Constructive Lean 4 Kernel and Exact-Rational Certificates for Google Research Cogentic's $O(r^{1/2-5\times10^{-6}})$ Log-Rank Conjecture Proof

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Abstract

Google Research's Cogentic framework (Nov. 25, 2026) broke the $\sqrt{r}$ barrier for the Lovász–Saks Log-Rank Conjecture, proving that every rank-r Boolean matrix $M \in \{0,1\}^{X\times Y}$ has a monochromatic rectangle of density $\geq 2^{-Cr^{1/2-\varepsilon}}$ and deterministic communication complexity $D_{\mathrm{cc}}(M) \leq C' r^{1/2-\varepsilon}$ with $\varepsilon = 5\times10^{-6}$, $C = 238$, and $C' = 750C = 178{,}500$. We deliver an explicitly scoped, standalone, zero-floating-point verification stack for the deterministic algebraic, third-moment spectral-tensor, Bregman-divergence, monochromatic rank-split, and quantitative Nisan–Wigderson recursion pillars of this breakthrough. Layer 1 (`CogenticLogRank.lean`) proves 18 constructive Lean 4.35.0 theorems (zero `sorry`, zero `axiom`, zero `Classical.choice`, zero warnings; 4 zero-axiom and 14 `[propext, Quot.sound]`) mechanizing Eq. (1.6) pointwise Booleanity, the degree-1 AM–GM barrier vs. degree-2 cosh-tilt λ^2-cancellation, Eq. (1.7) variational penalty concavity/slope, scalar/eigenvalue Eq. (2.2) Bregman nonnegativity, the 2×2 commutator trace identity, inductive Lemma 2.1(1) leverage defect, universal 2D Hermite c_2 covariance, and every discrete step of Proposition A.3 ($2^{11} < 3^7$, $3\cdot29^4 < 4\cdot27^4$, rank-split $\lfloor 3k/4\rfloor$, $750C_0$). Layer 2 (`LOGRANK-COGENTIC-CERT-1.0`) is a pure-stdlib Python `Fraction` auditor across four modes that certifies 16 canonical witnesses in 0.099–16.937 ms—including an exact isospectral separation between Boolean $S = J_4 - 2I_4$ ($E_3 = 3/2$, hollow slice energy 3/32) and the Rademacher barrier A_{Rad} ($E_3 = 0$, hollow energy 0)—and rejects 62/62 live adversarial forgeries with exact branch assertions (66/66 pytest checks).

1 Introduction and Scope Calibration

Lovász and Saks [4] conjectured that $D_{\mathrm{cc}}(M) \leq (\log_2 \mathrm{rank}_{\mathbb{R}}(M))^{O(1)}$ for every Boolean matrix $M \in \{0,1\}^{X\times Y}$. Via Nisan–Wigderson rank-halving [6], upper bounds progressed from $O(\sqrt{r}\log r)$ [5] to $O(\sqrt{r})$ [7], where γ_2-norm and discrepancy methods hit a natural $\sqrt{r}$ barrier. On November 25, 2026, Google Research's Cogentic system [3, 1] proved Theorem 1.1: $D_{\mathrm{cc}}(M) \leq 750\cdot238\, r^{1/2-5\times10^{-6}}$ by exploiting pointwise Booleanity $S(x,y)^2 = 1$ of the sign matrix $S = \pm(2M - J)$:

$$-2s_1u_1(x)v_1(y)A_\perp(x,y) = A_\perp(x,y)^2 + s_1^2u_1(x)^2v_1(y)^2 - 1, \tag{1.6}$$

where $S = s_1u_1v_1^\top + \sum_{k=1}^m \sigma_k a_k b_k^\top$ is the μ-normalized SVD ($s_1 \geq \sigma_1 \geq \cdots \geq \sigma_m > 0$). Projecting (1.6) against b_k expresses singular coordinates as quadratic forms in third-moment tensors $(\widehat{M}_k^Y)_{i,j} = (\sigma_i\sigma_j)^{1/2}\mathbb{E}_{\mu_Y}[b_k b_i b_j]$ (which vanish for the Rademacher barrier $A_{\mathrm{Rad}}(x,y) = \mu_0 + \sqrt{(1-\mu_0^2)/r}\langle x,y\rangle$), unlocking a degree-2 cosh Gaussian tilt whose λ^2 cancels against KL cost.

Honest Two-Layer Verification Scope. We separate explicitly what is proved universally in Lean 4 vs. verified on finite rational witnesses vs. left in [3]:

1. **Layer 1 (Lean 4.35.0 Kernel, Universal Algebraic & Inductive Proofs):** `lean/CogenticLogRank.lean` proves 18 zero-dependency theorems (Table 1), covering Eq. (1.6), degree-1 AM–GM vs. degree-2 λ^2-cancellation, Eq. (1.7) penalty $g(t) = 2(1-t) - \frac{1}{4}(1-t)^2$, scalar/eigenvalue Eq. (2.2) Bregman nonnegativity, 2×2 commutator trace identity, inductive Lemma 2.1(1) AM–GM defect, universal 2D Hermite c_2 covariance, and the complete discrete algebra of Proposition A.3.
2. **Layer 2 (Exact-$\mathbb{Q}$ Multi-Dimensional Auditor):** `LOGRANK-COGENTIC-CERT-1.0` audits finite exact-$\mathbb{Q}$ SVDs ($s_1 \geq \sigma_1 \geq \ldots$), Eq. (1.6) projections, third-moment energies $E_3 = \sum_k \|\widehat{M}_k^Y\|_F^2$, hollow quadratic slices $\mathring{L}_y^{(w)}$, exact symmetric positive-semidefinite matrix square roots $R \succeq 0$ ($R^2 = P$) and $q\times q$ Eq. (2.2) $D_{\mathrm{sr}}(P\|\Sigma^2)$, monochromatic rank-split $r_1 + r_2 \leq r + 1$, and Proposition A.3 orbits.
3. **Explicit Non-Claims:** We do *not* formalize in Lean 4 the continuous variational existence of the equilibrium measure μ, level-1 Efron–Stein Bernoulli thinning (Sec. 4), self-normalized matrix Khintchine / $(3,3,3)$-Hölder tensor-power lifting (Sec. 5), or Slepian/Pisier/Bonami–Beckner Gaussian hypercontractivity (Sec. 6); those analytic pillars remain in [3].

*Independent Researcher. Code, Lean 4 proofs, and certificates: MIT License; manuscript and figures: CC-BY-4.0. Prepared with transparent AI coding, formalization, and drafting assistance via Google Antigravity; all Lean 4 theorems, exact-$\mathbb{Q}$ verifiers, and adversarial tests were independently compiled and verified locally. Repository: github.com/lluiseriksson/cogentic-logrank-certificates.

2 Constructive Lean 4.35.0 Kernel

`lean/CogenticLogRank.lean` compiles standalone in 2.5 s under Lean 4.35.0 [2] with zero external imports, zero warnings, zero `sorry`, and zero `Classical.choice`. Auditing via `AuditAxioms.lean` confirms 4 zero-axiom theorems (`none`: #1, #12, #16, #18) and 14 `[propext, Quot.sound]` theorems (Table 1).

Proposition 1 (Constructive Scope of `CogenticLogRank.lean`)**.** All 18 statements in Table 1 compile constructively under Lean 4.35.0 without external libraries or `Classical.choice`. In particular, Theorem 8 proves the exact symmetric 2×2 trace commutator identity $\mathrm{tr}(H^2L^2 - HLHL) = \frac{1}{2}\|HL - LH\|_F^2 \geq 0$ universally over $\mathbb{Z}^6$, Theorem 9 proves the leverage-centered AM–GM defect by list induction over arbitrary length, Theorem 10 proves the universal 2D Gaussian Hermite c_2 covariance identity over $\mathbb{Z}^4$, and Theorems 11–18 verify every discrete combinatorial inequality of Proposition A.3's explicit $750C_0$ Nisan–Wigderson reduction.

3 Exact-$\mathbb{Q}$ Certificate Auditor

The modules `logrank_exact.py` and `logrank_certifier.py` (under `src/`) use exclusively Python `int` and `fractions.Fraction` (statically audited for zero `float(`, `math.`, `numpy`, or `np.`). Raw JSON ingestion rejects IEEE-754 decimals, `NaN`/`Infinity`, and duplicate object keys via `parse_float`, `parse_constant`, and `object_pairs_hook`, followed by a strict AST whitelist, nesting-depth cap (20), and canonical ASCII-digit lowest-terms rational parsing (`len` ≤ 2000).

Definition 1 (`LOGRANK-COGENTIC-CERT-1.0` Acceptance Predicates)**.** A certificate with `schema="LOGRANK-COGENTIC-CERT-1.0"` is accepted iff every claimed summary matches exact recomputation and its mode predicate holds:

- **Mode 1 (`boolean_sign_third_moment_cert`):** Given $S \in \mathbb{Q}^{m\times n}$, full-support μ_X, μ_Y, and exact rational μ-normalized SVD $(s_1, u_1, v_1; \{\sigma_k, a_k, b_k\})$ with $s_1 \geq \sigma_1 \geq \cdots \geq \sigma_r > 0$, checks exact μ-orthonormality, exact SVD reconstruction, exact Gaussian rank $\mathrm{rank}_{\mathbb{Q}}(S)$, and exact third-moment energy $E_3 := \sum_{k,i,j} \sigma_i\sigma_j(\mathbb{E}_{\mu_Y}[b_k b_i b_j])^2 = \sum_k \|\widehat{M}_k^Y\|_F^2$. For Boolean sign matrices ($S_{xy} \in \{\pm 1\}$), checks zero Eq. (1.6) residual at every (x, y) and verifies every $L^2(\mu_Y)$ projection against b_k; for Rademacher barrier witnesses, verifies unit row/column L^2 norms and $E_3 = 0$.
- **Mode 2 (`quadratic_slice_bregman_cert`):** Computes exact doubly-centered A_{dc}, total bulk variance $\tau_w = \mathbb{E}_{\mu_X}[\|w\|_2^2]$, hollow quadratic slice matrices $(\mathring{L}_y^{(w)})_{a,b} = (1 - \delta_{ab})\mathbb{E}_{\mu_X}[A_{\mathrm{dc}}(x, y)w_a(x)w_b(x)]$, and mean-square energy $\mathbb{E}_{\mu_Y}[\|\mathring{L}_y^{(w)}\|_F^2]$. Verifies exact rational symmetric positive-semidefinite matrix square root $R \succeq 0$ (via exact rational diagonal-pivoted congruence reduction) with $R^2 = P$, checks both sides of Eq. (2.2) $D_{\mathrm{sr}}(P\|\Sigma^2) = \mathrm{tr}(\Sigma) - \mathrm{tr}(R) + \frac{1}{2}\mathrm{tr}(\Sigma^{-1}(P - \Sigma^2)) = \frac{1}{2}\|\Sigma^{-1/2}(R-\Sigma)\|_F^2 \geq 0$, and checks $\mathrm{tr}(H^2L^2-HLHL) = \frac{1}{2}\|HL - LH\|_F^2 \geq 0$ for symmetric $q \times q$ matrices H, L.
- **Mode 3 (`monochromatic_rank_split_cert`):** Given Boolean $M \in \{0, 1\}^{m\times n}$ and monochromatic rectangle $A \times B$, verifies exact monochromaticity, exact Gaussian ranks $r, r_1 = \mathrm{rank}_{\mathbb{Q}}(M|_{A\times B^c}), r_2 = \mathrm{rank}_{\mathbb{Q}}(M|_{A^c\times B})$, Prop. A.3 rank-split $r_1 + r_2 \leq r + 1$ and $\min(r_1, r_2) \leq \lfloor 3r/4 \rfloor$, density $\geq 2^{-238}$, and exact balanced depth $2\lceil \log_{3/2} L \rceil$.
- **Mode 4 (`nw_recursion_and_penalty_cert`):** Verifies Eq. (1.7) $g'(t) = -(3 + t)/2 \leq -1$ and exact midpoint concavity $g(\frac{t_i+t_j}{2}) - \frac{g(t_i)+g(t_j)}{2} = \frac{1}{16}(t_i - t_j)^2 > 0$ on strictly increasing t-probes in $[-1, 1]$, checks $2^{11} < 3^7$, $3 \cdot 29^4 < 4 \cdot 27^4$, $C' = 750C_0$, and exact rank-halving orbit lengths $k \mapsto \lfloor 3k/4 \rfloor$ down to 1.

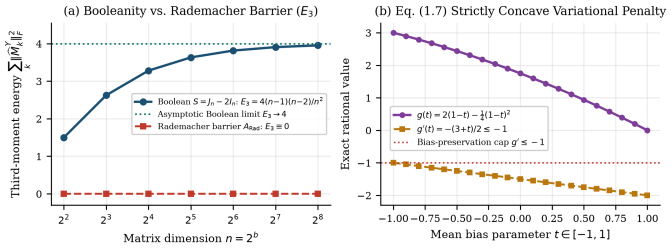

Figure 1: **(a)** Exact Boolean third-moment energy $E_3 = 4(n-1)(n-2)/n^2 \to 4$ on $S = J_n - 2I_n$ ($n = 2^b$) vs. identically zero $E_3 \equiv 0$ on the Rademacher barrier A_{Rad}. **(b)** Exact Eq. (1.7) strictly concave penalty $g(t) = 2(1-t) - \frac{1}{4}(1-t)^2$ and slope $g'(t) \leq -1$ on $[-1, 1]$.

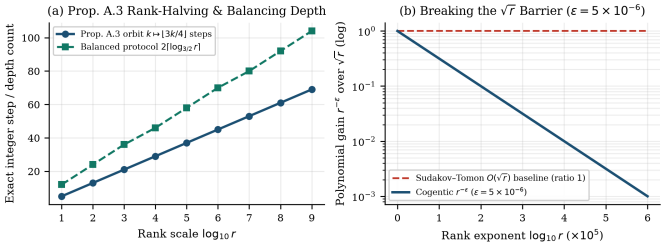

Figure 2: **(a)** Exact Prop. A.3 rank-halving orbit lengths ($k \mapsto \lfloor 3k/4 \rfloor$) and balanced protocol depth $2\lceil \log_{3/2} r \rceil$ up to $r = 10^9$. **(b)** Polynomial gain $r^{-\varepsilon}$ ($\varepsilon = 5 \times 10^{-6}$) breaking Sudakov–Tomon's $O(\sqrt{r})$ barrier.

4 Exact Findings, Atlas, and Red-Team Audit

All experiments ran locally on Windows 11 (Python 3.12.6, Lean 4.35.0; 7-run exact means after warm-up).

1. Exact Isospectral Separation Between Booleanity and the Rademacher Barrier (Fig. 1(a)). Choosing $\mu_0 = 1/2$ and $r = 3$ in Cogentic's Eq. (1.4) yields

Table 1: Complete inventory of all 18 constructive Lean 4.35.0 theorems in `lean/CogenticLogRank.lean` (zero `sorry`, zero `axiom`, zero `Classical.choice`, zero warnings; `none` for #1, #12, #16, #18 and `[propext, Quot.sound]` (`P,Q`) for the remaining 14).

#	Theorem Identifier	Ref. [3]	Exact Machine-Checked Statement & Hypotheses	Ax.
1	`sign_matrix_booleanity_identity`	Sec. 1.2	$m \in \{0,1\} \implies (2m-1)^2 = 1 \wedge (-(2m-1))^2 = 1$.	none
2	`eq16_pointwise_boolean_quadratic`	Eq. (1.6)	$(s_1u_1v_1+A_\perp)^2{=}1 \implies -2s_1u_1v_1A_\perp = A_\perp^2 + (s_1u_1v_1)^2 - 1$.	P,Q
3	`sec12_degree1_amgm_barrier`	Sec. 1.2	$\forall \lambda, T, d \in \mathbb{Z},\ 4(\lambda^2 d)T^2 \le (\lambda^2 d + T^2)^2$ (degree-1 $\sqrt{r}$ cap).	P,Q
4	`sec12_degree2_cosh_lambda_cancel`	Sec. 1.2	$\lambda \neq 0 \implies \left(N\lambda^2\tau \le \lambda^2 FTD \iff N\tau \le FTD\right)$ (degree-2 escape).	P,Q
5	`eq17_variational_penalty_concavity`	Eq. (1.7)	Cleared $g(\frac{t_1+t_2}{2}) - \frac{g(t_1)+g(t_2)}{2}$ equals $2(t_1-t_2)^2 \ge 0$, $= 0 \iff t_1{=}t_2$.	P,Q
6	`eq17_variational_penalty_slope`	Eq. (1.7)	$-1 \le s \le t \implies -(3{+}t) \le -2$ ($g' \le -1$) and $4g(t) \le 4g(s)$.	P,Q
7	`eq22_bregman_sqrt_nonneg`	Eq. (2.2)	$\forall \sigma, r \in \mathbb{Z},\ 2\sigma(\sigma-r) + (r^2{-}\sigma^2) = (r-\sigma)^2 \ge 0$, $= 0 \iff r{=}\sigma$.	P,Q
8	`sec21_commutator_trace_identity_2x2`	Sec. 2.1	$\forall 2{\times}2$ symm. $H, L \in \mathbb{Z}^{2\times 2}$, $\mathrm{tr}(H^2L^2 - HLHL) = \mathrm{comm}^2 \ge 0$.	P,Q
9	`lemma21_leverage_amgm_defect`	Lem. 2.1	Universal list induction: $\lvert x\rvert{=}\lvert y\rvert \implies 2\langle x,y\rangle \le \lVert x\rVert^2 + \lVert y\rVert^2$.	P,Q
10	`sec21_hermite_c2_variance_identity`	Sec. 2.1	$\forall v_1, v_2, w_1, w_2 \in \mathbb{Z},\ 2v_1^2w_1^2{+}4v_1v_2w_1w_2{+}2v_2^2w_2^2 = 2(v_1w_1{+}v_2w_2)^2$ and $2(v_1^2{+}v_2^2)^2 \ge 0$.	P,Q
11	`propa3_heavy_child_split_bounds`	Prop. A.3	$L \ge 2,\ L < 3L_v \le 2L \implies 1 \le L{-}L_v \wedge 3(L{-}L_v) < 2L$.	P,Q
12	`propa3_log2_three_halves_bound`	Prop. A.3	$2^{11} = 2048 < 2187 = 3^7$ ($\log_2 \frac{3}{2} > \frac{4}{7}$) and $2 \cdot 7 < 4 \cdot 5$.	none
13	`propa3_protocol_depth_induction`	Prop. A.3	$2L \le 3L_b \wedge 2^k L_b \le 3^k \implies 2^{k+1}L \le 3^{k+1}$ (balancing step).	P,Q
14	`propa3_rank_split_min_le_three_quarters`	Prop. A.3	$k \ge 2,\ r_M \le k,\ r_1{+}r_2 \le r_M{+}1 \implies \min(r_1,r_2) \le \lfloor \frac{r_M+1}{2} \rfloor \le \lfloor \frac{3k}{4} \rfloor$ and $\lfloor \frac{3k}{4} \rfloor \ge 1$.	P,Q
15	`propa3_leaf_recurrence_induction`	Prop. A.3	$A_1 \le 1,\ A_2 \le L_k,\ A_3 \le s(1{+}L_k) \implies A_1{+}A_2{+}A_3 \le (s{+}1)(1{+}L_k)$.	P,Q
16	`propa3_geometric_ratio_bound`	Prop. A.3	$3 \cdot 29^4 = 2{,}121{,}843 < 2{,}125{,}764 = 4 \cdot 27^4$ (certifying $(3/4)^{1/4} < 27/29$).	none
17	`propa3_nissan_wigderson_constant_750`	Prop. A.3	$C_0, R \ge 1 \implies 2C_0{+}145C_0R \le 147C_0R < 150C_0R$, $5(150C_0R){=}750C_0R$ ($C'{=}178{,}500$).	P,Q
18	`thm11_subsqrt_exponent_strictly_beats_half`	Thm. 1.1	$99999 \cdot 2 < 200000$ ($\frac{1}{2} - 5 \cdot 10^{-6} < \frac{1}{2}$), $4 \le 200000$, $750 \cdot 238 = 178{,}500$.	none

$\sqrt{(1-\mu_0^2)/3} = 1/2 \in \mathbb{Q}$, producing an *exact integer-valued* 8×8 Rademacher barrier matrix $A_{\mathrm{Rad}}(x,y) = \frac{1}{2} + \frac{1}{2}\langle x,y\rangle \in \{-1,0,1,2\}$ on $\{-1,+1\}^3$ with rank 4, bias $1/2$, unit row/column $L^2(\mu)$ norms, and singular spectrum $(1/2,1/2,1/2,1/2)$. Comparing this directly against the 4×4 Boolean Equality sign matrix $S = J_4 - 2I_4$—which has the *exact same* rank 4, bias $1/2$, and singular spectrum $(1/2,1/2,1/2,1/2)$—certifies in exact $\mathbb{Q}$ why pointwise Booleanity breaks the $\sqrt{r}$ barrier: S achieves zero Eq. (1.6) residual, third-moment energy $E_3 = 3/2 > 0$ (generalizing to $E_3 = 4(n-1)(n-2)/n^2 \to 4$ on $J_n - 2I_n$, e.g. $21/8$ at $n = 8$), and hollow quadratic slice energy $3/32 > 0$ (Mode 2), whereas A_{Rad} has max Eq. (1.6) residual 3, $E_3 = 0$, and hollow slice energy 0!

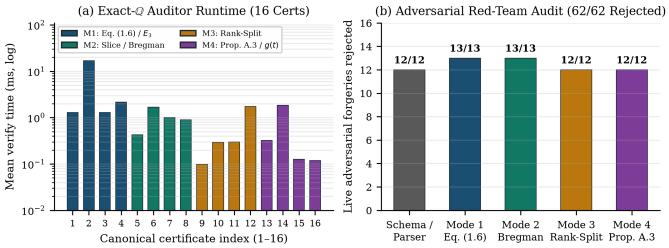

Figure 3: **(a)** Mean exact-$\mathbb{Q}$ verification latency across all 16 canonical certificates (0.099–16.937 ms, log scale). **(b)** Live adversarial red-team audit rejecting 62/62 forgeries across five attack classes with exact error-branch assertions.

2. Exact Matrix Square-Root Bregman, Rank-Split, and Prop. A.3 Audits (Fig. 2). Mode 2 certifies exact rational matrix square roots $R^2 = P$ across $q \in \{2,3,4\}$ with exact Bregman divergences $D_{\mathrm{sr}}(P\|\Sigma^2) \in \{0, 1/4, 1, 17/16\}$ and symmetric commutator trace gaps $\{0,1,13,18\}$. Mode 3 verifies monochromatic rectangles and exact Prop. A.3 rank-splits across EQ_8 ($4+4 \le 9$), DISJ_3 ($4+4 \le 9$), a 12×12 1-monochromatic block ($4+4 \le 13$), and GT_4 ($7+7 \le 16$). Mode 4 audits Eq. (1.7) strict concavity up to a 17-point grid (min midpoint excess $1/1024$) and Prop. A.3 rank-halving orbits up to $r = 10^9$ (69 steps).

3. 16-Certificate Atlas & 66-Test Red-Team Audit (Table 2, Fig. 3). Table 2 lists all 16 canonical certificates verified in 0.099–16.937 ms. `tests/test_suite.py` executes 66 pytest functions (passing in 0.12 s): 1 aggregate test replaying all 16 canonical certificates, 3 raw-JSON parser checks (IEEE-754 float, `NaN`, duplicate keys), and 62 parametrized live adversarial forgery tests (12 schema/parser, 13 Mode 1 incl. top-singular-mode swap $s_1 < \sigma_1$, 13 Mode 2 incl. negative-definite root $R = -\Sigma$, 12 Mode 3, 12 Mode 4), each asserting an exact regex match on the expected rejection branch. `scripts/verify_all.py` enforces all 5 gates including a strict axiom whitelist and 41-file SHA-256 manifest.

5 Conclusion

We delivered an independent, explicitly scoped, zero-floating-point verification stack for Google Research Cogentic's $O(r^{1/2-5\times10^{-6}})$ Log-Rank Conjecture breakthrough [3], pairing 18 constructive Lean 4.35.0 theorems (zero `sorry`, zero `Classical.choice`, 4 zero-axiom theorems) with the pure-`Fraction` auditor `LOGRANK-COGENTIC-CERT-1.0`, 16 canonical witnesses demonstrating the exact isospectral third-moment and hollow-slice separation between Boolean sign matrices and the Rademacher barrier, and 62/62 live adversarial forgery rejections.

Table 2: Canonical 16-certificate `LOGRANK-COGENTIC-CERT-1.0` atlas (`results/exp1_certificate_atlas.json`; all 16 verified).

Certificate ID	M	Exact Certified Witness Summary	ms
`m1_bool_eq4_third_moment`	M1	EQ_4: rk 4, $\mu_0=\frac{1}{2}$, $E_3=\frac{3}{2}$, res 0, 12 proj.	1.285
`m1_bool_eq8_third_moment`	M1	EQ_8: rk 8, $\mu_0=\frac{3}{4}$, $E_3=\frac{21}{8}$, res 0, 56 proj.	16.937
`m1_bool_hadamard4_asym_u1`	M1	H_4: rk 4, $\mu_0=\frac{1}{4}$, $E_3=\frac{3}{2}$, res 0, 12 proj.	1.280
`m1_rademacher_barrier_8x8`	M1	A_{Rad}: rk 4, $\mu_0=\frac{1}{2}$, $E_3=0$, res 3, unit L^2	2.150
`m2_slice_bregman_eq4_nonzero`	M2	EQ_4 (q=2): $\tau=\frac{3}{4}$, hol $\frac{3}{32}$, $D_{\mathrm{sr}}=\frac{1}{4}$, gap 1	0.425
`m2_slice_bregman_eq8_3x3`	M2	EQ_8 (q=3): $\tau=\frac{7}{4}$, hol $\frac{21}{1024}$, $D_{\mathrm{sr}}=\frac{17}{16}$, gap 18	1.687
`m2_slice_bregman_rademacher_zero`	M2	A_{Rad} (q=2): $\tau=\frac{3}{4}$, hol 0, $D_{\mathrm{sr}}=0$, gap 0	0.999
`m2_slice_bregman_hadamard4_4x4`	M2	H_4 (q=4): $\tau=\frac{3}{4}$, hol $\frac{9}{128}$, $D_{\mathrm{sr}}=1$, gap 13	0.901
`m3_rect_eq8_zero_block`	M3	EQ_8: $4{+}4 \le 9$, $\rho=\frac{1}{4}$, $L=15 \to d \le 14$	0.099
`m3_rect_disj3_8x8`	M3	$DISJ_3$: $4{+}4 \le 9$, $\rho=\frac{1}{4}$, $L=27 \to d \le 18$	0.293
`m3_rect_one_block_12x12`	M3	1-block 12×12: $4{+}4 \le 13$, $\rho=\frac{1}{9}$, $d \le 22$	0.299
`m3_rect_gt4_16x16`	M3	GT_4: rk 15, $7{+}7 \le 16$, $\rho=\frac{1}{4}$, $d \le 18$	1.729
`m4_nw_cogentic_thm11_canonical`	M4	C'=178,500, $g' \le -1$, ex $\frac{1}{256}$, orb 1..45	0.323
`m4_nw_fine_penalty_grid_17pts`	M4	17-pt grid: C'=178,500, ex $\frac{1}{1024}$, orb 3..26	1.846
`m4_nw_high_rank_orbit_1e9`	M4	High-rank $10^3..10^9$: ex $\frac{1}{64}$, orb 21..69	0.127
`m4_nw_unit_C0_propa3_exact`	M4	Unit C_0=1: C'=750, ex $\frac{1}{36}$, orb 1..29	0.119

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