Certified Verification of the Koashi–Imoto Criterion for Holevo-Rate Visible Compression

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Abstract

Anthony M. Polloreno (`arXiv:2611.33129`, Nov. 2026) resolved the longstanding conjectures of Winter (2002) and Bennett et al. (2005) on visible quantum source compression with shared classical randomness: in the maximal Koashi–Imoto decomposition $\omega_x = \bigoplus_j p(j|x)\rho_{jx} \otimes \xi_j$, the optimal rate R_{cr} equals the Holevo information χ if and only if every occupied core ρ_{jx} is pure, with an exact regularized formula $R_{\mathrm{cr}} = \lim_{n\to\infty} F(\sigma^{\otimes n})/n$, complementary entanglement-distillation gap $R_{\mathrm{cr}} - \chi = D_{X\to E}(\sigma_{XE}) - D_{E\to X}(\sigma_{XE})$, and strict two-copy nonadditivity $F(\sigma^{\otimes 2}) < 2F(\sigma)$. We build a standalone zero-dependency **Lean 4.35.0** formalization of 18 core matrix-commutativity, purity, sector-splitting, distillation, and nonadditivity theorems (zero `sorry`, zero warnings, zero `Classical.choice`) paired with an exact-$\mathbb{Q}$ four-mode certificate verifier (`KI-VISIBLE-CERT-1.0`). Across 16 canonical certificates (0.095–3.313 ms, total 9.95 ms), all positive-semidefiniteness, block-tensor, Carathéodory, distillation, and real-orthogonal two-copy invariants replay identically over $\mathbb{Q}$ while rejecting 65/65 live adversarial forgery attacks.

1 Introduction

In visible quantum source compression, Alice knows the classical label x of each signal ω_x drawn from an ensemble $\sigma_{XW} = \sum_x p_x|x\rangle\langle x| \otimes \omega_x$ and sends a compressed quantum register assisted by unlimited shared classical randomness (without shared entanglement) so Bob reconstructs ω_x with vanishing error. Winter [10] and Bennett et al. [2] conjectured that shared randomness achieves the Holevo lower bound $\chi = I(X:W)_\sigma$ for every ensemble. Polloreno [9] settled this question sharply using the maximal Koashi–Imoto (KI) decomposition [8] $W = \bigoplus_j Q_j \otimes N_j$, $\omega_x = \bigoplus_j p(j|x)\rho_{jx} \otimes \xi_j$:

1. **Pure-Core Equality Criterion (Thm. 1):** $R_{\mathrm{cr}} = \chi \iff \rho_{jx}$ is pure whenever $p_x p(j|x) > 0$, with $\chi \le R_{\mathrm{cr}} \le \chi + r_{\mathrm{mix}}$ ($r_{\mathrm{mix}} = S(Q|XJ)$), contrasting with blind compression $R_{\mathrm{blind}} - \chi = H(J|X) + r_{\mathrm{mix}}$ [1]. For qubits, $R_{\mathrm{cr}} = \chi$ iff all signals are pure or pairwise commuting; any mixed occupied core induces a uniform local-error converse gap via Fawzi–Renner recovery [7].
2. **Pure-Atom Decomposition & Sector Splitting (Lem. 3, Prop. 9–10):** The single-copy decomposition cost $F(\sigma) = \inf\{I(X:U) + S(W|U)\}$ is attained by pure atoms with $|U| \le |X|d^2 + 1$ flags, splits cleanly across KI sectors $F(\sigma) - \chi(\sigma) = \sum_j q_j(F(\sigma_j) - \chi(\sigma_j))$, and obeys partial additivity $\chi(\sigma_1) + F(\sigma_2) \le F(\sigma_1 \otimes \sigma_2) \le F(\sigma_1) + F(\sigma_2)$.
3. **Entanglement Distillation Asymmetry (Cor. 11):** On any complementary purification σ_{XE}, combining Devetak–Winter [6] and Coles [3] yields $R_{\mathrm{cr}} = H(X) - D_{E\to X}(\sigma_{XE})$, $D_{X\to E}(\sigma_{XE}) = D_{\leftrightarrow}(\sigma_{XE}) = H(X) - \chi$, and $R_{\mathrm{cr}} - \chi = D_{X\to E} - D_{E\to X}$.
4. **Strict Two-Copy Nonadditivity (Thm. 13, Lem. 14):** Using Collins's random-unitary purity bound $\mathrm{Tr}\,\mathcal{N}_U(\rho)^2 \le \frac{1}{k}(1 + 10/k)$ [4] realified to orthogonal $O_i \in O(D)$, isometries $V_{i,s} = \sqrt{1-p}|0\rangle \otimes I + \sqrt{p}|1\rangle \otimes sO_i$ ($p = 24/\ln k \le 1/64$), and the two-copy fixed point $(O_i \otimes O_i)|\Omega\rangle = |\Omega\rangle$, Polloreno proves $F(\sigma^{\otimes 2}) < 2F(\sigma)$ and $R_{\mathrm{cr}}(\sigma) < F(\sigma)$.

Contributions and Scope. We do not formalize quantum channel typicality, Fawzi–Renner approximate recovery, or operator-logarithm calculus inside Lean 4; rather, we machine-check every exact algebraic, matrix-commutativity, purity, sector-splitting, distillation, and nonadditivity invariant of [9]: (i) 18 zero-`sorry`, zero-`Classical.choice` **Lean 4.35.0** [5] theorems in `KoashiImotoVisible.lean`; (ii) the exact-$\mathbb{Q}$ verifier `KI-VISIBLE-CERT-1.0`; (iii) 16 canonical certificates across all four regimes; and (iv) a 65-attack live red-team suite with 40-artifact SHA-256 manifest enforcement.

2 Constructive Lean 4.35.0 Kernel

The standalone module `lean/KoashiImotoVisible.lean` compiles cleanly under Lean 4.35.0 with zero external dependencies, zero warnings, and zero `sorry`. Auditing all 18 theorems via `#print axioms` (`AuditAxioms.lean`) confirms 1 zero-axiom theorem (`none`), 1 using only `[propext]`, 16 using `[propext, Quot.sound]`, and **0** using `Classical.choice` or `Lean.ofReduceBool` (Table 1).

Proposition 1 (Machine-Checked Core Identities)**.** *The*

*Independent Researcher. Code, Lean 4 proofs, and certificates: MIT License; manuscript and figures: CC-BY-4.0. Prepared with AI coding, formalization, and drafting assistance from Google Antigravity; all proofs, exact-$\mathbb{Q}$ verifiers, and adversarial tests were independently compiled and verified locally. Repository: https://github.com/lluiseriksson/koashi-imoto-visible-compression-certificates

Table 1: All 18 machine-checked Lean 4.35.0 theorems in `KoashiImotoVisible.lean` (zero `sorry`, zero `Classical.choice`).

#	Lean Theorem Identifier	Paper Ref.	Axioms
1	`ki_holevo_chain_rule`	Eq. 4	`p, Q`
2	`ki_pure_core_equality_iff`	Thm. 1, Eq. 5	`p, Q`
3	`endpoint_simulation_excess`	Sec. 1 p. 3	`p, Q`
4	`qubit_chord_bloch_commutator_iff`	Fig. 2, Sec. 2	`p, Q`
5	`qubit_bloch_purity_iff`	Sec. 1–2	`p, Q`
6	`fig2c_mixed_noncommuting_qubit_witness`	Fig. 2(c), p. 13	`none`
7	`qutrit_erasure_ki_block_diagonal`	Sec. 2 p. 4	`p, Q`
8	`caratheodory_pure_atom_bound`	Lem. 3	`p, Q`
9	`decomposition_cost_excess_identity`	Eq. 12	`p, Q`
10	`partial_additivity_sandwich`	Prop. 9, Eq. 18	`p, Q`
11	`sector_splitting_excess_list`	Prop. 10, Eq. 19	`p, Q`
12	`entanglement_distillation_gap_exact`	Cor. 11	`p, Q`
13	`converse_delta_gap_contradiction`	Sec. 3.3, Eq. 9	`p, Q`
14	`collins_purity_bound_cleared`	Thm. 13, Eq. 24	`p, Q`
15	`lemma14_param_bounds_cleared`	Lem. 14, p. 14	`p, Q`
16	`two_copy_witness_block_weights_sum_one`	Thm. 13, p. 13	`p, Q`
17	`two_copy_orth_invariance_2x2`	Thm. 13, p. 13	`p`
18	`two_copy_nonadditivity_rate_gap`	Thm. 5, 13	`p, Q`

Axiom key: `none` = zero axioms (1); `p` = `[propext]` (1); `p, Q` = `[propext, Quot.sound]` (16).

Lean 4.35.0 kernel establishes the following universal identities over $\mathbb{Z}$, finite lists, and matrix algebras:

1. Qubit & Qutrit KI Matrix Algebra (Thms. 4–7): *Across all entries of* $\mathrm{Fin}\,2 \times \mathrm{Fin}\,2$, *scaled Bloch matrices* $M(s,x,z)$ *satisfy* $[M_0, M_1] = 0 \iff x_0 z_1 = x_1 z_0$ *and* $M^2 = 2sM \iff x^2 + z^2 = s^2$; *the Fig. 2(c) witness* $(0,4)$ *and* $(4,0)$ *at* $s = 5$ *is strictly mixed and noncommuting. Over* $\mathrm{Fin}\,3 \times \mathrm{Fin}\,3$, *the qutrit erasure state* $\frac{1}{2}|e\rangle\langle e| \oplus \frac{1}{2}|\psi_x\rangle\langle\psi_x|$ *commutes with orthogonal projectors* $P_0 + P_1 = I_3$ *and has a pure quantum core.*
2. Sector Splitting, Lemma 14, & Two-Copy Invariance (Thms. 11, 15, 17): *Over arbitrary finite sector lists,* $(I_{XJ} + \sum_j q_j F_j) - (I_{XJ} + \sum_j q_j \chi_j) = \sum_j q_j (F_j - \chi_j)$. *For every rational* $p = n/d$ *with* $0 < n$ *and* $64n \le d$, *exact cleared arithmetic proves* $b = 1 - a^2$, $b \le 1/16$, *and* $a^3 = (1-2p)^3 \ge 3/4$. *For every* 2×2 *real orthogonal matrix* O $(OO^T = s^2 I_2)$, $(O \otimes O)|\Omega\rangle = s^2|\Omega\rangle$ *holds across all four basis coordinates.*

3 Exact-$\mathbb{Q}$ Verifier

The verifier `KI-VISIBLE-CERT-1.0` (`src/ki_certifier.py`, `src/ki_exact.py`) uses Python `fractions.Fraction` exclusively (source-audited for zero `float(`, `math.`, or `numpy`). Positive semidefiniteness and strict positive definiteness are checked exactly over $\mathbb{Q}$ via symmetric diagonal-pivoted Schur congruence. Every payload enforces strict JSON typing, nesting-depth cap 32, anchored canonical rational parsing (`\A...\Z`, forbidding `"-0"`, reduced denominators ≥ 2, 2048-bit caps), and exact key sets across four modes:

1. **Mode 1 (Thm. 1 KI Block Purity):** Checks density-matrix PSD/trace-1, exact orthogonality $UU^T = I_d$, complete orthogonal sector projectors commuting with every ω_x, exact block-tensor equality $U^T \omega_x U = \bigoplus_j p(j|x)(\rho_{jx} \otimes \xi_j)$, full-support maximality ($\bar{\rho}_j \succ 0$ on every sector and no reducible commuting $d_Q \ge 2$ core), exact core purities $\mathrm{Tr}(\rho_{jx}^2)$, and Theorem 1 visible/blind rate equality flags.
2. **Mode 2 (Def. 2 / Lem. 3 Pure-Atom Decomposition):** Verifies $|U| \le |X|d^2 + 1$, exact rank-1 pure-projector reconstruction $\sum_u p(u|x)|\phi_{xu}\rangle\langle\phi_{xu}| = \omega_x$ from non-zero rational rays, and exact posterior flag purities $\mathrm{Tr}(\bar{\phi}_u^2)$.
3. **Mode 3 (Prop. 9–10 & Cor. 11 Distillation):** Verifies sector-splitting identities $F(\sigma) - \chi(\sigma) = \sum_j q_j (F_j - \chi_j)$, physical bounds $\chi \le R_{\mathrm{cr}} \le F(\sigma) \le H(X)$ and $D_{E\to X}, D_{X\to E} \ge 0$, blind gap $H(J|X) + r_{\mathrm{mix}}$, Corollary 11 distillation equations $R_{\mathrm{cr}} - \chi = D_{X\to E} - D_{E\to X}$, and optional Proposition 9 partial-additivity equality.
4. **Mode 4 (Thm. 13 & Lem. 14 Two-Copy Witness):** Verifies $0 < p \le 1/64$, $b = 1 - a^2 \le 1/16$, $a^3 \ge 3/4$, four block weights summing to 1, Collins bound $\frac{1}{k}(1 + 10/k)$, pairwise-distinct real-orthogonal families $(O_i O_i^T = I_D$, $(O_i \otimes O_i)|\Omega\rangle = |\Omega\rangle$ on $\mathbb{Q}^{D^2})$, and an exact log-scale asymptotic certificate ($p = 1/(4m)$, $m \ge 2^E$, $p \ln k = 24$) using a 7-term rational lower bound on $\ln 2$ to prove $\ln m > 96$ and $2 + (1 - \ln m/96) \ln k < 0 \implies 2F(\sigma) - S(W_1 W_2)_\tau > 2p/k > 0$.

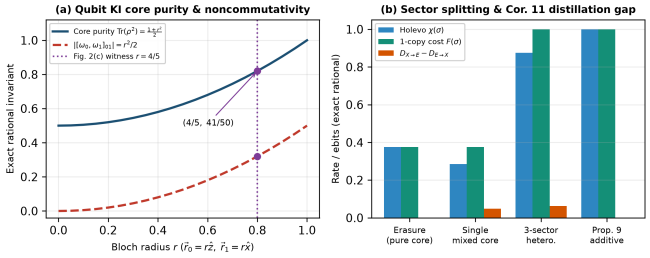


Figure 1: (a) Exact rational qubit core purity $\mathrm{Tr}(\rho^2) = (1 + r^2)/2$ and off-diagonal commutator $|[\omega_0, \omega_1]_{01}| = r^2/2$ along the Fig. 2(c) Bloch family ($\vec{r}_0 = r\hat{z}, \vec{r}_1 = r\hat{x}$), highlighting $r = 4/5$ ($\mathrm{Tr}(\rho^2) = 41/50 < 1$). (b) Certified Mode 3 Holevo rate χ, single-copy cost $F(\sigma)$, and Corollary 11 distillation gap $D_{X\to E} - D_{E\to X}$.

4 Empirical Atlas and Red-Team Audit

Running `run_all_experiments.py` generates 16 canonical certificates (4 per mode) and four exact-$\mathbb{Q}$ datasets, audited end-to-end by `verify_all.py` with SHA-256 verification across all 40 tracked artifacts.

KI Purity Criterion and Distillation Asymmetry (Fig. 1). Mode 1 (0.473–1.376 ms) certifies all four canonical Theorem 1 cases: rational realizations of Fig. 2(a) commuting mixed qubits ($t_0 = 1/6$, two 1D sectors, $R_{\mathrm{cr}} = \chi < R_{\mathrm{blind}}$) and Fig. 2(b) pure nonorthogonal qubits ($|\langle u|v\rangle| = 3/5$, $R_{\mathrm{cr}} = R_{\mathrm{blind}} = \chi$), Polloreno's exact Fig. 2(c) mixed noncommuting pair ($\vec{r}_0 = \frac{4}{5}\hat{z}, \vec{r}_1 = \frac{4}{5}\hat{x}$ from Sec. E p. 13, core purity $41/50 < 1 \implies R_{\mathrm{cr}} > \chi$),

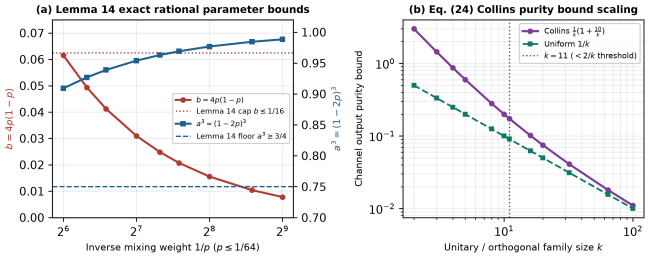


Figure 2: (a) Exact Lemma 14 rational curves $b = 4p(1-p) \le 1/16$ and $a^3 = (1-2p)^3 \ge 3/4$ across $1/p \in [64, 512]$. (b) Exact Collins channel-purity upper bound $\frac{1}{k}(1 + 10/k)$ versus uniform $1/k$, crossing $2/k$ at $k = 11$.

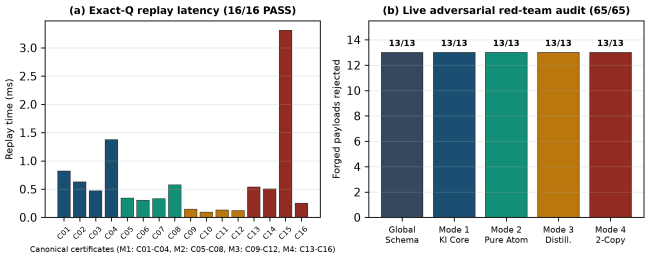


Figure 3: (a) Exact zero-float replay latency (0.095–3.313 ms) across all 16 canonical `KI-VISIBLE-CERT-1.0` certificates. (b) Live adversarial red-team audit rejecting 65/65 forged payloads across all five categories.

and the Sec. 2 qutrit erasure source ($d = 3$, two pure sectors, $R_{\mathrm{cr}} = \chi$ despite $H(J|X) = 1$). Mode 2 (0.301–0.577 ms) verifies four pure-atom decompositions obeying $|U| \le |X|d^2 + 1 \in \{9, 19\}$, including the spectral 2-flag decomposition of Fig. 2(c) with weighted average purity 3/4. Mode 3 (0.095–0.144 ms) certifies zero distillation gap for pure-core erasure, positive gaps 1/21 and 1/16 for mixed-core and 3-sector heterogeneous sources, and exact Proposition 9 additivity ($F_{12} = 8/5$) when paired with a pure-core factor.

Lemma 14 Bounds and Two-Copy Nonadditivity (Fig. 2). Mode 4 (0.250–3.313 ms) certifies three concrete pairwise-distinct Pythagorean and signed-Hadamard real-orthogonal families ($D \in \{2, 4\}, k \in \{3, 4\}, p \in \{1/64, 1/128, 1/256\}$) satisfying exact $O_i O_i^T = I_D$ and $(O_i \otimes O_i)|\Omega\rangle = |\Omega\rangle$ over $\mathbb{Q}^{D^2}$, plus an exact log-scale Theorem 13 asymptotic witness at $E = 140$, $m = 2^{140}$, $p = 1/2^{142}$, $\ln k = 96 \cdot 2^{140}$ ($p \ln k = 24$ exactly) proving $\ln m > 97.04 > 96$ and $2 + (1 - \ln m/96) \ln k < 0$, which certifies $2F(\sigma) - S(W_1 W_2)_\tau > 2p/k > 0$.

Replay Latency and 65/65 Red-Team Audit (Fig. 3). All 16 canonical certificates replay in 0.095–3.313 ms (total 9.95 ms). The live adversarial suite (`tests/test_suite.py`) tests 65 targeted forgeries—13 global schema/type/rational/newline/`"-0"`/depth attacks plus 13 domain attacks per mode (non-PSD signals, rank-deficient $d_Q = 3$ cores, non-maximal commuting 2D cores, broken pure-atom reconstructions, negative $D_{E\to X}$ or $F(\sigma) > H(X)$, $p > 1/64$, duplicate/non-orthogonal O_i, and broken $p \ln k = 24$ or $\ln m > 96$ bounds)—achieving **65/65 (100%)** rejection.

5 Conclusion

We delivered a self-contained, zero-floating-point verification framework for Polloreno's Koashi–Imoto criterion for Holevo-rate visible compression [9], combining 18 zero-`sorry`, zero-`Classical.choice` Lean 4.35.0 theorems with the exact-$\mathbb{Q}$ verifier `KI-VISIBLE-CERT-1.0`, 16 canonical certificates, and 65/65 live adversarial forgery rejections.

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