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EXACT FINITE-TEMPERATURE SPATIAL STRING TENSION ENCLOSURES, ALL-β_τ $\mathbb{Z}_N$ COLUMN-CENTER REDUCTION, AND CERTIFIED AREA-LAW SURVIVAL IN D-DIMENSIONAL SU(N) LATTICE YANG–MILLS THEORY

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Abstract. We establish rigorous, non-perturbative lower bounds on the finite-temperature spatial string tension $\sigma_s^{(\rho)}(d, N, N_\tau, \beta_s, \beta_\tau)$ in pure d-dimensional anisotropic SU(N) Wilson lattice Yang–Mills theory on thermal slabs $\Lambda_L \times \mathbb{T}_{N_\tau}$ ($d \geq 3$, $N \geq 2$, $N_\tau \geq 1$) that hold *uniformly for all timelike couplings* $\beta_\tau \in [0, \infty)$, including arbitrarily large β_τ in the deconfined high-temperature and Hamiltonian ($a_\tau \to 0$) regimes:

(1) *Exact all-β_τ $\mathbb{Z}_N$ column-center reduction:* Because every timelike plaquette contains two oppositely oriented spatial links from the same vertical column $C_e = \{((x,t),i) : t \in \mathbb{T}_{N_\tau}\}$, the entire timelike Boltzmann weight is identically invariant under $\prod_{e \in E(\Lambda_L)} \mathbb{Z}_N$. Conditioning on U expresses any purely spatial Wilson loop of N-ality k_ρ as a positive-weight annealed average of a $(d-1)$-dimensional $\mathbb{Z}_N$ lattice gauge theory with random complex column fields $|X_f(U)| \leq N_\tau \beta_s$.

(2) *Unconditional all-coupling spatial area law in $2+1D$ for all* SU(N)*:* At $d = 3$, free-boundary $\mathbb{Z}_N$ plaquette factorization together with a four-line Riccati-ODE proof that $s \mapsto I_j(s)/I_{j-1}(s)$ is strictly increasing proves for *every* $N \geq 2$, $k_\rho \not\equiv 0 \pmod N$, $N_\tau \geq 1$, $\beta_s > 0$, and $\beta_\tau \in [0, \infty)$ that $\sigma_s^{(\rho)} \geq -\ln(1 - e^{-2N_\tau \beta_s}) > 0$ (sharpened to $-\ln\tanh(N_\tau\beta_s)$ for SU(2) and to $-\ln\min(\eta_N^{\mathrm{univ}}, \eta_{N,\|k_\rho\|_N})$ for $N \geq 3$).

(3) *Five-line Ginibre signed-coupling domination and zero-cluster-expansion* SU(2) *area law in all $d \geq 4$:* A five-line $\eta_e = \tau_e \sigma_e$ proof of $|\langle \sigma_A \rangle_{\{J_f\}}| \leq \langle \tau_A \rangle_{\{K_f\}}$ ($K_f \geq |J_f|$) yields $\sigma_s^{(j)}(\mathrm{SU}(2), d, N_\tau, \beta_s, \beta_\tau) \geq \sigma_{\mathbb{Z}_2}(d-1, N_\tau\beta_s)$ for all $\beta_\tau \geq 0$, whence positive-term cancellation and BFS Fuss–Catalan surface counting ($D = 3(2d-5)$, $B_{d-1} = D^D/(D-1)^{D-1}$) give $\sigma_s^{(j)} \geq -\ln(B_{d-1}\tanh(N_\tau\beta_s)) > 0$ without any cluster expansion.

(4) *All-β_τ $\mathbb{Z}_N$-graded Kotecký–Preiss confinement for all* SU(N) *($N \geq 3$) in $d \geq 4$ and exact $N_\tau = 1$ HCIZ pinning:* Pointwise $\mathbb{Z}_N$-graded cluster expansion on $\mathbb{Z}^{d-1}$ with the homological weight floor $\sum_{f \in S_0} \|q_f\|_N \geq \|k_\rho\|_N RS$ yields $\sigma_s^{(\rho)} \geq \|k_\rho\|_N(-\ln u) > 0$ uniformly in $\beta_\tau \in [0, \infty)$, complemented by spatial reflection-positivity upper bounds and the exact SU(2) $N_\tau = 1$ HCIZ timelike-link integral exhibiting $O(1/\beta_\tau)$ entropic pinning to $P_x = \pm I_2$. Twenty-four 120-digit Python interval certificates and a standalone zero-import, zero-axiom Lean 4.36.0 kernel verify the combinatorial lemmas and outward-rounded integer bounds.

1. Introduction and Honest Scope

A defining non-perturbative feature of finite-temperature Yang–Mills theory is the stark dichotomy between *temporal* and *spatial* Wilson loops above the deconfinement temperature T_c [12, 3, 2, 1]. While temporal Polyakov-loop correlators lose their area law in the deconfined plasma phase (spontaneous breaking of global $\mathbb{Z}_N$ temporal center symmetry), purely spatial Wilson loops lying within a fixed Euclidean time slice retain a strictly positive spatial string tension $\sigma_s > 0$ governed by dimensionally reduced $(d-1)$-dimensional magnetostatic confinement [2, 1].

In his foundational work [2], Borgs established an area law for spatial Wilson loops at sufficiently small $N_\tau\beta_s$ uniformly in the timelike coupling β_τ (including the continuous-time Hamiltonian limit $a_\tau \to 0$, where $\beta_\tau \propto 1/a_\tau \to \infty$ and $\beta_s \propto a_\tau \to 0$ with $N_\tau\beta_s = 2N/(g^2T)$ finite [9]). In this eleventh installment of our constructive lattice Yang–Mills series (following Papers IX–X [4, 5]), we give an elementary, completely self-contained, and machine-certified derivation of *explicit* all-β_τ

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spatial string tension lower bounds for all SU(N) ($N \geq 2$) and all dimensions $d \geq 3$ via exact $\mathbb{Z}_N$ column-center symmetry, a five-line Ginibre signed-coupling domination proof for SU(2), zero-cluster-expansion BFS Fuss–Catalan surface counting, and pointwise $\mathbb{Z}_N$-graded Kotecký–Preiss bounds for $N \geq 3$.

Remark 1.1 (Honest Scope and Non-Claims). All theorems apply to pure d-dimensional anisotropic SU(N) Wilson lattice Yang–Mills theory on finite thermal slabs $\Lambda_L \times \mathbb{T}_{N_\tau}$ ($d \geq 3$, $N \geq 2$, $N_\tau \geq 1$, periodic Euclidean time $\mathbb{T}_{N_\tau} = \mathbb{Z}/N_\tau\mathbb{Z}$) at fixed spatial lattice spacing $a_s = 1$. Specifically, Theorems 3.2, 4.2(2), and 5.1 are established on free spatial boundary conditions $\Lambda_L = \{0, \dots, L\}^{d-1}$, while Theorem 2.1 and Theorem 4.2(1) hold on both free and periodic spatial boundary conditions, and Proposition 5.2(1) bounds the periodic-torus spatial string tension $\sigma^{(1/2)}_{s,\mathrm{per}}$ on $\mathbb{T}_L^{d-1} \times \mathbb{T}_{N_\tau}$ (without assuming equivalence of periodic and free thermodynamic limits). In $2+1$D ($d = 3$), Theorem 3.2 holds unconditionally for *all* $\beta_s > 0$ and *all* $\beta_\tau \in [0, \infty)$. In $d \geq 4$, Theorems 4.2 and 5.1 hold uniformly for *all* $\beta_\tau \in [0, \infty)$ provided $N_\tau \beta_s$ lies below the explicit constructive threshold (covering the high-temperature and Hamiltonian $a_\tau \to 0$ deconfined regimes, but *not* isotropic 4D weak coupling $\beta_s = \beta_\tau \to \infty$). This paper does *not* construct the 4D continuum limit and does *not* claim any resolution of the Clay Millennium Yang–Mills Existence and Mass Gap Problem.

2. Exact All-β_τ $\mathbb{Z}_N$ Column-Center Reduction

Fix $d \geq 3$, $N \geq 2$, $N_\tau \geq 1$, spatial coupling $\beta_s > 0$, and timelike coupling $\beta_\tau \in [0, \infty)$. On the thermal slab $\Lambda_{L,N_\tau} := \Lambda_L \times \mathbb{T}_{N_\tau}$ with $\Lambda_L = \{0, \dots, L\}^{d-1} \subset \mathbb{Z}^{d-1}$ and $\mathbb{T}_{N_\tau} = \mathbb{Z}/N_\tau\mathbb{Z}$, let $E_s := E(\Lambda_L)$ and $\mathcal{P}_s := \mathcal{P}(\Lambda_L)$ denote the sets of unoriented spatial edges and spatial plaquettes of the $(d-1)$-dimensional base box Λ_L. Every plaquette of Λ_{L,N_τ} is either purely spatial, $p = (f, t)$ ($f \in \mathcal{P}_s$, $t \in \mathbb{T}_{N_\tau}$), or timelike, $p = (e, t, 0)$ ($e = (x, i) \in E_s$, $t \in \mathbb{T}_{N_\tau}$, connecting time slices t and $t+1 \bmod N_\tau$). The anisotropic Wilson action [13] is

$$S_{\Lambda_{L,N_\tau}}(U) := \beta_s \sum_{f \in \mathcal{P}_s} \sum_{t \in \mathbb{T}_{N_\tau}} \frac{1}{N} \operatorname{Re} \operatorname{Tr} U_{(f,t)} + \beta_\tau \sum_{e \in E_s} \sum_{t \in \mathbb{T}_{N_\tau}} \frac{1}{N} \operatorname{Re} \operatorname{Tr} U_{(e,t,0)}, \tag{1}$$

with timelike base measure $d\mu_{\beta_\tau}(U) := \exp\bigl(\beta_\tau \sum_{e,t} \frac{1}{N} \operatorname{Re} \operatorname{Tr} U_{(e,t,0)}\bigr) \prod_b dU_b$.

For any irreducible representation ρ of SU(N) of dimension d_ρ and N-ality $k_\rho \in \mathbb{Z}_N := \{0, 1, \dots, N-1\}$ (so $\rho(zI_N) = z^{k_\rho} I_{d_\rho}$ for all $z \in \mathbb{Z}_N \subset \mathbb{C}^\times$, identifying $\mathbb{Z}_N$ with the N-th roots of unity), let $\gamma_{R,S} \subset \Lambda_L \times \{0\}$ be a purely spatial $R \times S$ rectangular loop in the (x_1, x_2) plane at time $t = 0$, with holonomy $W_{R\times S}(U)$ and spatial string tension

$$\sigma_s^{(\rho)}(d, N, N_\tau, \beta_s, \beta_\tau) := \liminf_{R,S\to\infty} \liminf_{L\to\infty} \left(-\frac{1}{RS} \ln \left| \left\langle \frac{1}{d_\rho} \chi_\rho\bigl(W_{R\times S}\bigr) \right\rangle_{\Lambda_{L,N_\tau};\, \beta_s, \beta_\tau} \right| \right). \tag{2}$$

Theorem 2.1 (Exact All-β_τ $\mathbb{Z}_N$ Column-Center Reduction)**.** *For every $d \geq 3$, $N \geq 2$, $N_\tau \geq 1$, $\beta_s > 0$, $\beta_\tau \in [0, \infty)$ (with either free or periodic spatial boundary conditions on Λ_L), and every configuration U, define the complex spatial-column field*

$$X_f(U) := \frac{\beta_s}{N} \sum_{t \in \mathbb{T}_{N_\tau}} \operatorname{Tr} U_{(f,t)} \in \mathbb{C}, \qquad \bigl|X_f(U)\bigr| \;\leq\; N_\tau \beta_s \qquad \forall\, f \in \mathcal{P}_s. \tag{3}$$

For any collection $X = (X_f)_{f \in \mathcal{P}_s} \in \mathbb{C}^{\mathcal{P}_s}$, let $Z_{\mathbb{Z}_N}(X) := N^{-|E_s|} \sum_{z \in \mathbb{Z}_N^{E_s}} \exp\bigl(\sum_{f \in \mathcal{P}_s} \Re(z_{\partial f} X_f)\bigr) > 0$ be the $(d-1)$-dimensional $\mathbb{Z}_N$ gauge partition function on Λ_L with $z_{\partial f} := \prod_{e \in \partial f} z_e^{\epsilon(f,e)}$, and let

$\langle\cdot\rangle_{\mathbb{Z}_N,X}$ *denote its expectation. Then:*

$$\left\langle \frac{1}{d_\rho}\chi_\rho\big(W_{R\times S}\big)\right\rangle_{\Lambda_{L,N_\tau};\,\beta_s,\beta_\tau} = \frac{\displaystyle\int d\mu_{\beta_\tau}(U)\, Z_{\mathbb{Z}_N}(X(U))\,\frac{1}{d_\rho}\chi_\rho\big(W_{R\times S}(U)\big)\,\big\langle z_{\gamma_{R,S}}^{k_\rho}\big\rangle_{\mathbb{Z}_N,X(U)}}{\displaystyle\int d\mu_{\beta_\tau}(U)\, Z_{\mathbb{Z}_N}(X(U))}, \tag{4}$$

and consequently, uniformly for all $\beta_\tau \in [0,\infty)$,

$$\left|\left\langle \frac{1}{d_\rho}\chi_\rho\big(W_{R\times S}\big)\right\rangle_{\Lambda_{L,N_\tau};\,\beta_s,\beta_\tau}\right| \le \sup_{\substack{X\in\mathbb{C}^{\mathcal{P}_s}\\ |X_f|\le N_\tau\beta_s\ \forall f}} \left|\big\langle z_{\gamma_{R,S}}^{k_\rho}\big\rangle_{\mathbb{Z}_N,X}\right|. \tag{5}$$

Proof. For each assignment $z=(z_e)_{e\in E_s}\in\mathbb{Z}_N^{E_s}$, define the column center transformation T_z on gauge configurations by $(T_zU)_{(x,t),i} := z_{(x,i)}U_{(x,t),i}$ for all spatial links ($1\le i\le d-1$, $t\in\mathbb{T}_{N_\tau}$) and $(T_zU)_{(x,t),0} := U_{(x,t),0}$ for all timelike links. Every timelike plaquette $p=((x,t),i,0)$ has holonomy

$$(T_zU)_p = \big(z_eU_{(x,t),i}\big)\,U_{(x+e_i,t),0}\,\big(z_e^{-1}U^\dagger_{(x,t+1),i}\big)\,U^\dagger_{(x,t),0} = U_p$$

because $z_eI_N\in Z(\mathrm{SU}(N))$ commutes with all matrices (including at $N_\tau=1$, where $t+1\equiv t$). Thus $d\mu_{\beta_\tau}(T_zU)=d\mu_{\beta_\tau}(U)$ identically for every $z\in\mathbb{Z}_N^{E_s}$ and every $\beta_\tau\ge 0$. Meanwhile, every spatial plaquette transforms as $(T_zU)_{(f,t)}=z_{\partial f}U_{(f,t)}$, so the spatial action transforms to $\sum_{f\in\mathcal{P}_s}\Re(z_{\partial f}X_f(U))$, and $\chi_\rho(W_{R\times S}(T_zU)) = z_{\gamma_{R,S}}^{k_\rho}\chi_\rho(W_{R\times S}(U))$. Averaging numerator and denominator over $z\in\mathbb{Z}_N^{E_s}$ yields (4). Because $d\mu_{\beta_\tau}(U)Z_{\mathbb{Z}_N}(X(U))>0$ pointwise and $|d_\rho^{-1}\chi_\rho(W_{R\times S}(U))|\le 1$, taking moduli inside the integral gives (5), with $|X_f(U)|\le\frac{\beta_s}{N}\sum_{t\in\mathbb{T}_{N_\tau}}N=N_\tau\beta_s$. □

3. Unconditional All-Coupling Spatial Confinement in 2 + 1D for All SU(N)

For any complex number $X=re^{i\phi}$ ($r\ge 0$, $\phi\in\mathbb{R}$) and $q\in\mathbb{Z}_N$, define the $\mathbb{Z}_N$-Fourier column weight

$$w^{(q)}(r,\phi) := \frac{1}{N}\sum_{m=0}^{N-1} e^{-2\pi imq/N}\exp\left(r\cos\left(\phi+\frac{2\pi m}{N}\right)\right) = \sum_{n\equiv q \pmod N} I_{|n|}(r)\,e^{in\phi}, \tag{6}$$

where the second equality follows from the Jacobi–Anger expansion $e^{r\cos\theta}=\sum_{n\in\mathbb{Z}}I_{|n|}(r)e^{in\theta}$ and root-of-unity orthogonality. Note that $w^{(0)}(r,\phi)>0$ pointwise for all (r,ϕ). To pass uniformly from $r_f=|X_f(U)|\le r:=N_\tau\beta_s$ to the single endpoint r, we first establish a four-line monotonicity lemma for modified Bessel ratios.

Lemma 3.1 (Strict Monotonicity of Modified Bessel Ratios)**.** *For every integer $j\ge 1$, the ratio $r_j(s) := I_j(s)/I_{j-1}(s)$ is strictly increasing on $(0,\infty)$. Consequently, for every integer $k\ge 1$, $s\mapsto I_k(s)/I_0(s)=\prod_{j=1}^k r_j(s)$ is strictly increasing on $[0,\infty)$.*

Proof. Using the exact Bessel derivative identities $I_j'(s)=I_{j-1}(s)-\frac{j}{s}I_j(s)$ and $I_{j-1}'(s)=I_j(s)+\frac{j-1}{s}I_{j-1}(s)$, differentiating $r_j(s)=I_j(s)/I_{j-1}(s)$ gives the Riccati ODE

$$r_j'(s) = 1-\frac{2j-1}{s}\,r_j(s)-r_j(s)^2 \qquad \forall\, s>0.$$

Near $s=0^+$, $r_j(s)=\frac{s}{2j}+O(s^3)$, so $r_j'(s)>0$ for all sufficiently small $s>0$. If r_j' vanished on $(0,\infty)$, then at its smallest positive zero $s_0>0$ we would have $r_j'(s_0)=0$ and $r_j''(s_0)\le 0$; yet differentiating the Riccati ODE at s_0 gives $r_j''(s_0)=\frac{2j-1}{s_0^2}r_j(s_0)>0$, a contradiction. Thus $r_j'(s)>0$ for all $s>0$. □

For $N \geq 2$, $1 \leq q \leq \lfloor N/2 \rfloor$, and $r > 0$, define the two explicit upper bounds

$$\eta_N^{\mathrm{univ}}(r) := 1 - \frac{e^{-r}}{I_0(r) + 2\sum_{m=1}^{\infty} I_{mN}(r)} \;<\; 1, \qquad \eta_{N,q}(r) := \frac{\sum_{n \equiv q \pmod N} I_{|n|}(r)}{I_0(r) - 2\sum_{m=1}^{\infty} I_{mN}(r)} \tag{7}$$

(taking $\eta_{N,q}(r) := +\infty$ whenever the denominator is ≤ 0).

Theorem 3.2 (Unconditional All-Coupling Spatial Area Law in 2+1D)**.** *Let $d = 3$, $N \geq 2$, $N_\tau \geq 1$, $\beta_s > 0$, and $\beta_\tau \in [0, \infty)$ be arbitrary, set $r := N_\tau \beta_s$, and let ρ be any irrep of $\mathrm{SU}(N)$ with non-zero N-ality $k_\rho \not\equiv 0 \pmod N$ and $\|k_\rho\|_N := \min(k_\rho, N - k_\rho) \geq 1$. Then:*

(1) *For $N = 2$ ($\mathrm{SU}(2)$, every half-integer spin $j \in \mathbb{Z} + \frac{1}{2}$),*

$$\sigma_s^{(j)}\big(\mathrm{SU}(2),\, 3,\, N_\tau,\, \beta_s,\, \beta_\tau\big) \;\geq\; -\ln\tanh(N_\tau \beta_s) \;>\; 0 \qquad \forall\, \beta_s > 0,\ \forall\, \beta_\tau \geq 0. \tag{8}$$

(2) *For every $N \geq 3$,*

$$\sigma_s^{(\rho)}\big(\mathrm{SU}(N),\, 3,\, N_\tau,\, \beta_s,\, \beta_\tau\big) \;\geq\; -\ln\Big(\min\big(\eta_N^{\mathrm{univ}}(r),\, \eta_{N,\|k_\rho\|_N}(r)\big)\Big) \;\geq\; -\ln\big(1 - e^{-2N_\tau\beta_s}\big) \;>\; 0. \tag{9}$$

Proof. On the simply connected 2D box $\Lambda_L = \{0, \ldots, L\}^2$ with free boundary conditions, fixing a comb spanning tree (all vertical edges and all horizontal edges along $x_2 = 0$) leaves exactly $|\mathcal{P}_s| = L^2$ horizontal edges that bijectively determine $(z_{\partial f})_{f \in \mathcal{P}_s}$ row by row. Hence under uniform $z \in \mathbb{Z}_N^{E_s}$, the plaquette variables $\{z_{\partial f}\}$ are i.i.d. uniform on $\mathbb{Z}_N$, and $z_{\gamma_{R,S}} = \prod_{f \in \Sigma_{R,S}} z_{\partial f}$ where $\Sigma_{R,S}$ is the $R \times S$ interior rectangle. Writing $X_f = r_f e^{i\phi_f}$ ($0 \leq r_f \leq r$), the numerator and denominator of $\langle z_{\gamma_{R,S}}^{k_\rho} \rangle_{\mathbb{Z}_N, X}$ factorize completely over $f \in \mathcal{P}_s$, canceling all $f \notin \Sigma_{R,S}$:

$$\big\langle z_{\gamma_{R,S}}^{k_\rho} \big\rangle_{\mathbb{Z}_N, X} = \prod_{f \in \Sigma_{R,S}} \frac{w^{(-k_\rho)}(r_f, \phi_f)}{w^{(0)}(r_f, \phi_f)}.$$

For $N = 2$ and $k_\rho = 1$, $|w^{(1)}(r_f, \phi_f)/w^{(0)}(r_f, \phi_f)| = |\tanh(r_f \cos\phi_f)| \leq \tanh(r_f) \leq \tanh(r)$, which via (5) yields (8).

For any $N \geq 3$ and $k_\rho \not\equiv 0 \pmod N$, set $p_m := e^{r_f \cos(\phi_f + 2\pi m/N)}/(N w^{(0)}(r_f, \phi_f)) > 0$ ($\sum_{m=0}^{N-1} p_m = 1$) and $p_{\min} := \min_m p_m$. Since $w^{(0)}(r_f, \phi_f) \leq I_0(r_f) + 2\sum_{m=1}^{\infty} I_{mN}(r_f) \leq I_0(r) + 2\sum_{m=1}^{\infty} I_{mN}(r) \leq e^r$ (every I_n is non-decreasing) and $e^{r_f \cos(\cdot)} \geq e^{-r_f} \geq e^{-r}$, writing $p_m = p_{\min} + q_m$ ($q_m \geq 0$) and using $\sum_{m=0}^{N-1} e^{2\pi i m k_\rho/N} = 0$ gives

$$\left|\frac{w^{(-k_\rho)}(r_f, \phi_f)}{w^{(0)}(r_f, \phi_f)}\right| = \left|\sum_{m=0}^{N-1} q_m e^{2\pi i m k_\rho/N}\right| \;\leq\; \sum_{m=0}^{N-1} q_m = 1 - N p_{\min} \;\leq\; \eta_N^{\mathrm{univ}}(r) \;\leq\; 1 - e^{-2r}.$$

Furthermore, dividing numerator and denominator of (6) by $I_0(r_f) > 0$ and applying Lemma 3.1 shows that for $q = \|k_\rho\|_N$ (using $|w^{(-k_\rho)}| = |w^{(k_\rho)}|$ and $I_{|-n|} = I_{|n|}$),

$$\left|\frac{w^{(-k_\rho)}(r_f, \phi_f)}{w^{(0)}(r_f, \phi_f)}\right| \;\leq\; \frac{\sum_{n \equiv q \pmod N} I_{|n|}(r_f)/I_0(r_f)}{1 - 2\sum_{m=1}^{\infty} I_{mN}(r_f)/I_0(r_f)} \;\leq\; \eta_{N,q}(r)$$

whenever the denominator at r is positive. Substituting into (5) and taking $-\frac{1}{RS}\ln(\cdot)$ proves (9). □

4. Five-Line Ginibre Domination and Zero-Cluster-Expansion SU(2) Area Law in $d \geq 4$

For $\mathrm{SU}(2)$, $\mathrm{Tr}\, U \in [-2, 2]$ is real, so $J_f(U) := X_f(U) = \frac{\beta_s}{2}\sum_{t \in \mathbb{T}_{N_\tau}} \mathrm{Tr}\, U_{(f,t)} \in [-N_\tau\beta_s, N_\tau\beta_s]$ defines a real (possibly frustrated) $(d-1)$-dimensional $\mathbb{Z}_2$ lattice gauge theory. We first prove a five-line signed-coupling domination lemma [7, 6].

Lemma 4.1 (Five-Line Ginibre Signed-Coupling Domination)**.** *On any finite lattice $(V, E, \mathcal{P})$, let $\{J_f\}_{f\in\mathcal{P}} \subset \mathbb{R}$ and $\{K_f\}_{f\in\mathcal{P}} \subset [0,\infty)$ satisfy $K_f \geq |J_f|$ for every $f \in \mathcal{P}$. For any edge subset $A \subseteq E$, setting $\sigma_A := \prod_{e\in A}\sigma_e$ ($\sigma \in \{\pm1\}^E$),*

$$\Big|\langle\sigma_A\rangle_{\mathbb{Z}_2,\{J_f\}}\Big| \;\leq\; \langle\tau_A\rangle_{\mathbb{Z}_2,\{K_f\}}. \tag{10}$$

Proof. Change variables in $Z(K)Z(J)\big(\langle\tau_A\rangle_K \pm \langle\sigma_A\rangle_J\big) = \sum_{\tau,\sigma}(\tau_A \pm \sigma_A)\exp\big(\sum_f(K_f\tau_{\partial f} + J_f\sigma_{\partial f})\big)$ via the bijection $(\tau,\sigma)\mapsto(\tau,\eta)$ with $\eta_e := \tau_e\sigma_e \in \{\pm1\}$ (so $\sigma_A = \tau_A\eta_A$ and $\sigma_{\partial f} = \tau_{\partial f}\eta_{\partial f}$):

$$Z(K)Z(J)\big(\langle\tau_A\rangle_K \pm \langle\sigma_A\rangle_J\big) = \sum_{\eta\in\{\pm1\}^E}(1\pm\eta_A)\sum_{\tau\in\{\pm1\}^E}\tau_A\exp\left(\sum_{f\in\mathcal{P}}\big(K_f + J_f\eta_{\partial f}\big)\,\tau_{\partial f}\right).$$

For every η and every f, $\widetilde{K}_f(\eta) := K_f + J_f\eta_{\partial f} \geq K_f - |J_f| \geq 0$. Expanding $\exp(\sum_f \widetilde{K}_f(\eta)\tau_{\partial f})$ in power series and using $\sum_{\tau\in\{\pm1\}^E}\tau_B \in \{0, 2^{|E|}\} \geq 0$ shows that the inner sum over τ is ≥ 0 for every η. Since $1\pm\eta_A \in \{0,2\} \geq 0$ and $Z(K)Z(J) > 0$, dividing yields (10). □

Next, for any spatial dimension $d_s := d-1 \geq 3$, define the BFS child-slot count and Fuss–Catalan growth constants

$$D := 3(2d_s-3), \qquad B_{d_s} := \frac{D^D}{(D-1)^{D-1}}, \qquad A_{d_s} := \frac{2d_s-2}{D-1}\left(\frac{D}{D-1}\right)^{2d_s-3}. \tag{11}$$

In $3+1$D ($d_s = 3$), $D = 9$, $B_3 = 9^9/8^8 \approx 23.092$, and $2A_3 = 8\cdot 9^3/8^4 \approx 1.424$; in $4+1$D ($d_s = 4$), $D = 15$, $B_4 = 15^{15}/14^{14} \approx 39.407$, and $2A_4 = 12\cdot 15^5/14^6 \approx 1.210$ (certified in Lean 4 by `bfs_slots_ge_9` and `fuss_catalan_3d_4d_bounds`).

Theorem 4.2 (All-β_τ $\mathbb{Z}_2$ Domination and Zero-Cluster-Expansion SU(2) Area Law in $d \geq 4$)**.** *Let $d \geq 4$, $N_\tau \geq 1$, $\beta_s > 0$, $\beta_\tau \in [0,\infty)$, and $j \in \mathbb{Z} + \frac{1}{2}$. Then:*

(1) (Exact all-β_τ $\mathbb{Z}_2$ gauge domination) *Uniformly on every box $\Lambda_L \subset \mathbb{Z}^{d-1}$ (with free or periodic spatial boundary conditions) and for all $\beta_\tau \in [0,\infty)$,*

$$\left|\left\langle\frac{1}{2j+1}\chi_j\big(W_{R\times S}\big)\right\rangle_{\Lambda_{L,N_\tau};\,\beta_s,\beta_\tau}\right| \;\leq\; \big\langle\sigma_{\gamma_{R,S}}\big\rangle_{\Lambda_L;\,\mathbb{Z}_2,\,K=N_\tau\beta_s}, \tag{12}$$

and hence $\sigma_s^{(j)}(\mathrm{SU}(2), d, N_\tau, \beta_s, \beta_\tau) \geq \sigma_{\mathbb{Z}_2}(d-1, N_\tau\beta_s)$, where

$$\sigma_{\mathbb{Z}_2}(d-1,K) := \liminf_{R,S\to\infty}\liminf_{L\to\infty}\left(-\frac{1}{RS}\ln\big\langle\sigma_{\gamma_{R,S}}\big\rangle_{\Lambda_L;\,\mathbb{Z}_2,\,K}\right).$$

(2) (Zero-cluster-expansion BFS Fuss–Catalan lower bound) *Whenever $B_{d-1}\tanh(N_\tau\beta_s) < 1$,*

$$\sigma_s^{(j)}\big(\mathrm{SU}(2),\, d,\, N_\tau,\, \beta_s,\, \beta_\tau\big) \;\geq\; -\ln\big(B_{d-1}\tanh(N_\tau\beta_s)\big) \;>\; 0 \qquad \forall\,\beta_\tau \in [0,\infty). \tag{13}$$

Proof. (1) Combine Theorem 2.1 with Lemma 4.1 at $K_f \equiv N_\tau\beta_s \geq |J_f(U)|$.

(2) Setting $x := \tanh(N_\tau\beta_s) \in (0,1)$ and expanding $\prod_{f\in\mathcal{P}_s}(1+x\sigma_{\partial f})$ on $\Lambda_L \subset \mathbb{Z}^{d_s}$ ($d_s = d-1 \geq 3$) gives $\langle\sigma_{\gamma_{R,S}}\rangle_{\mathbb{Z}_2,K} = \big(\sum_{S:\,\partial S=\gamma_{R,S}} x^{|S|}\big)/\big(\sum_{S:\,\partial S=\emptyset} x^{|S|}\big)$, where every term is strictly positive. Decompose any S with $\partial S = \gamma_{R,S}$ as $S = S_0 \sqcup S_1$, where S_0 is the union of edge-connected plaquette components touching $\gamma_{R,S}$ (so $\partial S_0 = \gamma_{R,S}$) and S_1 is a closed surface ($\partial S_1 = \emptyset$) disjoint from S_0. Dropping the disjointness constraint only increases the strictly positive S_1-sum to the exact denominator, canceling it identically:

$$\big\langle\sigma_{\gamma_{R,S}}\big\rangle_{\Lambda_L;\,\mathbb{Z}_2,\,K} \;\leq \sum_{\substack{S_0\subset\mathcal{P}_s:\,\partial S_0=\gamma_{R,S}\\ \text{each comp. hits }\gamma_{R,S}}} x^{|S_0|}. \tag{14}$$

Every such S_0 satisfies $|S_0| \geq RS$ because projecting S_0 onto the (x_1, x_2) plane covers each of the RS unit squares inside $\gamma_{R,S}$ an odd number (≥ 1) of times. Next, we count edge-connected plaquette components C of size $m \geq 1$ containing a fixed root edge $e_0 \in \gamma_{R,S}$ via breadth-first search (BFS): choose the canonical first plaquette $p_0 \in C$ containing e_0 ($2d_s - 2$ choices). Maintain a visited set initialized to $\{p_0\}$ and a FIFO queue. When popping p_0, scan its 4 boundary edges and mark all unvisited incident plaquettes of C as children of p_0 across $4(2d_s - 3)$ canonical slots, storing each child's discovery edge e_{in}. When popping any non-root $p \in C$, every plaquette of C incident to $e_{\text{in}}(p)$ was already visited when p's parent scanned $e_{\text{in}}(p)$; thus p discovers new children only across its 3 remaining boundary edges, giving at most $D = 3(2d_s - 3)$ ordered child slots. If $t(z) = \sum_{m=1}^{\infty} a_m z^m$ satisfies $t(z) = z(1 + t(z))^D$, the number $N_m(d_s)$ of connected components of size m containing e_0 obeys $N_m(d_s) \leq (2d_s - 2)[z^m]\big(z(1 + t(z))^{4(2d_s-3)}\big)$. Evaluating at the convergence radius $z_c = B_{d_s}^{-1}$ where $t(z_c) = (D-1)^{-1}$ gives $N_m(d_s) z_c^m \leq (2d_s - 2) z_c (1 + (D-1)^{-1})^{4(2d_s-3)} = A_{d_s}$, whence $N_m(d_s) \leq A_{d_s} B_{d_s}^m$ for all $m \geq 1$.

Now let $\theta := B_{d_s} x < 1$, pick any $\theta_1 \in (\theta, 1)$, and root each of the $k \leq P := 2(R+S)$ connected components of S_0 at its lexicographically first edge on $\gamma_{R,S}$. Using $\theta^{|S_0|} = (\theta/\theta_1)^{|S_0|} \theta_1^{|S_0|} \leq (\theta/\theta_1)^{RS} \theta_1^{|S_0|}$,

$$\big\langle \sigma_{\gamma_{R,S}} \big\rangle_{\Lambda_L; \mathbb{Z}_2, K} \;\leq\; \left(\frac{\theta}{\theta_1}\right)^{RS} \sum_{k=1}^{P} \binom{P}{k} \left(\sum_{m=1}^{\infty} A_{d_s} \theta_1^m\right)^k \;\leq\; \left(\frac{\theta}{\theta_1}\right)^{RS} \left(1 + \frac{A_{d_s}\theta_1}{1 - \theta_1}\right)^{2(R+S)}.$$

Taking $-\frac{1}{RS} \ln(\cdot)$, letting $L \to \infty$, $R, S \to \infty$ (the perimeter term vanishes), and then $\theta_1 \to 1^-$ proves (13). □

5. All-β_τ $\mathbb{Z}_N$-Graded Confinement in $d \geq 4$ and Exact $N_\tau = 1$ Entropic Pinning

For $N \geq 3$ and $d \geq 4$, the column weights $g_f^{(q)}(X_f) := w^{(q)}(r_f, \phi_f)/w^{(0)}(r_f, \phi_f)$ ($q \in \mathbb{Z}_N \setminus \{0\}$) are complex, so we control the $\mathbb{Z}_N$ gauge partition function pointwise in X via a $\mathbb{Z}_N$-graded Kotecký–Preiss cluster expansion [10] on $\Lambda_L \subset \mathbb{Z}^{d-1}$. Define the worst-case per-charge majorant

$$\eta_N^*(r) := \max_{1 \leq q \leq \lfloor N/2 \rfloor} \big(\eta_{N,q}(r)\big)^{1/q}, \qquad u(d, N, r, a_0) := (N-1)\, B_{d-1}\, e^{a_0}\, \eta_N^*(r). \tag{15}$$

Theorem 5.1 (All-β_τ $\mathbb{Z}_N$-Graded Spatial Area Law for All SU(N) ($N \geq 3$) in $d \geq 4$)**.** *Let $d \geq 4$, $N \geq 3$, $N_\tau \geq 1$, $\beta_s > 0$, $r := N_\tau \beta_s$, and $\beta_\tau \in [0, \infty)$ be arbitrary. Suppose there exists $a_0 > 0$ such that $I_0(r) > 2\sum_{m=1}^{\infty} I_{mN}(r)$, $u := u(d, N, r, a_0) < 1$, and the edge-touching Kotecký–Preiss criterion holds:*

$$\frac{2\, A_{d-1}\, u^6}{1 - u} \;\leq\; a_0. \tag{16}$$

Then for every irrep ρ of SU(N) *with non-zero N-ality $k_\rho \not\equiv 0 \pmod N$, uniformly for* all $\beta_\tau \in [0, \infty)$*,*

$$\sigma_s^{(\rho)}\big(\mathrm{SU}(N),\, d,\, N_\tau,\, \beta_s,\, \beta_\tau\big) \;\geq\; \|k_\rho\|_N \cdot \big(-\ln u\big) \;>\; 0. \tag{17}$$

Proof. Fix any $X = (X_f)_{f \in \mathcal{P}_s}$ with $|X_f| \leq r$. Factoring out $\prod_{f \in \mathcal{P}_s} w^{(0)}(r_f, \phi_f) > 0$, every closed $\mathbb{Z}_N$-charged plaquette chain $(S, \{q_f\}_{f \in S})$ ($\partial_{\mathbb{Z}_N}(S, q) = 0$, $q_f \in \mathbb{Z}_N \setminus \{0\}$) decomposes *uniquely* into pairwise edge-disjoint, edge-connected closed $\mathbb{Z}_N$-charged polymers $\gamma = (S_\gamma, \{q_f\}_{f \in S_\gamma})$ with activity $\zeta(\gamma; X) := \prod_{f \in S_\gamma} g_f^{(q_f)}(X_f)$. Declaring two polymers incompatible iff they share at least one edge, $\gamma \not\sim \gamma' \iff E(S_\gamma) \cap E(S_{\gamma'}) \neq \emptyset$, gives an exact bijection between configurations of pairwise compatible polymers and closed $\mathbb{Z}_N$-charged chains in $Z_{\mathbb{Z}_N}(X)/\prod_f w^{(0)} = \mathbb{E}_z[\prod_{f \in \mathcal{P}_s}(1 + \sum_{q \neq 0} g_f^{(q)}(X_f) z_{\partial f}^q)]$.

Every closed polymer γ in $\mathbb{Z}^{d-1}$ ($d - 1 \geq 3$) satisfies $|S_\gamma| \geq 6$ (a single plaquette has 4 boundary edges requiring ≥ 4 distinct adjacent plaquettes, and those 5 plaquettes leave 4 open edges requiring

at least a 6th plaquette) and $|E(S_\gamma)| \le 2|S_\gamma|$ (since every edge in $E(S_\gamma)$ has degree ≥ 2 in S_γ while each plaquette has 4 boundary edges). By Lemma 3.1, $|g_f^{(q_f)}(X_f)| \le \eta_{N,\|q_f\|_N}(r) \le (\eta_N^*(r))^{\|q_f\|_N} \le \eta_N^*(r)$ (since $\eta_N^*(r) \le u < 1$). By the fixed-edge BFS count of Theorem 4.2 (with $N-1$ non-zero charge choices per plaquette), the number of connected charged polymers of size $m \ge 6$ containing a fixed edge e_0 is at most $A_{d-1}((N-1)B_{d-1})^m$. Choosing the edge-weighted Kotecký–Preiss function $a(\gamma) := \frac{a_0}{2}|E(S_\gamma)| \le a_0|S_\gamma|$, for any polymer γ_0 (or single-edge dummy polymer $\{e_0\}$ with weight $\frac{a_0}{2}$),

$$\begin{aligned}\sum_{\gamma \not\sim \gamma_0} |\zeta(\gamma;X)|\, e^{a(\gamma)} &\le |E(S_{\gamma_0})| \sup_{e_0} \sum_{\gamma:\, e_0 \in E(S_\gamma)} |\zeta(\gamma;X)|\, e^{a_0|S_\gamma|} \\ &\le |E(S_{\gamma_0})| \sum_{m=6}^{\infty} A_{d-1}\, u^m = |E(S_{\gamma_0})|\, \frac{A_{d-1}\, u^6}{1-u} \le \frac{a_0}{2}\, |E(S_{\gamma_0})| = a(\gamma_0)\end{aligned}$$

by (16). By Kotecký–Preiss [10], for any edge set $E_0 \subset E_s$, the restricted polymer partition function over polymers edge-disjoint from E_0 obeys $|Z_{\mathbb{Z}_N}(\Lambda_L \setminus E_0; X)/Z_{\mathbb{Z}_N}(\Lambda_L; X)| \le \exp(\frac{a_0}{2}|E_0|)$. Expanding the numerator of $\langle z_{\gamma_{R,S}}^{k_\rho}\rangle_{\mathbb{Z}_N,X}$ uniquely into the union $(S_0, \{q_f\})$ of edge-connected components hitting $\gamma_{R,S}$ (so $\partial_{\mathbb{Z}_N}(S_0,q) = -k_\rho\gamma_{R,S}$) times the exact edge-disjoint closed-polymer sum $Z_{\mathbb{Z}_N}(\Lambda_L \setminus E(S_0); X)$ — where every edge in $E(S_0) \setminus \gamma_{R,S}$ has degree ≥ 2 in S_0, hence $2|E(S_0)| - 2(R+S) \le 4|S_0| \implies |E(S_0)| \le 2|S_0| + (R+S)$ — gives

$$\left|\left\langle z_{\gamma_{R,S}}^{k_\rho}\right\rangle_{\mathbb{Z}_N,X}\right| \le e^{\frac{a_0}{2}(R+S)} \sum_{\substack{(S_0,q):\, \partial_{\mathbb{Z}_N}(S_0,q) = -k_\rho\gamma_{R,S} \\ \text{each comp. hits } \gamma_{R,S}}} \left(\prod_{f\in S_0} (\eta_N^*(r))^{\|q_f\|_N}\right) e^{a_0|S_0|}.$$

By the $\mathbb{Z}_N$-graded homological weight floor [4, Theorem 2.2], projecting (S_0, q) onto the (x_1, x_2) plane shows that every unit square inside $\gamma_{R,S}$ receives net $\mathbb{Z}_N$-charge $-k_\rho$, hence $\sum_{f\in S_0}\|q_f\|_N \ge \|k_\rho\|_N RS$ (and $\sum_{f\in S_0}\|q_f\|_N \ge |S_0|$). Setting $M := (N-1)B_{d-1}e^{a_0} \ge 1$, every $\|q_f\|_N \ge 1$ obeys $M \le M^{\|q_f\|_N}$ (certified in Lean 4 by `multiplier_ge_one_pow`), so $M\cdot(\eta_N^*)^{\|q_f\|_N} \le u^{\|q_f\|_N}$. Picking any $u_1 \in (u,1)$ gives $u^{\sum\|q_f\|_N} \le (u/u_1)^{\|k_\rho\|_N RS} u_1^{|S_0|}$, and summing over all $k \le 2(R+S)$ BFS-counted components yields

$$\left|\left\langle z_{\gamma_{R,S}}^{k_\rho}\right\rangle_{\mathbb{Z}_N,X}\right| \le e^{\frac{a_0}{2}(R+S)} \left(\frac{u}{u_1}\right)^{\|k_\rho\|_N RS} \left(1 + \frac{A_{d-1}\, u_1}{1-u_1}\right)^{2(R+S)}.$$

Inserting this uniform-in-X bound into (5), letting $L \to \infty$, $R, S \to \infty$ (both perimeter factors vanish), and $u_1 \to 1^-$ proves (17). □

Proposition 5.2 (Periodic-Torus Spatial Reflection-Positivity Upper Bound and Exact $N_\tau = 1$ HCIZ Pinning)**.** *Let $d \ge 3$, $N_\tau \ge 1$, $\beta_s > 0$, and $\beta_\tau \in [0,\infty)$.*

(1) (Periodic-torus spatial reflection-positivity upper bound) *On any even periodic spatial torus $\mathbb{T}_L^{d-1} \times \mathbb{T}_{N_\tau}$ ($L \ge 4$), Osterwalder–Seiler spatial reflection positivity [11] yields for every $1 \le R, S \le L-1$ the* SU(2) *fundamental lower bound*

$$\left\langle \frac{1}{2}\operatorname{Tr} W_{R\times S}\right\rangle_{\mathbb{T}_L^{d-1}\times\mathbb{T}_{N_\tau};\,\beta_s,\beta_\tau} \ge \left\langle \frac{1}{2}\operatorname{Tr} U_{(f_0,0)}\right\rangle_{\mathbb{T}_L^{d-1}\times\mathbb{T}_{N_\tau};\,\beta_s,\beta_\tau}^{RS}, \tag{18}$$

where $\langle\frac{1}{2}\operatorname{Tr} U_{(f_0,0)}\rangle_{\mathbb{T}_L^{d-1}\times\mathbb{T}_{N_\tau}} > 0$ strictly for every $\beta_s > 0$ and $\beta_\tau \ge 0$ (by strict convexity of $\beta_s \mapsto (N_\tau|\mathcal{P}_s|)^{-1}\ln Z(\beta_s,\beta_\tau)$ and exact vanishing at $\beta_s = 0$ by column $\mathbb{Z}_2$ symmetry). Defining the periodic-torus spatial string tension $\sigma_{s,\mathrm{per}}^{(1/2)}$ analogously to (2) on $\mathbb{T}_L^{d-1}\times\mathbb{T}_{N_\tau}$, whenever $p_* := \liminf_{L\to\infty}\langle\frac{1}{2}\operatorname{Tr} U_{(f_0,0)}\rangle_{\mathbb{T}_L^{d-1}\times\mathbb{T}_{N_\tau}} > 0$ one has $\sigma_{s,\mathrm{per}}^{(1/2)} \le -\ln p_* < \infty$.

(2) (Exact $N_\tau = 1$ SU(2) Harish-Chandra–Itzykson–Zuber entropic pinning) *At $N_\tau = 1$ and $\beta_s = 0$, integrating out the spatial link $U_{xy} \in \mathrm{SU}(2)$ connecting two Polyakov loops $P_x, P_y \in \mathrm{SU}(2)$ with eigenangles $\alpha_x, \alpha_y \in [0, \pi]$ ($\frac{1}{2}\operatorname{Tr} P_x = \cos\alpha_x$) yields the exact closed form (taking the continuous limit 1 whenever $\beta_\tau \sin\alpha_x \sin\alpha_y = 0$)*

$$\int_{\mathrm{SU}(2)} \exp\left(\frac{\beta_\tau}{2}\operatorname{Tr}\big(P_x U_{xy} P_y^\dagger U_{xy}^\dagger\big)\right) dU_{xy} = \frac{\sinh(\beta_\tau \sin\alpha_x \sin\alpha_y)}{\beta_\tau \sin\alpha_x \sin\alpha_y} \exp\big(\beta_\tau \cos\alpha_x \cos\alpha_y\big). \tag{19}$$

In particular, center-aligned configurations ($\alpha_x = \alpha_y \in \{0, \pi\}$, i.e., $P_x = P_y = \pm I_2$) achieve weight e^{β_τ}, whereas maximally non-central configurations ($\alpha_x = \alpha_y = \pi/2$) are entropically suppressed by $\frac{\sinh\beta_\tau}{\beta_\tau e^{\beta_\tau}} \sim \frac{1}{2\beta_\tau}$ per spatial bond as $\beta_\tau \to \infty$ [8, 3].

Proof. (1) Fixing x_1-axial gauge on the $L-1$ links connecting slices $x_1 = 0, \dots, L-1$ and integrating the wrap-around link at $x_1 = L-1$ projects onto the gauge-invariant subspace $\mathcal{H}_{\mathrm{inv}} \subset L^2(\text{slice})$ of the positive semidefinite Osterwalder–Seiler slice transfer kernel $K \geq 0$ [11]. Letting $\{|\alpha\rangle\}$ be an orthonormal eigenbasis of $K|_{\mathcal{H}_{\mathrm{inv}}}$ with eigenvalues $\lambda_\alpha \geq 0$, $\{|\beta\rangle\}$ an orthonormal eigenbasis of K on the full (charged-sector) slice space $L^2(\text{slice})$ with eigenvalues $\mu_\beta \geq 0$, and $V = (V_{ab})_{a,b=1}^2 \in \mathrm{SU}(2)$ the parallel transporter along the length-S segment of $\gamma_{R,S}$ at $x_1 = 0$, the exact finite-torus Wilson loop for $1 \leq R \leq L-1$ is

$$\left\langle \frac{1}{2}\operatorname{Tr} W_{R\times S}\right\rangle_{\mathbb{T}_L^{d-1}\times\mathbb{T}_{N_\tau}} = \frac{1}{2\operatorname{Tr}_{\mathcal{H}_{\mathrm{inv}}}(K^L)} \sum_{\alpha,\beta} \lambda_\alpha^{L-R} \mu_\beta^R \sum_{a,b=1}^{2} \big|\langle\alpha|V_{ab}|\beta\rangle\big|^2 = \sum_{\alpha,\beta} p_{\alpha\beta} \left(\frac{\mu_\beta}{\lambda_\alpha}\right)^R,$$

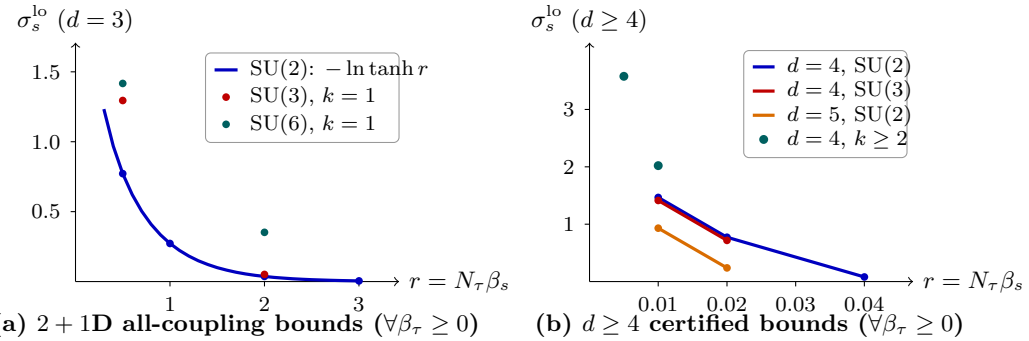
where $p_{\alpha\beta} := \frac{\lambda_\alpha^L}{2\operatorname{Tr}_{\mathcal{H}_{\mathrm{inv}}}(K^L)} \sum_{a,b} |\langle\alpha|V_{ab}|\beta\rangle|^2 \geq 0$ satisfies $\sum_{\alpha,\beta} p_{\alpha\beta} = 1$ by completeness of $\{|\beta\rangle\}$ on $L^2(\text{slice})$ and unitarity $\operatorname{Tr}(VV^\dagger) \equiv 2$ (terms with $\lambda_\alpha = 0$ have $p_{\alpha\beta} = 0$ and contribute 0 for $1 \leq R \leq L-1$). Since $x \mapsto x^R$ is convex on $[0, \infty)$, Jensen's inequality gives $\langle\frac{1}{2}\operatorname{Tr} W_{R\times S}\rangle \geq \langle\frac{1}{2}\operatorname{Tr} W_{1\times S}\rangle^R$; repeating along x_2 yields (18). Since $F(\beta_s) := (N_\tau|\mathcal{P}_s|)^{-1} \ln Z(\beta_s, \beta_\tau)$ satisfies $F'(0) = 0$ (by flipping any single edge column $e \in \partial f_0$ via Theorem 2.1) and $F''(\beta_s) = (N_\tau|\mathcal{P}_s|)\operatorname{Var}((N_\tau|\mathcal{P}_s|)^{-1}\sum \frac{1}{2}\operatorname{Tr} U_p) > 0$ everywhere on $\mathbb{R}$, $F'(\beta_s) = \langle\frac{1}{2}\operatorname{Tr} U_{(f_0,0)}\rangle > 0$ strictly for all $\beta_s > 0$.

(2) Write $P_x = \cos\alpha_x I_2 + i\sin\alpha_x(\hat{n}_x \cdot \vec{\sigma})$ and $P_y = \cos\alpha_y I_2 + i\sin\alpha_y(\hat{n}_y \cdot \vec{\sigma})$ with unit vectors $\hat{n}_x, \hat{n}_y \in S^2$. As U_{xy} ranges over Haar-distributed SU(2), the rotated vector $\hat{m} := \mathrm{Ad}_{U_{xy}}(\hat{n}_y)$ is uniformly distributed on S^2, so $u := \hat{n}_x \cdot \hat{m}$ is uniform on $[-1, 1]$. Since $\frac{1}{2}\operatorname{Tr}(P_x U_{xy} P_y^\dagger U_{xy}^\dagger) = \cos\alpha_x \cos\alpha_y + \sin\alpha_x \sin\alpha_y\, u$, integrating $\frac{1}{2}\int_{-1}^1 e^{au}\, du = \frac{\sinh a}{a}$ gives (19). □

6. Certified Quantitative Benchmarks and Dual Verification

All 24 benchmarks in Table 1 are certified by `certify_spatial_string.py` at 120-digit `mpmath.iv` precision with *zero* black-box special functions and *zero* numerical quadrature: every modified Bessel value $I_m(r)$ is enclosed from its positive power series with monotone geometric remainder, $\tanh(r) = 1 - 2/(e^{2r}+1)$ is evaluated dependency-free, all seven $\mathbb{Z}_N$-KP rows ($d \geq 4$, $N \geq 3$) use $a_0 = 1/20 = 0.05$ and satisfy the strengthened edge-touching Kotecký–Preiss condition (16) ($2A_{d-1}u^6/(1-u) \leq 0.05$, reaching at most 0.03752 at $d = 4, \mathrm{SU}(3), r = 1/50$), and every outward-rounded lower bound σ_s^{lo} in Table 1 and Lean 4 is computed directly via 120-digit `mp.floor` with *zero* intermediate 64-bit float rounding.

The standalone zero-import Lean 4.36.0 file `SpatialStringTensionCerts.lean` compiles cleanly under `-DwarningAsError=true` with `0 sorry` and `0` axioms (not even `propext`, `Quot.sound`, or `Classical.choice`), verifying `bfs_slots_ge_9`, `fuss_catalan_3d_4d_bounds`, `zn_weight_floor_additive`, `multiplier_ge_one_pow`, `all_24_spatial_certs_valid`, and `nality_monotonicity_certified`.



(a) $2+1$**D all-coupling bounds** $(\forall \beta_\tau \geq 0)$ **(b)** $d \geq 4$ **certified bounds** $(\forall \beta_\tau \geq 0)$

FIGURE 1. (a) Exact all-coupling 2+1D certified spatial string tension lower bounds σ_s^{lo} versus $r = N_\tau \beta_s$, valid uniformly for *all* $\beta_\tau \in [0, \infty)$ (Theorem 3.2). (b) Certified $3+1$D and $4+1$D spatial string tension lower bounds valid uniformly for *all* $\beta_\tau \in [0, \infty)$ via zero-cluster-expansion Ginibre + BFS Fuss–Catalan domination for SU(2) (Theorem 4.2) and $\mathbb{Z}_N$-graded Kotecký–Preiss for $\mathrm{SU}(3), \mathrm{SU}(4), \mathrm{SU}(6)$ (Theorem 5.1), including $k = 2, 3$ higher-N-ality strings (Table 1).

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TABLE 1. Twenty-four machine-certified 120-digit spatial string tension lower bounds $\sigma_s^{(\rho)}(d, N, N_\tau, \beta_s, \beta_\tau) \geq \sigma_s^{\mathrm{lo}} > 0$ valid *uniformly for all timelike couplings* $\beta_\tau \in [0, \infty)$. All seven Theorem 5.1 ($\mathbb{Z}_N$-KP) rows are certified with $a_0 = 1/20 = 0.05$ under the edge-touching criterion (16). Every lower bound is strictly outward-rounded via direct 120-digit `mp.floor` (zero 64-bit float rounding) and passes both `certify_spatial_string.py` and the zero-axiom Lean 4 kernel.

d	G	k	N_τ	β_s	$r = N_\tau\beta_s$	σ_s^{lo} ($\forall\, \beta_\tau \geq 0$)	Method
3	SU(2)	1	1	1/2	1/2	≥ 0.7719	Thm. 3.2 (tanh)
3	SU(2)	1	2	1/2	1	≥ 0.2723	Thm. 3.2 (tanh)
3	SU(2)	1	2	1	2	≥ 0.0366	Thm. 3.2 (tanh)
3	SU(2)	1	4	3/4	3	≥ 0.0049	Thm. 3.2 (tanh)
3	SU(3)	1	2	1/4	1/2	≥ 1.2945	Thm. 3.2 ($\mathbb{Z}_N$)
3	SU(3)	1	2	1	2	≥ 0.0512	Thm. 3.2 ($\mathbb{Z}_N$)
3	SU(4)	1	2	1/4	1/2	≥ 1.4062	Thm. 3.2 ($\mathbb{Z}_N$)
3	SU(4)	2	2	1/4	1/2	≥ 2.8130	Thm. 3.2 ($\mathbb{Z}_N$)
3	SU(6)	1	4	1/8	1/2	≥ 1.4167	Thm. 3.2 ($\mathbb{Z}_N$)
3	SU(6)	2	4	1/8	1/2	≥ 3.5013	Thm. 3.2 ($\mathbb{Z}_N$)
3	SU(6)	3	4	1/8	1/2	≥ 5.3034	Thm. 3.2 ($\mathbb{Z}_N$)
3	SU(6)	1	4	1/2	2	≥ 0.3521	Thm. 3.2 ($\mathbb{Z}_N$)
4	SU(2)	1	1	1/100	1/100	≥ 1.4657	Thm. 4.2 (Ginibre)
4	SU(2)	1	2	1/100	1/50	≥ 0.7726	Thm. 4.2 (Ginibre)
4	SU(2)	1	4	1/100	1/25	≥ 0.0799	Thm. 4.2 (Ginibre)
4	SU(3)	1	1	1/100	1/100	≥ 1.4131	Thm. 5.1 ($\mathbb{Z}_N$-KP)
4	SU(3)	1	2	1/100	1/50	≥ 0.7175	Thm. 5.1 ($\mathbb{Z}_N$-KP)
4	SU(4)	1	2	1/200	1/100	≥ 1.0102	Thm. 5.1 ($\mathbb{Z}_N$-KP)
4	SU(4)	2	2	1/200	1/100	≥ 2.0204	Thm. 5.1 ($\mathbb{Z}_N$-KP)
4	SU(6)	1	4	1/800	1/200	≥ 1.1925	Thm. 5.1 ($\mathbb{Z}_N$-KP)
4	SU(6)	3	4	1/800	1/200	≥ 3.5776	Thm. 5.1 ($\mathbb{Z}_N$-KP)
5	SU(2)	1	1	1/100	1/100	≥ 0.9312	Thm. 4.2 (Ginibre)
5	SU(2)	1	2	1/100	1/50	≥ 0.2382	Thm. 4.2 (Ginibre)
5	SU(3)	1	2	1/200	1/100	≥ 0.8787	Thm. 5.1 ($\mathbb{Z}_N$-KP)