Exact Finite Free Cumulant Additivity, Multiplicative Relative-Variance Contraction, Dyson Heat-Flow Discriminant Floors, and Derivative Intertwining for Real-Rooted Polynomials

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Abstract

We establish four self-contained, non-asymptotic algebraic, spectral, and dynamical theorems for monic real-rooted degree-d polynomials $p \in \mathcal{P}_d$ ($d \geq 2$) under the Marcus–Spielman–Srivastava finite free additive convolution $p \boxplus_d q$ and multiplicative convolution $p \boxtimes_d q$. First, via normalized elementary symmetric coefficients $\tilde{a}_k(p) := a_k(p)/\binom{d}{k}$, we prove exact additivity of the root mean, variance $\sigma_d^2(p)$, third central moment $\mu_3(p)$ ($d \geq 3$), and normalized fourth cumulant $\kappa_4^{(d)}(p) := \tilde{a}_4(p^{\mathrm{cent}}) - 3\tilde{a}_2(p^{\mathrm{cent}})^2$ ($d \geq 4$) under $\boxplus_d$, together with non-asymptotic two-sided extreme-root sandwiches combining Laguerre–Samuelson bounds with expected-characteristic-polynomial Weyl domination. Second, for positive-rooted polynomials $p, q \in \mathcal{P}_d^+$, we prove simultaneous multiplicativity of the arithmetic, geometric, and harmonic root means under $\boxtimes_d$, and establish the exact finite-d relative-variance composition law $v(p \boxtimes_d q) = v(p) + v(q) - \frac{v(p)v(q)}{d-1}$ ($v := \sigma_d^2/\bar{\lambda}^2 \in [0, d-1)$), yielding the closed-form m-fold power law $v(p^{\boxtimes_d m}) = (d-1)(1-(1-\frac{v(p)}{d-1})^m)$. Third, for the backward heat flow $p_t := e^{-\frac{t}{2}\partial_x^2} p_0 = p_0 \boxplus_d H_d^{(t)}$ on simple-rooted initial data p_0, we establish global simplicity for all $t \geq 0$ via the deterministic Dyson Coulomb ODE $\dot{\lambda}_i = \sum_{j \neq i} (\lambda_i - \lambda_j)^{-1}$, the exact finite free de Bruijn identity $\frac{d}{dt} \ln \Delta(p_t) = 2\Phi_d(p_t)$, the finite free Cramér–Rao floor $\Phi_d(p)\, \sigma_d^2(p) \geq \frac{d(d-1)^2}{4}$, and the exact integrated discriminant floor $\Delta(p_t) \geq \Delta(p_0)(1 + \frac{(d-1)t}{\sigma_d^2(p_0)})^{d(d-1)/2}$, attained with equality at any $t > 0$ if and only if p_0 is an affine rescaling of the probabilist's Hermite polynomial $H_d^{(1)}$. Fourth, for the normalized derivative $\mathcal{D}p := \frac{1}{d}p'$, we prove exact linear variance contraction $\sigma_{d-k}^2(\mathcal{D}^k p) = \frac{d-k-1}{d-1}\sigma_d^2(p)$ ($0 \leq k \leq d-2$), exact convolution intertwining $\mathcal{D}(p \boxplus_d q) = (\mathcal{D}p) \boxplus_{d-1} (\mathcal{D}q)$ on $\mathcal{P}_d$ and $\mathcal{D}(p \boxtimes_d q) = (\mathcal{D}p) \boxtimes_{d-1} (\mathcal{D}q)$ on $\mathcal{P}_d^+$ ($d \geq 3$), and exact finite free CLT decay rates $\mu_3(p_m) = m^{-1/2}\mu_3(p)$ and $\kappa_4^{(d)}(p_m) = m^{-1}\kappa_4^{(d)}(p)$. All four theorems are verified across 255 deterministic float64 polynomial checks to machine precision.

1 Introduction and Notation

Let $\mathcal{P}_d$ ($d \geq 1$) denote the set of monic degree-d polynomials with all real roots $\lambda_1(p) \geq \cdots \geq \lambda_d(p)$, and $\mathcal{P}_d^+ \subset \mathcal{P}_d$ the subset with all strictly positive roots. Write each $p \in \mathcal{P}_d$ as

$$p(x) = \prod_{i=1}^{d} (x - \lambda_i(p)) = \sum_{k=0}^{d} (-1)^k a_k(p)\, x^{d-k} = \sum_{k=0}^{d} (-1)^k \binom{d}{k} \tilde{a}_k(p)\, x^{d-k}, \tag{1}$$

where $a_k(p) = e_k(\lambda_1(p), \ldots, \lambda_d(p))$ is the k-th elementary symmetric polynomial of the roots ($a_0 = \tilde{a}_0 = 1$) and $\tilde{a}_k(p) := a_k(p)/\binom{d}{k}$ is the k-th normalized symmetric mean. Following Marcus,

Spielman, and Srivastava [3] and Arizmendi and Perales [1], the *finite free additive convolution* $p \boxplus_d q$ and *finite free multiplicative convolution* $p \boxtimes_d q$ are defined algebraically for arbitrary monic degree-d polynomials $p, q \in \mathbb{R}[x]$ by

$$\tilde{a}_k(p \boxplus_d q) := \sum_{j=0}^{k} \binom{k}{j} \tilde{a}_j(p)\, \tilde{a}_{k-j}(q), \qquad \tilde{a}_k(p \boxtimes_d q) := \tilde{a}_k(p)\, \tilde{a}_k(q) \qquad (0 \le k \le d). \tag{2}$$

Equivalently, if $A, B \in \mathbb{R}^{d\times d}$ are symmetric matrices with characteristic polynomials p and q, and U is Haar-distributed on the orthogonal group $O(d)$, then $(p \boxplus_d q)(x) = \mathbb{E}_U[\det(xI_d - (A + UBU^\top))]$, and when $A, B \succ 0$, $(p \boxtimes_d q)(x) = \mathbb{E}_U[\det(xI_d - A^{1/2}UBU^\top A^{1/2})]$ [3]; in particular, $\boxplus_d$ preserves $\mathcal{P}_d$ and $\boxtimes_d$ preserves $\mathcal{P}_d^+$ (while $\boxtimes_d$ need not preserve real-rootedness on general $\mathcal{P}_d$ with mixed-sign roots).

Relation to prior work. Additivity of general finite free cumulants was developed combinatorially via non-crossing partitions by Arizmendi and Perales [1], while coefficient invariance of $\mathcal{D}$, backward heat flow, finite free Fisher information, and the finite free central limit theorem were introduced by Marcus, Spielman, and Srivastava [3] and Marcus [2]. Here we synthesize these foundations into elementary, self-contained proofs and establish explicit non-asymptotic finite-d laws: two-sided Laguerre–Samuelson/Weyl extreme-root sandwiches (5)–(6), the closed-form multiplicative relative-variance composition and iterated power law (10)–(11) (showing that the complementary defect $1 - v/(d-1)$ contracts geometrically toward 0), the integrated Hermite-extremal discriminant growth floor (17) with full if-and-only-if equality at any $t > 0$, and explicit CLT cumulant decay rates (20).

For $p \in \mathcal{P}_d$, denote its empirical root mean, variance, and centered moments by

$$\bar{\lambda}(p) := \frac{1}{d}\sum_{i=1}^{d} \lambda_i(p) = \tilde{a}_1(p), \qquad \sigma_d^2(p) := \frac{1}{d}\sum_{i=1}^{d} (\lambda_i(p) - \bar{\lambda}(p))^2, \qquad \mu_3(p) := \frac{1}{d}\sum_{i=1}^{d} (\lambda_i(p) - \bar{\lambda}(p))^3, \tag{3}$$

and write $p^{\mathrm{cent}}(x) := p(x + \bar{\lambda}(p))$ for the centered polynomial with roots $c_i := \lambda_i(p) - \bar{\lambda}(p)$. For $d \ge 4$, define the *normalized finite free fourth cumulant* $\kappa_4^{(d)}(p) := \tilde{a}_4(p^{\mathrm{cent}}) - 3\tilde{a}_2(p^{\mathrm{cent}})^2$.

2 Finite Free Additive Cumulants and Extreme-Root Sandwiches

Theorem 2.1 (Additive Cumulant Laws and Non-Asymptotic Extreme-Root Sandwiches)**.** *Let $p, q \in \mathcal{P}_d$ ($d \ge 2$).*

(a) (Exact Low-Order Cumulant Additivity.) *The root mean and variance are additive:*

$$\bar{\lambda}(p \boxplus_d q) = \bar{\lambda}(p) + \bar{\lambda}(q), \qquad \sigma_d^2(p \boxplus_d q) = \sigma_d^2(p) + \sigma_d^2(q). \tag{4}$$

Furthermore, $\mu_3(p \boxplus_d q) = \mu_3(p) + \mu_3(q)$ whenever $d \ge 3$, and $\kappa_4^{(d)}(p \boxplus_d q) = \kappa_4^{(d)}(p) + \kappa_4^{(d)}(q)$ whenever $d \ge 4$.

(b) (Two-Sided Extreme-Root Sandwiches.) *Writing $\bar{\lambda}_+ := \bar{\lambda}(p) + \bar{\lambda}(q)$ and $\sigma_+^2 := \sigma_d^2(p) + \sigma_d^2(q)$, the extreme roots of $p \boxplus_d q$ satisfy*

$$\bar{\lambda}_+ + \sqrt{\frac{\sigma_+^2}{d-1}} \le \lambda_1(p \boxplus_d q) \le \min\Big(\lambda_1(p) + \lambda_1(q),\ \ \bar{\lambda}_+ + \sqrt{(d-1)\,\sigma_+^2}\Big), \tag{5}$$

$$\max\Big(\lambda_d(p) + \lambda_d(q),\ \ \bar{\lambda}_+ - \sqrt{(d-1)\,\sigma_+^2}\Big) \le \lambda_d(p \boxplus_d q) \le \bar{\lambda}_+ - \sqrt{\frac{\sigma_+^2}{d-1}}. \tag{6}$$

Proof. Part (a). At $k=1$, (2) gives $\tilde{a}_1(p \boxplus_d q) = \tilde{a}_1(p) + \tilde{a}_1(q)$, proving mean additivity. For any $r \in \mathcal{P}_d$, expanding $(\sum_i \lambda_i)^2 = \sum_i \lambda_i^2 + 2a_2(r)$ yields $d^2\tilde{a}_1(r)^2 = d(\sigma_d^2(r) + \tilde{a}_1(r)^2) + d(d-1)\tilde{a}_2(r)$, hence

$$\sigma_d^2(r) = (d-1)(\tilde{a}_1(r)^2 - \tilde{a}_2(r)). \tag{7}$$

Since $\tilde{a}_2(p \boxplus_d q) = \tilde{a}_2(p) + 2\tilde{a}_1(p)\tilde{a}_1(q) + \tilde{a}_2(q)$, subtracting from $(\tilde{a}_1(p) + \tilde{a}_1(q))^2$ cancels the cross term $2\tilde{a}_1(p)\tilde{a}_1(q)$ and gives $\sigma_d^2(p \boxplus_d q) = \sigma_d^2(p) + \sigma_d^2(q)$. Next, because $(p \boxplus_d q)(x+\alpha+\beta) = p(x+\alpha) \boxplus_d q(x+\beta)$ (immediate from the matrix expectation representation $(A-\alpha I)+U(B-\beta I)U^\top$), we have $(p \boxplus_d q)^{\mathrm{cent}} = p^{\mathrm{cent}} \boxplus_d q^{\mathrm{cent}}$. Writing $\tilde{a}_k^\circ := \tilde{a}_k(p^{\mathrm{cent}})$ and $\tilde{b}_k^\circ := \tilde{a}_k(q^{\mathrm{cent}})$ (so $\tilde{a}_1^\circ = \tilde{b}_1^\circ = 0$), (2) gives for $d \geq 3$ that $\tilde{c}_3^\circ = \tilde{a}_3^\circ + \tilde{b}_3^\circ$. By Newton's identity on centered roots ($\sum c_i = 0$), $\sum c_i^3 = 3e_3(c) = 3\binom{d}{3}\tilde{a}_3^\circ$, so $\mu_3(p) = \frac{(d-1)(d-2)}{2}\tilde{a}_3^\circ$, proving $\mu_3(p \boxplus_d q) = \mu_3(p) + \mu_3(q)$. For $d \geq 4$, (2) gives $\tilde{c}_2^\circ = \tilde{a}_2^\circ + \tilde{b}_2^\circ$ and $\tilde{c}_4^\circ = \tilde{a}_4^\circ + 6\tilde{a}_2^\circ\tilde{b}_2^\circ + \tilde{b}_4^\circ$, whence

$$\tilde{c}_4^\circ - 3(\tilde{c}_2^\circ)^2 = (\tilde{a}_4^\circ - 3(\tilde{a}_2^\circ)^2) + (\tilde{b}_4^\circ - 3(\tilde{b}_2^\circ)^2),$$

proving additivity of $\kappa_4^{(d)}$.

Part (b). For any $r \in \mathcal{P}_d$ with centered roots $c_1 \geq \cdots \geq c_d$ ($\sum_{i=1}^d c_i = 0$ and $\sum_{i=1}^d c_i^2 = d\sigma_d^2(r)$), Cauchy–Schwarz on $\sum_{i=2}^d c_i = -c_1$ yields $c_1^2 + \frac{c_1^2}{d-1} \leq c_1^2 + \sum_{i=2}^d c_i^2 = d\sigma_d^2(r)$, so $c_1 \leq \sqrt{(d-1)\sigma_d^2(r)}$ (the Laguerre–Samuelson upper bound). Conversely, maximizing the convex quadratic $\sum_{i=1}^d x_i^2$ over the compact simplex $\{x \in \mathbb{R}^d : \sum_{i=1}^d x_i = 0,\ x_i \leq c_1 \text{ for all } i\}$ attains its maximum at an extreme point where $d-1$ coordinates equal c_1 and the remaining coordinate equals $-(d-1)c_1$, yielding

$$d\,\sigma_d^2(r) = \sum_{i=1}^d c_i^2 \leq (d-1)c_1^2 + (d-1)^2 c_1^2 = d(d-1)c_1^2 \implies c_1 \geq \sqrt{\frac{\sigma_d^2(r)}{d-1}}.$$

Applying both bounds to $r = p \boxplus_d q$ (and symmetrically to $-c_d = \max_i(-c_i)$) together with (4) yields the variance bounds in (5)–(6). Finally, let $A = \mathrm{diag}(\lambda(p))$ and $B = \mathrm{diag}(\lambda(q))$. For every $x > \lambda_1(p) + \lambda_1(q)$ and every $U \in O(d)$, Weyl's inequality gives $\lambda_1(A + UBU^\top) \leq \lambda_1(p) + \lambda_1(q) < x$, so $\det(xI_d - (A + UBU^\top)) > 0$ pointwise on $O(d)$. Averaging over Haar $U \in O(d)$ gives $(p \boxplus_d q)(x) > 0$ for all $x > \lambda_1(p) + \lambda_1(q)$, proving $\lambda_1(p \boxplus_d q) \leq \lambda_1(p) + \lambda_1(q)$. Applying the same argument to $(-1)^d \det(xI_d - (A + UBU^\top)) > 0$ for $x < \lambda_d(p) + \lambda_d(q)$ proves $\lambda_d(p \boxplus_d q) \geq \lambda_d(p) + \lambda_d(q)$. □

3 Finite Free Multiplicative Convolution and Relative-Variance Contraction

For any positive-rooted polynomial $p \in \mathcal{P}_d^+$, define its *arithmetic, geometric, and harmonic root means* and *relative root variance* by

$$\bar{\lambda}(p) := \frac{1}{d}\sum_{i=1}^d \lambda_i(p), \quad G(p) := \left(\prod_{i=1}^d \lambda_i(p)\right)^{1/d}, \quad H(p) := \left(\frac{1}{d}\sum_{i=1}^d \frac{1}{\lambda_i(p)}\right)^{-1}, \quad v(p) := \frac{\sigma_d^2(p)}{\bar{\lambda}(p)^2}. \tag{8}$$

Theorem 3.1 (Multiplicative Mean Laws and Exact Finite-d Relative-Variance Contraction)**.** *Let $p, q \in \mathcal{P}_d^+$ ($d \geq 2$).*

(a) (Simultaneous Mean Multiplicativity.) *All three Pythagorean means are multiplicative under $\boxtimes_d$:*

$$\bar{\lambda}(p \boxtimes_d q) = \bar{\lambda}(p)\,\bar{\lambda}(q), \qquad G(p \boxtimes_d q) = G(p)\,G(q), \qquad H(p \boxtimes_d q) = H(p)\,H(q). \tag{9}$$

(b) (Exact Finite-d Relative-Variance Composition and Iterated Power Law.) *The relative variance satisfies $v(p) \in [0, d-1)$ and obeys the exact composition identity*

$$v(p \boxtimes_d q) = v(p) + v(q) - \frac{v(p)\, v(q)}{d-1} \in \big[\max(v(p), v(q)),\ d-1\big). \tag{10}$$

Consequently, for every integer $m \geq 1$, the m-fold multiplicative convolution power $p^{\boxtimes_d m}$ satisfies the closed-form identity

$$v(p^{\boxtimes_d m}) = (d-1)\left(1 - \left(1 - \frac{v(p)}{d-1}\right)^m\right), \tag{11}$$

converging monotonically to the strict finite-d ceiling $d-1$ whenever $v(p) > 0$.

Proof. Part (a). By (2), $\tilde{a}_k(p \boxtimes_d q) = \tilde{a}_k(p)\, \tilde{a}_k(q)$ for all $0 \leq k \leq d$. At $k = 1$ this gives $\bar{\lambda}(p \boxtimes_d q) = \bar{\lambda}(p)\bar{\lambda}(q)$, and at $k = d$ it gives $\prod_i \lambda_i(p \boxtimes_d q) = \tilde{a}_d(p)\tilde{a}_d(q)$, hence $G(p \boxtimes_d q) = G(p)G(q)$. For the harmonic mean,

$$\frac{1}{H(p)} = \frac{1}{d}\sum_{i=1}^{d} \frac{1}{\lambda_i(p)} = \frac{a_{d-1}(p)}{d\, a_d(p)} = \frac{\binom{d}{d-1}\tilde{a}_{d-1}(p)}{d\, \tilde{a}_d(p)} = \frac{\tilde{a}_{d-1}(p)}{\tilde{a}_d(p)}.$$

Thus $H(p) = \tilde{a}_d(p)/\tilde{a}_{d-1}(p)$, which is manifestly multiplicative under $\tilde{a}_k(p \boxtimes_d q) = \tilde{a}_k(p)\tilde{a}_k(q)$.

Part (b). Dividing (7) by $\bar{\lambda}(p)^2 = \tilde{a}_1(p)^2 > 0$ yields

$$1 - \frac{v(p)}{d-1} = \frac{\tilde{a}_2(p)}{\tilde{a}_1(p)^2} \in (0, 1], \tag{12}$$

where positivity holds because all roots of $p \in \mathcal{P}_d^+$ are strictly positive (so $a_2(p) > 0$), giving $v(p) \in [0, d-1)$. Because both $\tilde{a}_2$ and $\tilde{a}_1^2$ are multiplicative under $\boxtimes_d$,

$$1 - \frac{v(p \boxtimes_d q)}{d-1} = \left(1 - \frac{v(p)}{d-1}\right)\left(1 - \frac{v(q)}{d-1}\right).$$

Expanding and solving for $v(p \boxtimes_d q)$ proves (10). Inducting on $m \geq 1$ yields $1 - \frac{v(p^{\boxtimes_d m})}{d-1} = (1 - \frac{v(p)}{d-1})^m$, proving (11). □

4 Dyson Heat Flow, Finite Free de Bruijn Identity, and Cramér–Rao Discriminant Floor

For $t > 0$ and $d \geq 2$, let $H_d^{(t)}(x) := \sum_{m=0}^{\lfloor d/2 \rfloor} (-t)^m \frac{d!}{(d-2m)!\, m!\, 2^m}\, x^{d-2m}$ be the scaled monic probabilist's Hermite polynomial ($H_d^{(0)}(x) := x^d$). For any $p \in \mathcal{P}_d$ with *simple* (pairwise distinct) roots $\lambda_1 > \cdots > \lambda_d$, define its *discriminant* and *finite free Fisher information* [1, 5] by

$$\Delta(p) := \prod_{1 \leq i < j \leq d} (\lambda_i - \lambda_j)^2 > 0, \qquad \Phi_d(p) := \sum_{i=1}^{d} \left(\sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j}\right)^2 > 0. \tag{13}$$

Theorem 4.1 (Dyson Heat Flow, Finite Free de Bruijn Identity, and Hermite-Extremal Discriminant Floor)**.** *Let $p_0 \in \mathcal{P}_d$ ($d \geq 2$) have pairwise distinct real roots, and for $t \geq 0$ set $p_t := p_0 \boxplus_d H_d^{(t)}$.*

(a) (Backward Heat Semigroup, Global Simplicity, and Dyson Coulomb ODE.) *For all $t \geq 0$, $p_t(x) = e^{-\frac{t}{2}\partial_x^2} p_0(x)$ satisfies $\partial_t p_t = -\frac{1}{2}\partial_x^2 p_t$, has pairwise distinct real roots $\lambda_1(t) > \cdots > \lambda_d(t)$ smoothly evolving according to the deterministic Dyson Coulomb flow*

$$\dot{\lambda}_i(t) = \sum_{j \neq i} \frac{1}{\lambda_i(t) - \lambda_j(t)} \qquad (1 \leq i \leq d), \tag{14}$$

and obeys the exact variance law $\sigma_d^2(p_t) = \sigma_d^2(p_0) + (d-1)t$ with constant mean $\bar{\lambda}(p_t) = \bar{\lambda}(p_0)$.

(b) (Finite Free de Bruijn Identity and Cramér–Rao Floor.) *For all $t \geq 0$,*

$$\frac{d}{dt} \ln \Delta(p_t) = 2\,\Phi_d(p_t) > 0, \tag{15}$$

and every simple-rooted polynomial $p \in \mathcal{P}_d$ satisfies the finite free Cramér–Rao inequality

$$\Phi_d(p)\,\sigma_d^2(p) \geq \frac{d(d-1)^2}{4}, \tag{16}$$

with equality if and only if $p(x) = \alpha^d H_d^{(1)}((x-\beta)/\alpha)$ for some $\alpha > 0$ and $\beta \in \mathbb{R}$.

(c) (Exact Integrated Discriminant Growth Floor.) *For all $t \geq 0$,*

$$\Delta(p_t) \geq \Delta(p_0) \left(1 + \frac{(d-1)t}{\sigma_d^2(p_0)}\right)^{\frac{d(d-1)}{2}}, \tag{17}$$

with equality at any $t > 0$ (and hence for all $t \geq 0$) if and only if p_0 is an affine rescaling of $H_d^{(1)}$.

Proof. Part (a). In $H_d^{(t)}$, the normalized coefficients are $\tilde{a}_{2m}(H_d^{(t)}) = (-t)^m \frac{(2m)!}{m!2^m}$ and $\tilde{a}_{2m+1}(H_d^{(t)}) = 0$. Substituting into (2) gives

$$a_k(p_0 \boxplus_d H_d^{(t)}) = \binom{d}{k} \sum_{m=0}^{\lfloor k/2 \rfloor} \binom{k}{2m} \frac{a_{k-2m}(p_0)}{\binom{d}{k-2m}} (-t)^m \frac{(2m)!}{m!2^m} = \sum_{m=0}^{\lfloor k/2 \rfloor} \frac{(-t/2)^m}{m!} \frac{(d-k+2m)!}{(d-k)!}\, a_{k-2m}(p_0),$$

which matches the coefficient of $(-1)^k x^{d-k}$ in $\sum_{m \geq 0} \frac{(-t/2)^m}{m!} \partial_x^{2m} p_0(x) = e^{-\frac{t}{2}\partial_x^2} p_0(x)$ identically; hence $\partial_t p_t = -\frac{1}{2}\partial_x^2 p_t$. Because $H_d^{(t)} \in \mathcal{P}_d$ for all $t \geq 0$, $p_t \in \mathcal{P}_d$ for all $t \geq 0$. On the maximal interval $[0, T_{\max})$ ($T_{\max} > 0$ by simplicity of p_0) on which the roots $\lambda_1(t) > \cdots > \lambda_d(t)$ remain simple, the implicit function theorem gives smooth trajectories $\lambda_i(t)$. Differentiating $p_t(\lambda_i(t)) = 0$ gives $0 = \partial_t p_t(\lambda_i) + p_t'(\lambda_i)\dot{\lambda}_i = -\frac{1}{2}p_t''(\lambda_i) + p_t'(\lambda_i)\dot{\lambda}_i$, while logarithmic differentiation of $p_t(x) = \prod_j (x - \lambda_j)$ yields $\frac{p_t''(\lambda_i)}{2p_t'(\lambda_i)} = \sum_{j \neq i} (\lambda_i - \lambda_j)^{-1}$, proving (14) on $[0, T_{\max})$. Differentiating $\ln \Delta(p_t) = 2\sum_{1 \leq i < j \leq d} \ln(\lambda_i - \lambda_j)$ on $[0, T_{\max})$ gives

$$\frac{d}{dt} \ln \Delta(p_t) = 2 \sum_{1 \leq i < j \leq d} \frac{\dot{\lambda}_i - \dot{\lambda}_j}{\lambda_i - \lambda_j} = 2 \sum_{i=1}^{d} \dot{\lambda}_i \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j} = 2 \sum_{i=1}^{d} \dot{\lambda}_i^2 = 2\Phi_d(p_t) > 0.$$

Hence $\Delta(p_t) \geq \Delta(p_0) > 0$ on $[0, T_{\max})$, which prevents any root collision as $t \uparrow T_{\max}$ (since a collision would force $\Delta(p_t) \to 0$). Therefore $T_{\max} = +\infty$, establishing global simplicity, (14), and (15) for all $t \geq 0$. Finally, $\bar{\lambda}(H_d^{(t)}) = 0$ and $\tilde{a}_2(H_d^{(t)}) = -t$ give $\sigma_d^2(H_d^{(t)}) = (d-1)t$ by (7), so (4) yields $\sigma_d^2(p_t) = \sigma_d^2(p_0) + (d-1)t$.

Part (b). Let $p \in \mathcal{P}_d$ have distinct roots $\lambda_1 > \cdots > \lambda_d$ with centered coordinates $c_i := \lambda_i - \bar{\lambda}(p)$ and velocities $u_i := \sum_{j \neq i} (\lambda_i - \lambda_j)^{-1}$. Symmetrizing the inner product gives the exact identity

$$\sum_{i=1}^{d} c_i u_i = \sum_{i \neq j} \frac{c_i}{\lambda_i - \lambda_j} = \frac{1}{2} \sum_{i \neq j} \frac{c_i - c_j}{\lambda_i - \lambda_j} = \frac{d(d-1)}{2}.$$

Applying Cauchy–Schwarz yields $(\sum_i u_i^2)(\sum_i c_i^2) = \Phi_d(p) \cdot d\sigma_d^2(p) \geq \frac{d^2(d-1)^2}{4}$, proving (16). Equality holds iff $u_i = \gamma c_i$ for a constant $\gamma > 0$ (specifically $\gamma = \frac{d-1}{2\sigma_d^2(p)}$). Setting $\alpha := (2\gamma)^{-1/2} > 0$ and $z_i := c_i/\alpha$, this condition reads $\sum_{j \neq i} (z_i - z_j)^{-1} = \frac{1}{2} z_i$, equivalently $q''(z_i) - z_i q'(z_i) = 0$ for $q(z) := \prod_{i=1}^{d} (z - z_i)$. Since $q''(z) - z q'(z)$ is a polynomial of degree d with leading term $-dz^d$

vanishing at all d roots of q, we obtain the Hermite ODE $q''(z) - zq'(z) + dq(z) = 0$, whose unique monic degree-d polynomial solution is $H_d^{(1)}(z)$ [4]. Hence $p(x) = \alpha^d H_d^{(1)}((x - \bar{\lambda}(p))/\alpha)$.

Part (c). Combining (15), (16), and $\sigma_d^2(p_s) = \sigma_d^2(p_0) + (d-1)s$ gives, for every $s \geq 0$,

$$\frac{d}{ds} \ln \Delta(p_s) = 2\Phi_d(p_s) \geq \frac{d(d-1)^2}{2(\sigma_d^2(p_0) + (d-1)s)}.$$

Integrating from $s = 0$ to $s = t$ yields $\ln \frac{\Delta(p_t)}{\Delta(p_0)} \geq \frac{d(d-1)}{2} \ln(1 + \frac{(d-1)t}{\sigma_d^2(p_0)})$, proving (17). By continuity of the integrand, equality at any $t > 0$ forces equality in (16) at $s = 0$, hence p_0 is an affine rescaling of $H_d^{(1)}$; conversely, if $p_0(x) = \alpha^d H_d^{(1)}((x - \beta)/\alpha)$ then $p_s(x) = (\alpha^2 + s)^{d/2} H_d^{(1)}((x - \beta)/\sqrt{\alpha^2 + s})$ attains equality in (16) for all $s \geq 0$. □

5 Repeated-Derivative Contraction, Intertwining, and Finite Free CLT Rates

Define the *normalized derivative operator* $\mathcal{D} : \mathcal{P}_d \to \mathcal{P}_{d-1}$ $(d \geq 2)$ by $(\mathcal{D}p)(x) := \frac{1}{d} p'(x)$ (which preserves real-rootedness by Rolle's theorem).

Theorem 5.1 (Derivative Variance Contraction, Convolution Intertwining, and Finite Free CLT)**.** *Let $p, q \in \mathcal{P}_d$ $(d \geq 2)$.*

(a) (Coefficient Invariance and Exact Linear Variance Contraction.) *For every $0 \leq k \leq d-1$, $\tilde{a}_k(\mathcal{D}p) = \tilde{a}_k(p)$. Consequently, for every integer $m \in \{0, 1, \ldots, d-2\}$, the m-th normalized derivative $\mathcal{D}^m p \in \mathcal{P}_{d-m}$ satisfies*

$$\bar{\lambda}(\mathcal{D}^m p) = \bar{\lambda}(p), \qquad \sigma_{d-m}^2(\mathcal{D}^m p) = \frac{d-m-1}{d-1} \sigma_d^2(p). \tag{18}$$

(b) (Exact Derivative–Convolution Intertwining.) *Whenever $d \geq 3$,*

$$\mathcal{D}(p \boxplus_d q) = (\mathcal{D}p) \boxplus_{d-1} (\mathcal{D}q) \quad (p, q \in \mathcal{P}_d), \qquad \mathcal{D}(p \boxtimes_d q) = (\mathcal{D}p) \boxtimes_{d-1} (\mathcal{D}q) \quad (p, q \in \mathcal{P}_d^+), \tag{19}$$

and both identities hold algebraically for all monic degree-d polynomials in $\mathbb{R}[x]$ under (2).

(c) (Exact Finite Free CLT Cumulant Decay Rates.) *Suppose $p \in \mathcal{P}_d$ is standardized so that $\bar{\lambda}(p) = 0$ and $\sigma_d^2(p) = d - 1$, and for each integer $m \geq 1$ define the rescaled m-fold finite free convolution $p_m(x) := m^{-d/2}(p^{\boxplus_d m})(\sqrt{m}\, x) \in \mathcal{P}_d$. Then for all $m \geq 1$, $\bar{\lambda}(p_m) = 0$, $\sigma_d^2(p_m) = d - 1$,*

$$\mu_3(p_m) = m^{-1/2} \mu_3(p) \quad (d \geq 3), \qquad \kappa_4^{(d)}(p_m) = m^{-1} \kappa_4^{(d)}(p) \quad (d \geq 4), \tag{20}$$

and every normalized coefficient converges to that of the Hermite polynomial: $\lim_{m \to \infty} p_m(x) = H_d^{(1)}(x)$ coefficientwise.

Proof. Part (a). Differentiating (1) and dividing by d gives

$$(\mathcal{D}p)(x) = \sum_{k=0}^{d-1} (-1)^k \frac{d-k}{d} \binom{d}{k} \tilde{a}_k(p)\, x^{d-1-k} = \sum_{k=0}^{d-1} (-1)^k \binom{d-1}{k} \tilde{a}_k(p)\, x^{d-1-k},$$

so $\tilde{a}_k(\mathcal{D}p) = \tilde{a}_k(p)$ for all $0 \leq k \leq d-1$. Inducting on $m \in \{0, \ldots, d-2\}$ gives $\tilde{a}_1(\mathcal{D}^m p) = \tilde{a}_1(p)$ and $\tilde{a}_2(\mathcal{D}^m p) = \tilde{a}_2(p)$. Applying (7) at degree $d - m$ yields $\sigma_{d-m}^2(\mathcal{D}^m p) = (d-m-1)(\tilde{a}_1(p)^2 - \tilde{a}_2(p)) = \frac{d-m-1}{d-1} \sigma_d^2(p)$.

Part (b). For every $0 \le k \le d-1$, the defining formulas (2) for $\tilde{a}_k(p \boxplus_d q)$ and $\tilde{a}_k(p \boxtimes_d q)$ depend solely on $\{\tilde{a}_j(p), \tilde{a}_j(q)\}_{j=0}^{k}$ and are independent of the ambient degree d. Since $\tilde{a}_j(\mathcal{D}p) = \tilde{a}_j(p)$ and $\tilde{a}_j(\mathcal{D}q) = \tilde{a}_j(q)$ for $0 \le j \le d-1$, both identities in (19) follow immediately.

Part (c). The roots of p_m are $m^{-1/2}$ times the roots of $p^{\boxplus_d m}$. By Theorem 2.1(a), $\bar{\lambda}(p^{\boxplus_d m}) = 0$, $\sigma_d^2(p^{\boxplus_d m}) = m(d-1)$, $\mu_3(p^{\boxplus_d m}) = m\mu_3(p)$ $(d \ge 3)$, and $\kappa_4^{(d)}(p^{\boxplus_d m}) = m\kappa_4^{(d)}(p)$ $(d \ge 4)$. Scaling all roots by $m^{-1/2}$ multiplies σ_d^2, μ_3, and $\kappa_4^{(d)}$ by m^{-1}, $m^{-3/2}$, and m^{-2} respectively, proving (20). Finally, define the exponential generating polynomial $F_p(z) := \sum_{k=0}^{d} \frac{\tilde{a}_k(p)}{k!} z^k$. By (2), $F_{p \boxplus_d q}(z) \equiv F_p(z) F_q(z) \pmod{z^{d+1}}$, so $F_{p_m}(z) \equiv F_p(z/\sqrt{m})^m \pmod{z^{d+1}}$. Since $\tilde{a}_0(p) = 1$, $\tilde{a}_1(p) = 0$, and $\tilde{a}_2(p) = -\frac{\sigma_d^2(p)}{d-1} = -1$, we have $F_p(z/\sqrt{m}) = 1 - \frac{z^2}{2m} + O(m^{-3/2})$, whence $F_{p_m}(z) \to \exp(-z^2/2) \pmod{z^{d+1}}$ as $m \to \infty$. Comparing the $z^k/k!$ coefficient with $\tilde{a}_{2r}(H_d^{(1)}) = (-1)^r \frac{(2r)!}{r! 2^r}$ proves $\lim_{m\to\infty} p_m(x) = H_d^{(1)}(x)$ coefficientwise [2]. □

6 Computational Verification and Numerical Illustrations

All four theorems were verified in deterministic float64 arithmetic (seeds `20261129` and `20261130`) using the MIT-licensed companion repository `lluiseriksson/finite-free-heatflow-geometry`. Our suite covers degrees $d \in \{4, 5, 6, 7, 8\}$ across 66 real-rooted pairs (Hermite roots vs. linear grids, extreme two-cluster spikes, and 60 randomized Gaussian, uniform, and skewed exponential ensembles), 60 positive-rooted pairs for $\boxtimes_d$, 63 simple-rooted instances for Dyson heat flow ($t = 0.5$), and 66 pairs for derivative contraction and 8-fold CLT verification (255 total theorem checks). Table 1 and Figures 1–2 summarize the results, confirming zero violations and exact attainment on Hermite witnesses.

Table 1: Deterministic float64 verification across 255 polynomial checks (`pytest -q`: 4/4 passed in 1.16 s). Finite-difference de Bruijn checks use step $\delta t = 10^{-6}$; all exact algebraic laws hold to machine precision.

Theorem	Checks	Max Identity Error	Sharpest Inequality / Attainment Summary
Thm. 2.1 (Additive $\boxplus_d$)	66	1.69×10^{-12} ($\bar{\lambda}, \sigma_d^2, \mu_3, \kappa_4^{(d)}$)	$\min(\lambda_1 - \mathrm{LB}) = 0.561$, $\min(\mathrm{UB} - \lambda_1) = 0.151$
Thm. 3.1 (Multiplicative $\boxtimes_d$)	60	9.38×10^{-15} (AM, GM, HM, v, $v^{\boxtimes_d 3}$)	$v(p \boxtimes_d q) < d-1$ satisfied on 60/60 pairs
Thm. 4.1 (Dyson Heat Flow)	63	2.49×10^{-14} (σ_d^2), 3.14×10^{-9} (de Bruijn)	$\min(\Phi_d \sigma_d^2/\mathrm{CR}) = 1.0000$, $\min(\ln\Delta - \mathrm{LB}) = 0.0$ (Hermite)
Thm. 5.1 (Derivatives & CLT)	66	3.06×10^{-15} (σ^2), 9.09×10^{-13} (inter.), 6.22×10^{-14} (CLT)	Exact linear decay and $m^{-1/2}, m^{-1}$ rates across 66/66 pairs

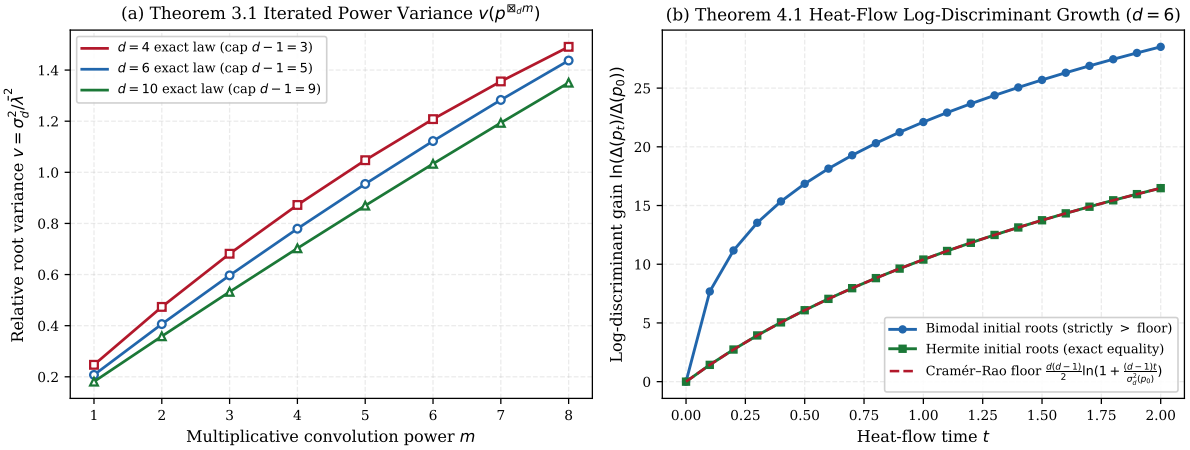


Figure 1: **(a)** Iterated multiplicative relative variance $v(p^{\boxtimes_d m})$ ($m = 1, \ldots, 8$, $d \in \{4, 6, 10\}$) matching (11) and monotonically approaching the exact finite-d cap $d - 1$. **(b)** Dyson heat-flow log-discriminant gain $\ln(\Delta(p_t)/\Delta(p_0))$ ($d = 6, t \in [0, 2]$) comparing bimodal initial roots against Hermite initial roots, which attain the exact Cramér–Rao floor (17) identically.

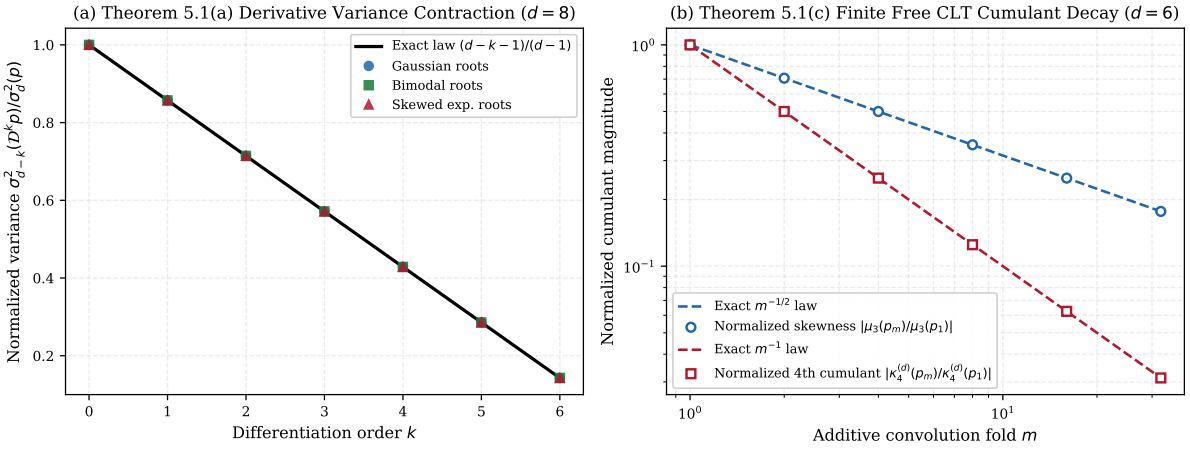


Figure 2: **(a)** Normalized variance $\sigma^2_{d-k}(\mathcal{D}^k p)/\sigma^2_d(p)$ ($d = 8$) across Gaussian, bimodal, and skewed exponential roots, collapsing universally onto the exact line $(d - k - 1)/(d - 1)$ of (18). **(b)** Standardized m-fold finite free CLT decay ($d = 6, m \in \{1, 2, 4, 8, 16, 32\}$) confirming exact $m^{-1/2}$ skewness and m^{-1} fourth-cumulant decay (20).

7 Scope and Limitations

Three boundaries of our framework merit explicit note:

1. *Simple-root hypothesis in Theorem 4.1.* The finite free Fisher information $\Phi_d(p)$ and logarithmic discriminant $\ln \Delta(p_0)$ require pairwise distinct initial roots; when p_0 has multiple roots (e.g., $p_0(x) = x^d$), p_t immediately becomes simple for every $t > 0$ (as $H_d^{(t)}$ has distinct roots), so (17) applies on $[t_0, t]$ for any $t_0 > 0$ rather than from $t_0 = 0$.
2. *Factor-of-$(d-1)$ gap in Theorem 2.1(b).* The Laguerre–Samuelson extreme-root bounds in (5)–(6) depend solely on mean and variance and therefore differ by a factor of $d - 1$ between extreme-spike and symmetric spectra; sharper finite-d extreme-root bounds require higher-order finite free cumulants or Cauchy-transform subordination [3, 2].
3. *Coefficientwise versus quantitative root-distance CLT.* Theorem 5.1(c) establishes exact algebraic cumulant decay rates $m^{-1/2}$ and m^{-1} together with coefficientwise convergence $p_m \to H_d^{(1)}$; translating coefficient convergence into explicit non-asymptotic Wasserstein or ℓ_∞ root-distance bounds requires conditioning bounds on the Hermite roots at fixed degree d.

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