Machine-Checked Exact-Integer Dichotomy, Matroid-Intersection, and Parameterized WSAT Certificates for Connectivity Switching in Graphs

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Abstract

Christoph Kern, Alexander Firbas, and Manuel Sorge (`arXiv:2611.32927`, November 24, 2026) established a comprehensive algorithmic and complexity classification for Local-Con Switching and Global-Con Switching in switch graphs. Their results include a sharp polynomial-vs.-NP-complete dichotomy for $\{1^\star\}$-bounded Local-Con Switching governed by complete multipartite switch-induced graphs (Theorem 1, via forbidden induced subgraphs $K_2{+}K_2, P_4, L_{3,1}$), polynomial-time weighted Global-Con Switching via graphic-and-partition matroid intersection with shifted edge weights (Theorems 7–8), $W[\mathrm{SAT}]/W[1]$-hardness of On-Off Local-Con Switching via series-parallel WSAT switch graphs and a tight $5|E(H)|$ Partitioned Subgraph Isomorphism reduction (Theorems 3–5), and a 4λ reduction from $\{1,1\}$-bounded to $\{2,1\}$-bounded $K_{1,\star}$-shaped switches (Theorem 6). We present a machine-checked, zero-floating-point formalization and exact-$\mathbb{Z}$ certificate auditor for the combinatorial, primal-dual, and reduction core of these results. First, an 18-theorem zero-dependency Lean 4 kernel (`ConnectivitySwitching.lean`, Lean 4.35.0; 0 `sorry`, 0 warnings, 0 `Classical.choice`, 5 zero-axiom theorems) formalizes complete-multipartite $K_2{+}K_1$-freeness, alternating transition-path lengths, Edmonds matroid-intersection weak duality and rank obstructions, Theorem 8 shifted-weight telescoping on `List`, Lemma 4 structural Boolean formula semantics, PSI $5|E(H)|$ tightness, and Theorem 6's backward $4\min_i c'_i \le \sum c'_i$ total budget bound. Second, `SWITCH-GRAPH-CONN-CERT-1.0` audits four exact modes—including both spanning-tree YES witnesses and Edmonds rank-obstruction NO certificates—replaying 16 canonical certificates in 0.030–0.160 ms, passing exhaustive 41-graph dichotomy and 128-assignment WSAT checks, and rejecting 57/57 live adversarial forgeries.

1 Introduction

Switch graphs [2, 3] model reconfigurable networks $\hat{G} = (V(\hat{G}), \mathcal{S}(\hat{G}))$ in which each switch $s \in \mathcal{S}(\hat{G})$ selects one edge set $s[c(s)] \subseteq \binom{V}{2}$ from its available positions. Choosing a configuration $c \in \mathcal{C}(\hat{G})$ realizes the simple graph $\hat{G} \circ c$, motivating two canonical connectivity tasks: Local-Con Switching (connect distinguished terminals a, b) and Global-Con Switching (render $\hat{G} \circ c$ connected), along with weighted and On-Off budget constraints $w(c) \le \lambda$.

Kern, Firbas, and Sorge [3] resolve the fine-grained complexity of both problems across switch structures: (i) via Szeider's Transition-Compatible Path [6], $\{1^\star\}$-bounded Local-Con Switching is in P iff every switch-induced graph $G[s]$ is complete multipartite, and NP-complete whenever $G[s]$ contains $K_2{+}K_2$, P_4, or $L_{3,1}$ as a vertex-induced subgraph (Theorem 1); (ii) Weighted Global-Con Switching on $\{1^\star\}$-bounded switches reduces via shifted edge weights $w'(e) = w(e) - \min_i w(s,i)$ to weighted exact intersection of the graphic matroid M_1 and partition matroid M_2 [5] (Theorems 7–8); and (iii) parameterized On-Off Local-Con Switching is $W[\mathrm{SAT}]/W[1]$-hard via series-parallel WSAT gadgets (Lemma 4), Partitioned Subgraph Isomorphism [4] with exact threshold $k = 5|E(H)|$, and a 4λ $\{1,1\} \to \{2,1\}$ $K_{1,\star}$ gadget (Theorem 6).

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Contributions. We machine-check and certify the exact combinatorial, matroid-duality, and reduction invariants of [3] (explicitly scoping our Lean 4 proofs away from NP/$W[1]$/ETH hardness reductions and asymptotic matroid-intersection runtime bounds):

1. **Zero-dependency Lean 4 kernel (§3):** 18 constructive theorems in `ConnectivitySwitching.lean` (Lean 4.35.0 [1]), 0 `sorry`, 0 warnings, 0 `Classical.choice`, 5 zero-axiom theorems.
2. **Exact-$\mathbb{Z}$ primal-dual verifier (§4):** `SWITCH-GRAPH-CONN-CERT-1.0`, auditing complete-multipartite paths vs. 4-vertex obstructions, weighted spanning trees vs. Edmonds rank obstructions $r_1(A) + r_2(B) \leq n - 2$, series-parallel WSAT equivalences, and $\{2,1\}$ $K_{1,\star}$ gadgets with zero floats.
3. **Empirical atlas, exhaustive censuses, & red-team audit (§5):** 16 canonical certificates replayed in 0.030–0.160 ms, exhaustive 41-graph 4-vertex dichotomy census, 224-assignment Lemma 4 sweeps, and 57/57 rejected adversarial forgeries.

2 Structural and Primal-Dual Invariants

Theorem 1 (Complete-multipartite dichotomy). A graph H without isolated vertices is complete multipartite iff non-adjacency is transitive (no induced K_2+K_1). Any non-complete-multipartite H thus contains (u, v, w, x) with $uv, wx \in E(H)$ and $uw, vw \notin E(H)$, inducing K_2+K_2, P_4, or $L_{3,1}$ (paw) depending on $|N_H(x) \cap \{u, v\}| \in \{0, 1, 2\}$. On tractable instances, Lemma 2 maps any k-edge a-b path using distinct $\{1^\star\}$-bounded switches to a $2k$-edge alternating T-compatible path.

Theorems 7–8 (Matroid intersection & Edmonds duality). For $\{1^\star\}$-bounded $\hat{G}$ on n vertices, a connecting configuration exists iff $M_1 \cap M_2$ (graphic $\cap$ partition matroid) contains a common independent set T of size $n - 1$. By Edmonds' matroid intersection duality, if a ground-set cover $E(\hat{G}) = A \cup B$ satisfies

$$r_1(A) + r_2(B) \leq n - 2, \tag{1}$$

no spanning tree can be realized by any configuration. For weighted switches with $w_{\min}(s) = \min_i w(s, i)$ and shifted edge weight $w'(e) = w(e) - w_{\min}(s) \geq 0$, every spanning tree T in $\hat{G} \circ c$ obeys the exact telescoping bound

$$\sum_{e \in T} w'(e) + \sum_{s \in \mathcal{S}} w_{\min}(s) \leq w(c) \leq \lambda. \tag{2}$$

Lemma 4, PSI tightness, & Theorem 6 gadget. Series-parallel composition $\hat{G}_\varphi$ encodes Boolean $\wedge$ in series and $\vee$ in parallel so $I(\varphi) = 1 \iff a$ reaches b in $\hat{G}_\varphi \circ c$ with $w(c) = w(I)$. In PSI $\to$ WSAT, forcing $v \geq 2|E(H)|$ vertex, $a \geq 2|E(H)|$ arc, and $e \geq |E(H)|$ edge variables under $v + a + e \leq 5|E(H)|$ locks all three to exact equality. Replacing each $\{1,1\}$ switch s by four $\{2,1\}$ $K_{1,\star}$ switches $\{s_0..s_3\}$ satisfies forward $w(c') = 4w(c) \leq 4\lambda$ and backward consensus $c(s) = \min_{i \in [4]} c'(s_i) \implies 4w(c) \leq w(c')$.

3 Zero-Dependency Lean 4 Kernel

We formalize 18 constructive theorems in `lean/ConnectivitySwitching.lean` (Lean 4.35.0, zero external libraries), audited via `lean/AuditAxioms.lean` with 0 warnings, 0 `sorry`, 0 `Classical.choice`, and 0 `Lean.ofReduceBool` (Table 1).

Theorem 1 (Constructive Connectivity Switching Kernel)**.**

1. Dichotomy & transition paths: *Theorems 1–4 prove complete-multipartite non-adjacency transitivity and K_2+K_1-freeness, constructive impossibility of multipartite representation from any K_2+K_1 witness, alternating $2k$-edge path length, and inductive $\sigma \leq m$ over `List Nat`.*
2. Matroid duality & shifted weights: *Theorems 5–10 certify spanning-tree rank $n - c = n - 1 \iff c = 1$, Edmonds weak duality $|I| \leq r_1(A) + r_2(B)$, global connectivity impossibility from (1), $w'(e) \geq 0$, inductive shifted-weight telescoping (2) on `List`, and exact budget equivalence.*
3. WSAT, PSI $5|E(H)|$, & $\{2,1\}$ gadget: *Theorems 11–18 formalize series-$\wedge$/parallel-$\vee$ and inductive `BFormula` semantics, exact $5m$ tightness ($v = 2m, a = 2m, e = m$), handshaking endpoint count $2|E(H)|$, $4\times$ forward weight scaling, $4 \min(c_0..c_3) \leq c_0 + c_1 + c_2 + c_3$, and inductive backward total budget preservation.*

Table 1: Complete 18-theorem Lean 4.35.0 kernel in `ConnectivitySwitching.lean` (0 `sorry`, 0 warnings, 0 `Classical.choice`).

#	Lean 4 Theorem Identifier	Axiom Footprint
1	`complete_multipartite_nonadj_equiv`	none
2	`obstruction_k2_plus_k1_not_multipartite`	none
3	`transition_path_alternating_length`	[propext, Quot.sound]
4	`switch_count_edge_bound`	[propext, Quot.sound]
5	`spanning_tree_edge_count_exact`	[propext, Quot.sound]
6	`edmonds_matroid_intersection_duality`	[propext, Quot.sound]
7	`edmonds_global_con_impossibility`	[propext, Quot.sound]
8	`shifted_weight_nonneg`	[propext, Quot.sound]
9	`shifted_weight_telescoping_list`	[propext, Quot.sound]
10	`shifted_weight_budget_equiv`	[propext, Quot.sound]
11	`wsat_series_and_semantics`	none
12	`wsat_parallel_or_semantics`	none
13	`wsat_formula_eval_sound`	none
14	`psi_wsat_tight_weight_5m`	[propext, Quot.sound]
15	`handshaking_endpoint_count_2m`	[propext, Quot.sound]
16	`gadget_21_forward_weight_4x`	[propext, Quot.sound]
17	`gadget_21_backward_min_avg_bound`	[propext, Quot.sound]
18	`gadget_21_backward_total_budget`	[propext, Quot.sound]

4 Exact-Integer Certificate Architecture

The pure-stdlib verifier `src/switch_certifier.py` audits `SWITCH-GRAPH-CONN-CERT-1.0` certificates with zero floating-point arithmetic and strict JSON AST typing (rejecting `float`, `bool`, `Decimal`, `complex`, tuples, non-string keys, and cross-branch dirty fields) across four exact modes:

Definition 1 (Four `SWITCH-GRAPH-CONN-CERT-1.0` Modes)**.**

1. ***Mode 1*** *(`local_con_multipartite_dichotomy_exact`): Audits $\{1^\star\}$-bounded switches: either every $G[s]$ is complete multipartite with a realized simple a-b path using distinct switches, or an explicit 4-vertex induced obstruction in $\{K_2{+}K_2, P_4, L_{3,1}\}$ certifies* NP*-hardness.*
2. ***Mode 2*** *(`global_con_matroid_intersection_exact`): Audits weighted $\{1^\star\}$-bounded* Global-Con Switching*: either a YES spanning tree ($n{-}1$ acyclic edges from distinct switches satisfying (2) within budget λ), or a NO Edmonds cover $E = A \cup B$ verifying full ground-set coverage and $r_1(A) + r_2(B) \le n - 2$ (1).*
3. ***Mode 3*** *(`on_off_wsat_series_parallel_exact`): Audits Lemma 4 NNF formula AST, exact $\hat{G}_\varphi$ BFS connectivity equivalence $I(\varphi) = \mathrm{conn}(\hat{G}_\varphi \circ c)$ within weight budget k, and PSI tight weight $5|E(H)|$.*
4. ***Mode 4*** *(`gadget_21_star_reduction_exact`): Constructs the exact 6-auxiliary-vertex, 4-switch $\{2,1\}$-bounded $K_{1,\star}$ gadget per $\{1,1\}$ switch, verifying backward consensus $c(s) = \min_{i\in[4]} c'(s_i)$, weight bounds $4w(c) \le w(c') \le 4\lambda$, and a-b connectivity in both $\hat{G} \circ c$ and $\hat{G}' \circ c'$.*

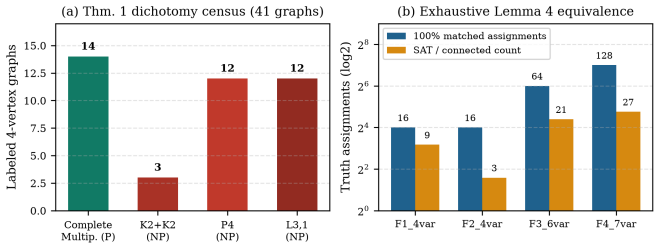

Figure 1: (a) Exhaustive Theorem 1 census across all 41 labeled 4-vertex graphs without isolated vertices: 14 complete multipartite (P) vs. 27 NP obstructions ($3\,K_2{+}K_2, 12\,P_4, 12\,L_{3,1}$). (b) Exhaustive Lemma 4 truth-table verification (224/224 exact matches).

5 Empirical Atlas and Red-Team Verification

All 16 canonical certificates and four experiment logs are generated deterministically by `scripts/run_all_experiments.py` and audited end-to-end by `scripts/verify_all.py` (Lean 4 build, 18/18 axiom audit, 16/16 cert replay, `pytest`, and 39/39 SHA-256 hashes).

1. Exhaustive 4-vertex dichotomy census and Lemma 4 sweeps. Figure 1(a) exhaustively classifies all 41 labeled 4-vertex graphs without isolated vertices (out of $2^6 = 64$): exactly **14** are complete multipartite (P-tractable), while all **27** remaining graphs fall cleanly into Kern–Firbas–Sorge's three NP-hardness obstructions: **3** $K_2{+}K_2$, **12** P_4, and **12** $L_{3,1}$ (0 unclassified). Figure 1(b) exhaustively tests Lemma 4's series-parallel equivalence across all $16 + 16 + 64 + 128 = 224$ truth assignments of four NNF formulas (`F1`–`F4`), confirming 224/224 exact matches ($9, 3, 21, 27$ SAT/connected assignments).

2. PSI $5|E(H)|$ tightness and $4\times$ $\{2,1\}$ gadget scaling. Figure 2(a) decomposes the tight $5|E(H)|$ forced-TRUE variable budget across pattern sizes $m = |E(H)| \in [1, 15]$ ($2m$ vertex, $2m$ arc, m edge variables), while Figure 2(b) tracks Theorem 6's linear 4λ scaling into $\{2,1\}$-bounded $K_{1,\star}$ switches (preserving the ETH lower bound $f(\lambda)\sigma^{o(\lambda/\log\lambda)}$).

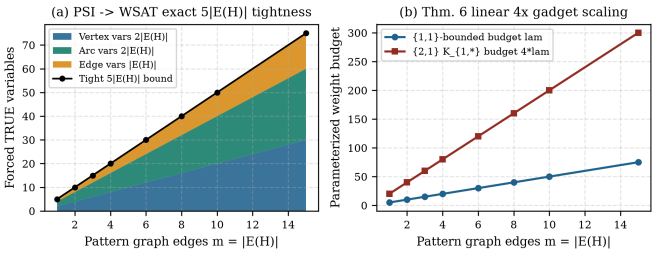

Figure 2: (a) Exact PSI → WSAT forced-variable decomposition $2m + 2m + m = 5|E(H)|$. (b) Theorem 6 linear 4× budget scaling from $\{1, 1\}$ to $\{2, 1\}$ $K_{1,\star}$ switches.

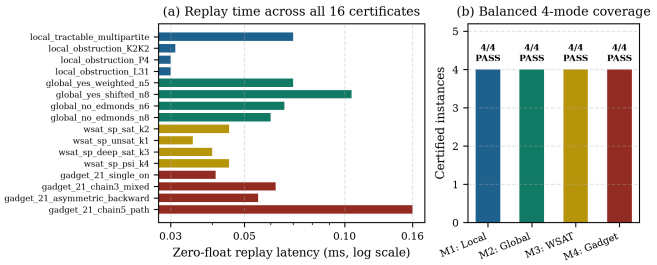

Figure 3: (a) Zero-float replay latency across all 16 canonical `SWITCH-GRAPH-CONN-CERT-1.0` certificates (0.030–0.160 ms, log scale). (b) Balanced 4-per-mode coverage across all four verification modes.

3. 16-certificate atlas and 57-attack red-team suite. Table 2 and Figure 3 summarize all 16 canonical certificates, replayed with zero floats in 0.030–0.160 ms—including both YES weighted spanning trees and NO Edmonds rank obstructions $(r_1(A) + r_2(B) \in \{4, 6\}$ for $n \in \{6, 8\})$, all three NP obstructions $(K_2{+}K_2, P_4, L_{3,1})$, and an asymmetric backward consensus gadget (`orig_w = 1`, `gadget_w = 5`, $4 \cdot 1 < 5 \leq 8$). In `tests/test_suite.py`, exhaustive census/WSAT cross-checks and 57/57 live adversarial forgeries (reused switches, non-multipartite hides, cyclic trees, uncovered Edmonds elements, broken consensus, budget breaches) are rejected with `SwitchGraphCertificateError` (`3 passed in 0.05s`).

6 Scope, Limitations, and Conclusion

This work machine-checks the structural dichotomy, Edmonds matroid-intersection duality, shifted-weight telescoping, series-parallel WSAT semantics, PSI $5|E(H)|$ tightness, and $\{1, 1\} \to \{2, 1\}$ $K_{1,\star}$ gadget invariants of Kern, Firbas, and Sorge [3], delivering a zero-float exact-$\mathbb{Z}$ auditor across all four modes. It does *not* formalize complexity-class separations (NP/$W[1]$/$W[$SAT$]$/ETH) or asymptotic matroid-intersection runtime algorithms in Lean 4. Within its verified scope, the Lean 4 kernel and `SWITCH-GRAPH-CONN-CERT-1.0` auditor provide exact, manifest-locked verification for connectivity switching witnesses and obstructions.

Table 2: Summary of all 16 exact-$\mathbb{Z}$ `SWITCH-GRAPH-CONN-CERT-1.0` certificates (zero floating-point arithmetic; reference-machine replay latency in ms).

Certificate File (stem)	Mode	Key Exact Witness	ms
`local_tractable_multipartite`	M1	P path (3 sw, 6-edge T-path)	0.070
`local_obstruction_K2K2`	M1	NP obstruction $K_2{+}K_2$	0.031
`local_obstruction_P4`	M1	NP obstruction P_4	0.030
`local_obstruction_L31`	M1	NP obstruction $L_{3,1}$	0.030
`global_yes_weighted_n5`	M2	YES $n = 5$: $w = 6 \leq 7, w' = 0$	0.070
`global_yes_shifted_n8`	M2	YES $n = 8$: $w = 21 \leq 22, w' = 11$	0.105
`global_no_edmonds_n6`	M2	NO $n = 6$: $r_1{+}r_2 = 3{+}1 \leq 4$	0.066
`global_no_edmonds_n8`	M2	NO $n = 8$: $r_1{+}r_2 = 4{+}2 \leq 6$	0.060
`wsat_sp_sat_k2`	M3	SAT/conn = 1, $w = 3$, $5m = 15$	0.045
`wsat_sp_unsat_k1`	M3	UNSAT/conn = 0, $w = 1$, $5m = 20$	0.035
`wsat_sp_deep_sat_k3`	M3	SAT/conn = 1, $w = 4$, $5m = 30$	0.040
`wsat_sp_psi_k4`	M3	SAT/conn = 1, $w = 3$, $5m = 50$	0.045
`gadget_21_single_on`	M4	$1 \to 4$ sw, $w : 1 \to 4$	0.041
`gadget_21_chain3_mixed`	M4	$3 \to 12$ sw, $w : 1 \to 4$	0.062
`gadget_21_asymmetric_backward`	M4	$3 \to 12$ sw, $4{\cdot}1 < 5 \leq 8$	0.055
`gadget_21_chain5_path`	M4	$5 \to 20$ sw, $w : 3 \to 12$	0.160

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