2);$$

- (b) *there is a vector subspace $W \subset V$, $\dim W = d+1$, such that*

$$Z \subset X \cap \mathbf{P}(W), \quad \widehat{T}_p X = W \quad (p \in Z),$$

and

$$\dim S_Z, |Z| \geq \binom{d+m}{d}.$$

Consequently,

$$\dim S_Z, |Z| \geq \min \left\{ (d+1)(d+2), \binom{d+m}{d} \right\}. \quad (1)$$

The next bound is independent of the Gauss alternative and grows with m .

Theorem 1.3 (Growing jet-block bound). *Under the hypotheses of Theorem 1.2, put*

$$J(d, m) := \left\lfloor \frac{m+1}{2} \right\rfloor (d+1) + \begin{cases} 1, & m \text{ even,} \\ 0, & m \text{ odd.} \end{cases}$$

Then

$$\dim S_Z, |Z| \geq J(d, m).$$

In particular, combining this with (1) gives the bound in the abstract, and $J(d, m) > (d+1)(d+2)$ for every $m \geq 2d+4$.

Our first geometric corollary rules out the exceptional branch for a broad class of original embeddings.

Corollary 1.4 (Factorized polarizations). *Assume in addition that the original embedding is the complete H -embedding, so $V = H^0(X, H)^*$, and that $H = A \otimes B$, where A and B each separate every ordered pair of distinct closed points. Then the Gauss map γ_H is injective on closed points, alternative (a) of Theorem 1.2 holds, and*

$$\dim S_Z, |Z| \geq (d+1)(d+2).$$

In particular this applies when $H = L^r$ for a very ample line bundle L and $r \geq 2$.

Remark 1.5 (Why completeness is necessary). *The factorization alone does not control the Gauss map of an arbitrary incomplete subsystem. Indeed*

$$[s : t] \mapsto [s^5 + st^4 : s^4t + t^5 : s^3t^2 : s^2t^3]$$

embeds $\mathbf{P}^1$ in $\mathbf{P}^3$ by a subsystem of $H^0(\mathbf{P}^1, \mathcal{O}(5))$. The four coordinates have no common zero: at either endpoint one of the first two is nonzero, and on $st \neq 0$ the last two are nonzero. On $st \neq 0$ the ratio of the last two coordinates recovers t/s ; near $[1 : 0]$ and $[0 : 1]$, respectively, the ratios of the second to the first and the first to the second recover local parameters. Thus the map is injective and immersive. Nevertheless the affine tangents at the two endpoints are both $\text{Span}(e_0, e_1)$. Here $\mathcal{O}(5) = \mathcal{O}(2) \otimes \mathcal{O}(3)$ with both factors very ample. Hence the complete embedding hypothesis in Corollary 1.4 cannot simply be dropped.

The opposite behavior also occurs. The following statement makes exact only the estimate inside branch (b); it does not say that the binomial is the global minimum when the other branch has the smaller bound.

Theorem 1.6 (Sharp exceptional Gauss branch). *For every $d \geq 2$ and $m \geq 3$, set*

$$N = \binom{d+m}{d}, \quad e = 3N.$$

There exist a smooth integral complex hypersurface $X^d \subset \mathbf{P}^{d+1}$ of degree e , a hyperplane $\Lambda \cong \mathbf{P}^d$, and a reduced set $Z \subset X \cap \Lambda$ of N points such that

$$\widehat{T}_p X = \widehat{\Lambda} \quad (p \in Z)$$

and Z is point-span tangent-absorbing for the complete $\mathcal{O}_X(m)$ -embedding, with

$$\dim_{\mathbf{C}} S_Z = |Z| = N.$$

Thus the binomial lower bound in alternative (b) of Theorem 1.2 is attained for every such (d, m) .

The proofs are self-contained apart from standard Bertini and cohomology of projective space. The finite-jet constructions are related to jet ampleness [3]; the rank defect sits at the extreme end of Terracini-type interpolation phenomena [2, 8]. Earlier public working-note forms of the hypersurface and factorization arguments appear in [5, 6]. Those notes are antecedents, not external validation. The companion ARR paper [7] proves the sharper exact universal binomial floor in characteristic zero by an adapted finite projection and a symbolic-power argument. The present paper neither supersedes nor reproves that sharper characteristic-zero statement: it extends the rank framework to arbitrary characteristic, adds the growing bound $J(d, m)$, and isolates complementary Gauss-map criteria and examples. No exhaustive novelty or priority assertion is made.

2 First jets and the annihilator criterion

For $p \in X$, let $2p$ be the first infinitesimal neighborhood defined by $\mathfrak{m}_p^2$. It has length $d + 1$. Restriction to $2p$ is surjective for a very ample line bundle, and the dual of its target is the affine tangent space to the cone over the associated embedding.

Let $2Z = \coprod_{p \in Z} 2p$. The point span and the tangent span have annihilators

$$S_Z^\perp = H^0(X, \mathcal{I}_Z \otimes H^m),$$

and

$$\left(\sum_{p \in Z} \hat{T}_{\varphi_m(p)} \varphi_m(X) \right)^\perp = H^0(X, \mathcal{I}_{2Z} \otimes H^m).$$

Since each evaluation line lies in its affine tangent space, double annihilators give the following exact criterion.

Lemma 2.1 (Kernel criterion). *The set Z is point-span tangent-absorbing if and only if*

$$H^0(X, \mathcal{I}_Z \otimes H^m) = H^0(X, \mathcal{I}_{2Z} \otimes H^m).$$

Equivalently, restriction to Z and restriction to $2Z$ have the same rank.

This formulation is scheme-theoretic and characteristic-free.

3 Independent first-jet blocks

We use two kinds of finite interpolation. The first exploits projective independence and is efficient already in degree three.

Lemma 3.1 (Projectively independent supports). *Let $S \subset X$ be a finite set whose representatives in V are linearly independent. If $m \geq 3$, then $2S$ imposes independent conditions on H^m .*

Proof. Order $S = \{p_1, \dots, p_r\}$. A single first neighborhood is separated by H^m . Suppose the first neighborhoods at $p_1, \dots, p_{j-1}$ have been separated. Projective independence supplies a linear form a vanishing on their representative span and nonzero at p_j . Then a^2 vanishes on all the previous first neighborhoods and is a unit on $2p_j$. Since H^{m-2} is very ample, its sections restrict surjectively to $2p_j$; multiplication by the unit a^2 preserves that local surjectivity without changing the old data. Induction proves the claim. $\square$

This direct proof is the characteristic-free specialization needed here. Over $\mathbf{C}$, the same conclusion also follows from Ballico–Chiantini’s product criterion with $L_1 = H^2$, $L_2 = H^{m-2}$ and their condition $\dagger$ [2, Definition 3.1 and Theorem 3.5].

The second interpolation statement works for arbitrary distinct supports and is the direct input for Theorem 1.3.

Lemma 3.2 (Arbitrary-support jet blocks). *Let $p_1, \dots, p_r$ be distinct points of X .*

- (i) *The union $2p_1 \cup \dots \cup 2p_r$ imposes independent conditions on H^{2r-1} .*
- (ii) *If q is a further point, then $2p_1 \cup \dots \cup 2p_r \cup q$ imposes independent conditions on H^{2r} .*

Both conclusions remain true after increasing the exponent of H .

Proof. For every ordered pair of distinct supports choose a section of H that vanishes at the second support and is nonzero at the first.

For part (i), fix i and multiply the squares of separators vanishing at all p_j with $j \neq i$. The product $P_i \in H^0(X, H^{2r-2})$ is a unit at p_i and vanishes on $2p_j$ for $j \neq i$. Multiplying P_i by sections of H , whose first jets span at p_i , realizes arbitrary data on $2p_i$ and zero data on every other block. Summing over i proves surjectivity.

For part (ii), multiply the preceding P_i by one separator vanishing at q and nonzero at p_i , then multiply by a section of H to prescribe the first jet at p_i . This has total degree $2r$ and kills the simple value at q . Conversely, the product of squared separators vanishing at each

p_i and nonzero at q is a section of H^{2r} that vanishes on all double blocks and is a unit at q . These sections separate the whole mixed scheme.

To raise the degree, multiply the constructed sections by a section of a positive power of H that is nonzero at every selected support. Such a section exists because $\mathbf{k}$ is infinite and a finite union of proper linear subspaces cannot exhaust the space of sections. Multiplication by this local unit preserves the prescribed truncated jets. $\square$

Lemma 3.2 is also a concrete instance of the tensor product behavior of jet-ampleness established in [3, Proposition 2.3].

4 Escape or concentration in a Gauss fiber

Lemma 4.1 (Tangent escape separator). *Let $0 \neq v \in W \subsetneq V$ represent $p = [v] \in X$, and let $u \in \widehat{T}_p X \setminus W$. For every $m \geq 1$ there is a section of H^m that vanishes on $X \cap \mathbf{P}(W)$ but has nonzero first jet at p in the direction u .*

Proof. Choose $\lambda \in V^*$ with $\lambda|_W = 0$ and $\lambda(u) \neq 0$, and choose $\mu \in V^*$ with $\mu(v) \neq 0$. The restriction to X of $f = \lambda\mu^{m-1}$ vanishes on $X \cap \mathbf{P}(W)$, while

$$df_v(u) = \lambda(u)\mu(v)^{m-1} \neq 0.$$

No numerical factor depending on m is used. $\square$

Lemma 4.2 (Support alternative). *Assume Z is point-span tangent-absorbing. Let $p_1, \dots, p_r \in Z$ have linearly independent representatives and put $W = \text{Span}(v_1, \dots, v_r)$.*

- (i) *If $r \leq d$, some point of Z lies outside $\mathbf{P}(W)$.*
- (ii) *If $r = d + 1$, either some point of Z lies outside $\mathbf{P}(W)$, or*

$$Z \subset \mathbf{P}(W), \quad \widehat{T}_p X = W \quad (p \in Z).$$

Proof. Suppose $Z \subset \mathbf{P}(W)$. If $r \leq d$, then $\dim W = r < d + 1 = \dim \widehat{T}_{p_1} X$, so Lemma 4.1 produces a section annihilating S_Z but not the tangent space at p_1 , contrary to absorption.

If $r = d + 1$, the spaces W and $\widehat{T}_p X$ have the same dimension. If they differ for some $p \in Z$, choose a tangent direction outside W and apply the same separator. Hence the no-escape case forces equality at every support. $\square$

5 Unisolvence in the Gauss branch in all characteristics

Lemma 5.1 (Gauss-branch unisolvence). *Let W have dimension $d + 1$, let $Z \subset \mathbf{P}(W)$ be a nonempty finite reduced set, and assume*

$$\widehat{T}_p X = W \quad (p \in Z).$$

If Z is point-span tangent-absorbing for H^m , then, for $R = \text{Sym}(W^)$ and $I = I(Z) \subset R$,*

$$I_m = 0.$$

Consequently,

$$\dim S_Z, |Z| \geq \binom{d+m}{d}.$$

Proof. Let $F \in I_m$. The inclusion $W \subseteq V$ induces a surjection $\text{Sym}^m(V^*) \twoheadrightarrow \text{Sym}^m(W^*)$, so F extends to a degree- m form on V . Its restriction to X vanishes on Z , hence pairs to zero with S_Z . Absorption forces its first jet along X to vanish at every $p \in Z$. Since $\widehat{T}_p X = W$, this is the full projective first jet of F on $\mathbf{P}(W)$.

Choose coordinates $x_0, \dots, x_d$ on W with $p = [1 : 0 : \dots : 0]$. On $x_0 = 1$, a zero value and zero directional derivatives mean that $F(1, x_1, \dots, x_d)$ has no terms of degree zero or one. By homogeneity this is equivalent to $F \in I(p)^2$. Therefore

$$I_m \subseteq \bigcap_{p \in Z} I(p)^2. \quad (2)$$

Suppose $I_m \neq 0$ and choose a nonzero homogeneous $F \in I$ of minimal positive degree $\alpha \leq m$. For each $p \in Z$, choose $G_p \in R_{m-\alpha}$ with $G_p(p) \neq 0$. Then $G_p F \in I_m$, so (2) and local invertibility of G_p give $F \in I(p)^2$ locally at every support. All first partial derivatives of F consequently vanish on Z and belong to the radical ideal I .

If at least one partial derivative is nonzero, it has degree $\alpha - 1$ and contradicts the minimality of α . Otherwise the characteristic is $p > 0$ and every monomial of F has each exponent divisible by p . Because $\mathbf{k}$ is algebraically closed, hence perfect, its coefficients have p th roots and

$$F = G^p$$

for a homogeneous polynomial G of degree $\alpha/p < \alpha$. Radicality of I and $F \in I$ imply $G \in I$, again a contradiction. Hence $I_m = 0$.

The evaluation map $R_m \rightarrow \mathbf{k}^Z$ is injective and has rank $\binom{d+m}{d}$. Every form on W extends to V , so this value matrix is a submatrix of restriction for H^m and its rank is at most both $\dim S_Z$ and $|Z|$. $\square$

Remark 5.2. *The reducedness of Z , equivalently radicality of $I(Z)$ in this setting, is essential to this proof in the Frobenius case. The argument is elementary but is consistent with the differential characterization of symbolic powers over perfect fields; see [4] for a broader modern framework.*

6 Proofs of the rank bounds

Proof of Theorem 1.2. Starting from any support, repeated application of Lemma 4.2(i) selects $d + 1$ points with linearly independent representatives. Lemma 3.1 makes their first neighborhoods independent. Their dual tangent blocks, each of dimension $d + 1$, lie in S_Z , so

$$\dim S_Z \geq (d + 1)^2, \quad |Z| \geq \dim S_Z.$$

Apply Lemma 4.2(ii). If a further point escapes, the resulting $d + 2$ supports are projectively independent and Lemma 3.1 gives alternative (a). If no point escapes, the same lemma gives a common W and common Gauss fiber; then Lemma 5.1 gives alternative (b). Taking the smaller branch bound proves (1). $\square$

Proof of Theorem 1.3. Write first $m = 2q + 1$. If $|Z| \leq q$, part (i) of Lemma 3.2, followed by degree raising, makes $2Z$ independent in degree m . Its dual tangent span would have dimension $|Z|(d + 1)$ but would lie by absorption in S_Z , whose dimension is at most $|Z|$, a contradiction. Hence Z contains $q + 1$ supports. Their first neighborhoods are independent in $H^{2(q+1)-1} = H^m$, so

$$\dim S_Z, |Z| \geq (q + 1)(d + 1) = J(d, m).$$

If $m = 2q$, the same argument excludes $|Z| \leq q$. Choose q supports and one further support. Lemma 3.2(ii) makes the first q double blocks and the last simple value independent in degree $2q = m$. All their dual spaces lie in S_Z , giving

$$\dim S_Z, |Z| \geq q(d+1) + 1 = J(d, m).$$

The threshold comparison is immediate in the two parity cases. $\square$

7 Injective Gauss maps from factorization

Proposition 7.1 (Point-separating factors). *Let $H = A \otimes B$ be very ample, and suppose A and B each separate every ordered pair of distinct closed points. Then the Gauss map of the complete H -embedding is injective on closed points.*

Proof. Fix $p \neq q$. Choose

$$a \in H^0(X, A), \quad a(p) = 0, \quad a(q) \neq 0, \quad b \in H^0(X, B), \quad b(p) = 0, \quad b(q) \neq 0.$$

The product $s = ab \in H^0(X, H)$ belongs to $H^0(X, H \otimes \mathfrak{m}_p^2)$ but satisfies $s(q) \neq 0$. Thus the hyperplane defined by s contains the affine tangent space at p but not the evaluation line at q , which lies in the tangent space at q . Consequently $\hat{T}_p X \neq \hat{T}_q X$. $\square$

Proof of Corollary 1.4. Proposition 7.1 makes every Gauss fiber a singleton. Alternative (b) of Theorem 1.2 would require $|Z| \geq \binom{d+m}{d} > 1$ inside one fiber, so it is impossible. If $H = L^r$ with $r \geq 2$, use the point-separating factors L and L^{r-1} . $\square$

8 Smooth hypersurfaces realizing the exceptional bound

We now work over $\mathbf{C}$. We first record two elementary interpolation facts.

Lemma 8.1 (Simplex-lattice unisolvence). *For $n, r \geq 0$, the lattice*

$$\Delta_{n,r} = \{(a_1, \dots, a_n) \in \mathbf{Z}_{\geq 0}^n : a_1 + \dots + a_n \leq r\}$$

is unisolvent for polynomials of total degree at most r .

Proof. If $n = 0$ or $r = 0$, the assertion is the constant-polynomial case. Now induct on $n + r$. If f vanishes on $\Delta_{n,r}$, its restriction to $x_n = 0$ vanishes on $\Delta_{n-1,r}$, so $f = x_n g$ by induction. Then $g(x_1, \dots, x_{n-1}, t+1)$ vanishes on $\Delta_{n,r-1}$ and is zero by induction. Thus $f = 0$. The number of lattice points is $\binom{n+r}{n}$, the dimension of the polynomial space. $\square$

Lemma 8.2 (Prescribed triple jets). *Let $p_1, \dots, p_N$ be distinct points of $\mathbf{P}^d$ and set $e = 3N$. Then*

$$H^0(\mathbf{P}^d, \mathcal{O}(e)) \longrightarrow \bigoplus_{i=1}^N \mathcal{O}_{\mathbf{P}^d, p_i} / \mathfrak{m}_{p_i}^3$$

is surjective. Moreover, $H^0(\mathbf{P}^d, \mathcal{I}_{3Z}(e))$ is basepoint-free on $\mathbf{P}^d \setminus Z$.

Proof. For fixed i , choose for each $j \neq i$ a linear form vanishing at p_j and nonzero at p_i . The product of their cubes has degree $3(N-1)$, is a unit at p_i , and vanishes to order at least three at every other support. In affine coordinates centered at p_i , the classes of monomials of total degree at most two form a basis of the local algebra modulo $\mathfrak{m}_{p_i}^3$; they are restrictions of global quadrics. After multiplying by one further linear form nonzero at every support and adjusting the local datum by the resulting unit, degree- e forms realize arbitrary triple jets at p_i and zero jets elsewhere. Summing over i proves surjectivity.

For $q \notin Z$, choose a linear form ℓ_i vanishing at p_i and nonzero at q for every i . The product $\prod_i \ell_i^3$ has degree e , belongs to $\mathcal{I}_{3Z}$, and is nonzero at q . $\square$

Proposition 8.3 (A large special Gauss fiber). *For every $d \geq 2$ and $N \geq 1$ there is a smooth integral hypersurface $X^d \subset \mathbf{P}_{\mathbf{C}}^{d+1}$ of degree $3N$ and a hyperplane $\Lambda \cong \mathbf{P}^d$ such that one fiber of the Gauss map contains any prescribed set of N distinct points of Λ .*

Proof. Use coordinates with $\Lambda = (x_0 = 0)$ and let $Z = \{p_1, \dots, p_N\} \subset \Lambda$. By Lemma 8.2, choose a degree- e form f_0 on Λ whose constant and linear terms vanish at each p_i and whose quadratic term there is nondegenerate. Every member of

$$f_0 + H^0(\Lambda, \mathcal{I}_{3Z}(e))$$

has the same ordinary-double-point jets. The translation space is basepoint-free away from Z . The corresponding projective system has base locus exactly Z ; its nonempty Bertini open meets the affine chart where the coefficient of f_0 is one. Hence Bertini gives a member f whose hypersurface $Y = (f = 0) \subset \Lambda$ has ordinary double points at the p_i and is smooth away from them [9, III, Corollary 10.9].

Consider ambient degree- e forms

$$F = f + x_0 G, \quad G \in H^0(\mathbf{P}^{d+1}, \mathcal{O}(e-1)).$$

The projective linear system spanned by f and $x_0 H^0(\mathbf{P}^{d+1}, \mathcal{O}(e-1))$ has base locus exactly Y . Its Bertini-smooth locus outside Y is a nonempty open subset of the irreducible parameter space, and it meets the affine chart in which the coefficient of f is one. At a smooth point of Y , the differential of f in a direction tangent to Λ is nonzero. At a marked node, choose $G(p_i) \neq 0$; then

$$dF_{p_i} = G(p_i) dx_0,$$

so the ambient hypersurface is smooth there and has tangent hyperplane Λ . Away from Y , Bertini makes a general member smooth [9, III, Corollary 10.9]. The finitely many conditions $G(p_i) \neq 0$ are nonempty open conditions in the same irreducible affine chart, so their intersection with the Bertini open is nonempty. The resulting hypersurface is reduced by smoothness. If it had two positive-degree components, they would meet in $\mathbf{P}^{d+1}$ because $d \geq 2$ [9, I, Theorem 7.2], making their union singular; hence it is irreducible and therefore integral. All marked points lie in the Gauss fiber over Λ . $\square$

Proof of Theorem 1.6. Take $N = \binom{d+m}{d}$ and identify the affine chart of $\Lambda \cong \mathbf{P}^d$ with $\mathbf{C}^d$. Let Z be the homogenization of $\Delta_{d,m}$. Lemma 8.1 says that no nonzero degree- m form on Λ vanishes on all of Z . Equivalently, their degree- m evaluation vectors span the full vector space $\text{Sym}^m(\hat{\Lambda})$ of dimension N . Here we use the canonical dual identification $H^0(\Lambda, \mathcal{O}_{\Lambda}(m))^* \cong \text{Sym}^m(\hat{\Lambda})$.

Apply Proposition 8.3 to these supports, obtaining a smooth degree- $e = 3N$ hypersurface X . Since $e > m$ and $d+1 \geq 3$, the restriction sequence

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^{d+1}}(m-e) \longrightarrow \mathcal{O}_{\mathbf{P}^{d+1}}(m) \longrightarrow \mathcal{O}_X(m) \longrightarrow 0$$

and the vanishing of intermediate cohomology of projective space [9, III, Theorem 5.1] show that

$$H^0(\mathbf{P}^{d+1}, \mathcal{O}(m)) \xrightarrow{\sim} H^0(X, \mathcal{O}_X(m)).$$

Thus the complete $\mathcal{O}_X(m)$ -embedding is the restriction of the ambient m th Veronese embedding.

If $p = [v] \in Z$, Proposition 8.3 gives $\hat{T}_p X = \hat{\Lambda}$. The affine tangent space after the Veronese embedding is

$$v^{m-1} \hat{\Lambda} \subseteq \text{Sym}^m(\hat{\Lambda}).$$

The differential carries the scalar factor m , which is nonzero and irrelevant to this subspace because the construction is over $\mathbf{C}$. The right side is exactly the span of the evaluation vectors of Z . Therefore every tangent block lies in S_Z and Z is absorbing. The same unisolvence gives $\dim S_Z = N = |Z|$. $\square$

Remark 8.4. For $(d, m) = (2, 3)$ the construction gives ten absorbing points on a smooth degree-30 surface in $\mathbf{P}^3$, so the exact value ten is not confined to a linearly embedded plane. In higher parameters the example is sharp for the exceptional branch but need not attain the smaller global branch bound.

9 Relation to adjacent notions and limitations

The equality of value and first-jet kernels in Lemma 2.1 is stronger than an ordinary Terracini defect. It says that the full double-point rank has collapsed to the value rank. Conversely, a Terracini defect alone does not imply absorption [2, 8].

Under absorption one also has

$$S_Z = \text{Span}\{\hat{T}_{\varphi_m(p)}\varphi_m(X) : p \in Z\}.$$

If $A \subseteq Z$ is minimal with the same tangent span, then $Z \setminus A$ lies in the strong base locus of A . This places the problem inside the framework of [1], while the additional equality with a span of evaluations is the special constraint used here. Strong base loci and Terracini loci are not comparable in general [1]. For classical background on Gauss maps, tangent loci, and dual varieties, see [12]. Multitangent hyperplanes also have a substantial enumerative and dual-variety literature. Vainsencher counts n -fold tangent hyperplanes to surfaces in specific generic settings for $1 \leq n \leq 6$ [11], while Holweck studies bitangent components in singular loci of dual varieties [10]. Those works concern enumerative or dual-singularity questions; they do not by themselves establish the value-span absorption constraint used here or priority for the present construction.

The following limitations are material.

- Theorem 1.2 is stated over algebraically closed fields. A version over a nonclosed perfect field would require a careful formulation for closed points and residue fields; it is not asserted here.
- The hypersurface construction is over $\mathbf{C}$. It uses ordinary double points with nondegenerate quadratic parts and a standard Bertini argument; no positive-characteristic version is claimed.
- The construction proves existence, not an explicit globally smooth equation. Finite rank replays can certify interpolation and absorption matrices, but cannot replace Bertini or prove the universal theorem.
- Neither equality configurations nor varieties with large special Gauss fibers are classified. The simplex lattice is one unisolvent fixture among many.
- The cited literature comparison is selective. The paper claims the displayed deductions and constructions, not exhaustive priority for tangent containment, jet separation, or special Gauss fibers.

10 Conclusion

Point-span tangent absorption admits a uniform characteristic-free structure: support escape gives independent tangent blocks, while failure of escape puts all supports into a single original Gauss fiber and forces degree- m unisolvence there. Arbitrary-support interpolation adds the growing bound $J(d, m)$. Factorization eliminates the Gauss branch when the original H -embedding is complete, whereas smooth complex hypersurfaces with prescribed nodal

hyperplane sections realize its binomial estimate exactly. These results sharpen the rank picture in arbitrary characteristic without asserting a classification or relying on computational fixtures as proof. In characteristic zero the stronger exact global floor is supplied by the companion work [7].

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