Bayesian Matroid-Union Bounds for Quantum List Discrimination

Support Congestion, Process Compression, and Exact Adaptive–Parallel Phases

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August 2026

Abstract

In quantum list discrimination a measurement returns at most ℓ candidate labels and succeeds when the true hypothesis belongs to the returned list. We derive a hierarchy of upper bounds that retains the complete labeled dependence pattern of mixed quantum hypotheses and arbitrary priors. To the support subspaces of the hypotheses we associate their Rado independent-transversal matroid $\mathcal{R}$. We prove that the vector of true-label inclusion probabilities of every ℓ-list measurement belongs to the independence polytope of the ℓ-fold matroid union $\mathcal{R}^{\vee \ell}$. Its rank has the closed support-congestion formula

$$r_\ell(A) = \min_{C \subseteq A} \big(|A \setminus C| + \ell \dim \sum_{i \in C} \operatorname{ran} \rho_i \big).$$

Thus the Bayes list success is at most the maximum prior weight of a set partitionable into ℓ independent support transversals. The result also covers soft lists whose fractional rewards have total budget at most ℓ per outcome. A prefix formula gives the bound for arbitrary priors and an exact rank-saturation audit for equality of any proposed strategy.

For uniform priors the rank formula yields an integer failure certificate $\Delta_\ell = \max_C(|C| - \ell d(C))$ and the precise list size at which this support obstruction disappears. Within any process-tester class satisfying the stated unnormalised-Choi trace interface, canonical compression transfers the hierarchy without adding a causal-order assumption. We exhibit exact attainment families. Orthogonal classes of repeated states attain the arbitrary-prior bound for every ℓ. More geometrically, m equiprobable regular-simplex states have exact one-guess optimum $(m-1)/m$ but perfect two-list success: a root-frame POVM reports an edge of the simplex and never omits the true vertex. Finally, an exact three-qubit-state example shows that a full union matroid does not imply the existence of a perfect-list measurement; its uniform two-list optimum is $(2 + 1/\sqrt{3})/3 < 1$. The theorem is therefore a support obstruction with exact attainments, not a general solution of state exclusion. The proof combines standard Rado–Edmonds theory with quantum support inequalities; the ingredients themselves are not claimed as new.

The process extension becomes exact for a complete input-dependent family whose channels dephase before preparing a classical output. For $M = 2^h$ equiprobable hypotheses, list size $\ell = 2^s$, and q uses, we prove against arbitrary entangled parallel inputs and adaptive quantum memories that

$$P^{(\ell)}_{\mathrm{par}}(q) = \min\{1, \ell(q+1)/M\}, \qquad P^{(\ell)}_{\mathrm{ad}}(q) = \min\{1, \ell 2^q/M\}.$$

Thus perfect list identification needs exactly $M/\ell - 1$ parallel calls but only $\log_2(M/\ell)$ adaptive calls. The fixed parallel transcripts induce partition matroids whose union caps are attained cell by cell. This is an exact list/feedback tradeoff in classical channels embedded quantumly, not a claim of quantum or indefinite-causal-order advantage.

For a complete unitary-error ensemble and fixed pure probe, we translate the known optimal approximate dense-coding law into the present process-list language, $P_{\mathrm{guess}} = (\operatorname{Tr} \sqrt{\rho_R})^2/D$.

It yields the sharp Schmidt-rank cap r/D and, for independent Weyl histories, exact list optima for carrier-only serial, unreferenced parallel, and Bell-assisted wirings. These channel theorems turn process compression into resource-resolved exact statements while retaining explicit boundaries on prior art and scope.

1 Introduction

Minimum-error quantum discrimination normally asks a measurement to output one label. A different and useful task permits a set-valued decision: the measurement may return a short list, and it succeeds whenever the true label is contained in that list. Taking complements turns this into quantum state exclusion or elimination. That subject has a well-developed semidefinite formulation and includes unambiguous, minimum-error, and worst-case variants [1]. Multi-state elimination has also been implemented experimentally [2]. We do not claim the operational task as new.

The question addressed here is structural. Suppose the hypotheses are mixed and their support subspaces overlap in a complicated labeled pattern. How large can the total probability be that the correct label appears in a list of size at most ℓ? A total Hilbert-space dimension records only one number. It does not distinguish a dense cluster of dependent, high-prior labels from independent, low-prior outliers. In the one-label problem this loss is repaired by a Rado-matroid Bayes bound [3]. Here we show that the full list hierarchy is governed not by an ad hoc multiple of that bound, but by the successive union powers of the same matroid.

The connection is exact at the polyhedral level. If s_i is the conditional probability that label i appears in the reported list, then for each set A of labels,

$$s(A) := \sum_{i\in A} s_i \le \ell \dim \sum_{i\in A} \operatorname{ran} \rho_i.$$

Combining these support inequalities with $s_i \le 1$ and minimizing over a subset produces precisely the rank inequalities of the ℓ-fold union of the Rado matroid. Edmonds' union theorem and independence-polytope theorem then convert the quantum inequality into a maximum-weight independent-set bound. This provides, in one statement, arbitrary priors, every finite list budget, mixed supports, and soft fractional rewards.

Three consequences make the hierarchy operational rather than merely formal. First, for equal priors its failure deficit is the integer

$$\Delta_\ell = \max_C \big(|C| - \ell d(C)\big), \qquad d(C) = \dim \sum_{i\in C} \operatorname{ran} \rho_i.$$

Second, this matroid support obstruction vanishes at the explicit density threshold

$$\max_C \left\lceil \frac{|C|}{d(C)} \right\rceil.$$

Third, canonical inverse-square-root compression carries the argument unchanged to quantum-process testers. A fixed tester normalization may describe parallel circuits, adaptive combs, or more general process-matrix strategies [4, 5, 6].

Compression by itself is only a converse and does not compare architectures. We therefore solve one input-dependent channel family exactly. The unknown channel measures a query from a binary-prefix alphabet and returns membership in the corresponding right-child cylinder. These cylinders are laminar. For a fixed parallel query tuple, their response fibres form a partition matroid, so the union bound is attained by reporting the largest-prior labels in each fibre. A flag-genie argument proves that coherent or entangled inputs cannot improve this value, while backward induction reduces every adaptive quantum strategy to a classical decision tree. The resulting all-depth phase diagram separates the linear growth of parallel transcript cells from the doubling supplied by feedback. Adaptive advantages for channel discrimination, including entanglement-breaking examples, are established in the literature [7, 8]; our contribution is not the existence of adaptivity, but the closed list-valued law and its exact matroid-union realization.

The disappearance threshold of this obstruction is necessary, not sufficient, for perfect list discrimination. Geometry inside the support remains relevant. We make that limitation exact: three pairwise independent qubit states have a free two-fold union matroid but cannot be perfectly two-list discriminated because their Bloch vectors lie in one strict hemisphere. Conversely, regular simplices attain a striking sharp transition. Their one-guess optimum loses exactly $1/m$, whereas a two-list root-frame measurement succeeds with certainty for every $m \geq 3$. The channel sections then go beyond static attainment: one family isolates feedback exactly, while complete unitary-error ensembles expose how the full Schmidt spectrum and the multitime wiring alter the achievable list probability.

The paper is organized as follows. Section 2 defines list and soft reward decisions. Section 3 reviews only the Rado and union facts needed later. Section 4 proves the polytope theorem and the nonuniform Bayes formula. Section 5 gives the process-tester extension. Section 6 derives the exact parallel–adaptive channel phase. Section 7 derives the process-list consequences of the dense-coding spectrum law and proves the unitary-history wiring laws. Section 8 derives the congestion deficit and obstruction threshold. Section 9 proves exact attainment families, and Section 10 proves the insufficiency counterexample. We close with computational certificates, related work, and explicit limitations.

2 List decisions and soft rewards

Let $E = \{1, \ldots, M\}$ label density operators ρ_i on a finite-dimensional Hilbert space $\mathcal{K}$. Let $p_i \geq 0$ and $\sum_i p_i = 1$. Zero priors are allowed.

Definition 2.1 (Deterministic list measurement)**.** For an integer $1 \leq \ell \leq M$, an ℓ-list measurement is a POVM $(M_L)_{L \in \mathcal{L}}$ indexed by a family $\mathcal{L} \subseteq \{L \subseteq E : |L| \leq \ell\}$. Outcome L reports the labels in L. Its true-label inclusion probabilities and Bayes list success are

$$s_i = \sum_{L \ni i} \operatorname{Tr}(M_L \rho_i), \qquad P_{\text{list}} = \sum_i p_i s_i. \tag{1}$$

Empty lists may be omitted without changing any statement.

This convention includes ordinary minimum-error discrimination at $\ell = 1$. At $\ell = M - 1$, perfect list success is the usual conclusive exclusion of at least one label. Intermediate values quantify multi-label elimination.

The proof needs only a fractional reward budget, not deterministic membership.

Definition 2.2 (Soft-list reward rule)**.** Let $(M_a)_a$ be a POVM. A soft-list reward rule assigns numbers $w_{ia} \in [0, 1]$ such that for each outcome a,

$$\sum_i w_{ia} \leq \ell. \tag{2}$$

Define

$$s_i = \sum_a w_{ia} \operatorname{Tr}(M_a \rho_i), \qquad P = \sum_i p_i s_i. \tag{3}$$

A deterministic list is recovered from $w_{iL} = \mathbf{1}_{\{i \in L\}}$. Fractional w_{ia} allow randomized postprocessing or partial rewards. The budget (2) is strictly more general than requiring at most ℓ nonzero rewards. The assumption $w_{ia} \leq 1$ is essential for the singleton bound $s_i \leq 1$.

Lemma 2.3 (Lists and exclusion)**.** *For a nonnegative inclusion reward, an at-most-ℓ list may be padded to an exactly-ℓ list without decreasing success. Complementing that list is therefore equivalent to reporting exactly $k = M - \ell$ excluded labels. In particular, perfect at-most-ℓ list discrimination is equivalent to perfect weak k-state exclusion after padding.*

Proof. For every outcome list L with $|L| < \ell$, add arbitrary labels from $E \setminus L$ until its size is ℓ. A true label already in L remains present, and some previously omitted labels may become successful, so the inclusion reward cannot decrease. The complement of the padded list has cardinality $M - \ell$, and the true label lies in the list exactly when it is not among the excluded labels. Complementation is reversible. □

This elementary dictionary aligns our convention with k-state exclusion and zero-error list decoding [9, 10]. Stronger exclusion notions that require exhaustive elimination of every tuple impose additional conditions and are not identified with the present average-reward task [11].

3 Rado matroids and their union powers

Put $S_i = \operatorname{ran} \rho_i$ and, for $A \subseteq E$, define

$$d(A) = \dim \sum_{i \in A} S_i, \qquad d(\varnothing) = 0. \tag{4}$$

Definition 3.1 (Support Rado matroid)**.** The support Rado matroid $\mathcal{R}$ on E declares $I \subseteq E$ independent when vectors $v_i \in S_i$ can be chosen so that $(v_i)_{i \in I}$ is linearly independent.

Rado's independent-transversal theorem [12] gives the rank formula

$$r_{\mathcal{R}}(A) = \min_{C \subseteq A} \big(|A \setminus C| + d(C)\big). \tag{5}$$

Equivalently, I is independent iff $|C| \le d(C)$ for every $C \subseteq I$. For rank-one states, $\mathcal{R}$ is simply the vector matroid of the state vectors.

The union $\mathcal{M}_1 \vee \mathcal{M}_2$ of two matroids on the same ground set has as independent sets the unions $I_1 \cup I_2$ with I_j independent in $\mathcal{M}_j$. Let $\mathcal{R}^{\vee \ell}$ denote the union of ℓ copies of $\mathcal{R}$. Thus I is independent in $\mathcal{R}^{\vee \ell}$ iff it can be partitioned into at most ℓ $\mathcal{R}$-independent sets. Edmonds' matroid-union theorem [13, 14] states

$$r_{\mathcal{R}^{\vee \ell}}(A) = \min_{B \subseteq A} \big(|A \setminus B| + \ell r_{\mathcal{R}}(B)\big). \tag{6}$$

For the present subspace family this formula collapses to one minimization.

Lemma 3.2 (Closed union-rank law)**.** *For every $A \subseteq E$,*

$$r_\ell(A) := r_{\mathcal{R}^{\vee \ell}}(A) = \min_{C \subseteq A} \big(|A \setminus C| + \ell d(C)\big). \tag{7}$$

Proof. Insert (5) into (6). The resulting minimum is over $C \subseteq B \subseteq A$. Write $B = C \,\dot\cup\, F$. Then

$$|A \setminus B| + \ell |B \setminus C| + \ell d(C) = |A \setminus C| + (\ell - 1)|F| + \ell d(C).$$

For fixed C this is minimized at $F = \varnothing$. Minimizing over C proves the claim. The argument also covers $\ell = 1$, when every F gives the same value. □

For a matroid $\mathcal{M}$ with rank r, its independence polytope is

$$\mathcal{P}(\mathcal{M}) = \{x \in \mathbb{R}^E_{\ge 0} : x(A) \le r(A) \text{ for all } A \subseteq E\}. \tag{8}$$

Edmonds proved that this polytope is the convex hull of incidence vectors of independent sets [15]. Consequently a nonnegative linear functional is maximized by the matroid greedy algorithm.

4 The matroid-union list theorem

Theorem 4.1 (Support-congestion law)**.** *For every density ensemble (p_i, ρ_i), every $\ell \geq 1$, and every soft-list reward rule of budget ℓ, the vector $s = (s_i)$ in (3) belongs to $\mathcal{P}(\mathcal{R}^{\vee \ell})$. Equivalently, for all $A \subseteq E$,*

$$\sum_{i \in A} s_i \leq r_\ell(A) = \min_{C \subseteq A} \big(|A \setminus C| + \ell d(C)\big). \tag{9}$$

In particular, the statement holds for deterministic ℓ-list measurements.

Proof. Let P_A denote the orthogonal projector onto $\sum_{i \in A} S_i$. Since ρ_i is positive with trace one, all its eigenvalues lie in $[0, 1]$. Therefore

$$0 \leq \rho_i \leq P_A \qquad (i \in A). \tag{10}$$

For each outcome a, (10) and (2) give

$$0 \leq \sum_{i \in A} w_{ia} \rho_i \leq \left(\sum_{i \in A} w_{ia} \right) P_A \leq \ell P_A.$$

It follows that

$$\begin{aligned} s(A) &= \sum_a \operatorname{Tr}\left(M_a \sum_{i \in A} w_{ia} \rho_i \right) \\ &\leq \ell \sum_a \operatorname{Tr}(M_a P_A) = \ell \operatorname{Tr} P_A = \ell d(A). \end{aligned} \tag{11}$$

Also

$$0 \leq s_i = \operatorname{Tr}\left(\rho_i \sum_a w_{ia} M_a \right) \leq 1,$$

because $\sum_a w_{ia} M_a \leq \sum_a M_a = I$. Hence for arbitrary $C \subseteq A$,

$$s(A) = s(C) + s(A \setminus C) \leq \ell d(C) + |A \setminus C|.$$

Minimizing over C and applying Theorem 3.2 proves (9). Nonnegativity of s and the rank-inequality description (8) finish the proof. □

The raw inequality (11) alone is not generally a matroid rank inequality because it can exceed $|A|$. The singleton constraints are what perform the exact submodular truncation in (7).

Corollary 4.2 (Arbitrary-prior Bayes cap)**.** *Every soft-list rule of budget ℓ obeys*

$$P \leq \beta_\ell(p, S) := \max \left\{ \sum_{i \in I} p_i : I \in \mathcal{I}(\mathcal{R}^{\vee \ell}) \right\}. \tag{12}$$

Thus the cap is the maximum prior mass of a set partitionable into at most ℓ independent support transversals.

Proof. By Theorem 4.1, $s \in \mathcal{P}(\mathcal{R}^{\vee \ell})$. The vertices of this polytope are incidence vectors of independent sets, so maximizing $p \cdot s$ over it gives (12). □

The cap has a useful formula that avoids choosing a maximizing independent set explicitly. Order the labels so that $p_{(1)} \geq \cdots \geq p_{(M)}$ and put $E_j = \{(1), \ldots, (j)\}$, $E_0 = \varnothing$, and $p_{(M+1)} = 0$.

Proposition 4.3 (Prefix formula and equality audit)**.** *One has*

$$\beta_\ell(p,S) = \sum_{j=1}^{M} \big(p_{(j)} - p_{(j+1)}\big) r_\ell(E_j). \tag{13}$$

For a fixed soft-list rule with inclusion vector s, equality $p \cdot s = \beta_\ell$ holds iff

$$s(E_j) = r_\ell(E_j) \quad \text{whenever } p_{(j)} > p_{(j+1)}. \tag{14}$$

The condition is independent of how labels are ordered inside a prior tie.

Proof. The matroid greedy theorem selects label (j) precisely when $r_\ell(E_j) - r_\ell(E_{j-1}) = 1$. Summation by parts gives (13). The same identity for an arbitrary vector is

$$p \cdot s = \sum_{j=1}^{M} (p_{(j)} - p_{(j+1)}) s(E_j).$$

Every coefficient and every deficit $r_\ell(E_j) - s(E_j)$ is nonnegative. Their weighted sum vanishes exactly at the strict prior breakpoints, proving (14). At a tie the coefficient is zero, so rearranging tied labels has no effect. $\square$

Remark 4.4 (Hierarchy). The matroids satisfy $\mathcal{R} \preceq \mathcal{R}^{\vee 2} \preceq \cdots$, so $\beta_1 \le \beta_2 \le \cdots \le 1$. The case $\ell = 1$ is the previously derived one-label Rado bound [3]. The new content is the union hierarchy, its soft-decision meaning, and the exact obstruction threshold and attainment results below.

5 General quantum-process testers

We use unnormalised Choi operators throughout this section. This avoids an input-dimension factor in the tester normalization.

Let $J_i \ge 0$ be the Choi operators of a finite family of deterministic processes. A tester with list-valued outcomes consists of positive operators T_L whose sum $T = \sum_L T_L$ is a valid deterministic tester normalization. For every candidate process in the stated tester interface,

$$\mathrm{Tr}(T J_i) = 1, \tag{15}$$

and the conditional probability of outcome L is $\mathrm{Tr}(T_L J_i)$. The linear constraints selecting parallel, sequential, or general testers affect the admissible set of T, but not the following compression.

Lemma 5.1 (Canonical compression)**.** *Let $\mathcal{K} = \operatorname{supp} T$ and let $T^{-1/2}$ be the Moore–Penrose inverse on $\mathcal{K}$. Define*

$$\rho_i = T^{1/2} J_i T^{1/2}, \qquad M_L = T^{-1/2} T_L T^{-1/2}\big|_{\mathcal{K}}. \tag{16}$$

Then (ρ_i) are density operators on $\mathcal{K}$, (M_L) is a POVM on $\mathcal{K}$, and

$$\mathrm{Tr}(M_L \rho_i) = \mathrm{Tr}(T_L J_i) \tag{17}$$

for all i, L.

Proof. Positivity is immediate and (15) gives $\mathrm{Tr}\,\rho_i = 1$. Since $0 \le T_L \le T$, the support of every T_L is contained in $\mathcal{K}$, and

$$\sum_L M_L = T^{-1/2} T T^{-1/2} = I_{\mathcal{K}}.$$

Using the support projection $P_{\mathcal{K}} = T^{-1/2} T^{1/2}$, cyclicity of trace, and $T_L = P_{\mathcal{K}} T_L P_{\mathcal{K}}$ gives

$$\begin{aligned}\mathrm{Tr}(M_L \rho_i) &= \mathrm{Tr}(T^{-1/2} T_L T^{-1/2} T^{1/2} J_i T^{1/2}) \\ &= \mathrm{Tr}(P_{\mathcal{K}} T_L P_{\mathcal{K}} J_i) = \mathrm{Tr}(T_L J_i).\end{aligned}$$

$\square$

Theorem 5.2 (Process-list matroid bound)**.** *Fix any admissible tester normalization T satisfying* (15)*. Let $S_i^T = \operatorname{ran}(T^{1/2} J_i T^{1/2})$. Every ℓ-list tester with this normalization obeys Theorems 4.1 to 4.3 for the Rado matroid of (S_i^T).*

Let $S_i^0 = \operatorname{ran} J_i$ and $d_0(A) = \dim \sum_{i \in A} S_i^0$. Then a tester-independent bound is obtained by replacing d_T by d_0 in (7)*. In particular, it holds after optimizing over all admissible general testers.*

Proof. The fixed-T statement follows from Theorems 4.1 and 5.1. Moreover,

$$\operatorname{ran}(T^{1/2} J_i T^{1/2}) \subseteq T^{1/2} \operatorname{ran} J_i,$$

so a common linear map can only decrease the dimension of every support sum: $d_T(A) \le d_0(A)$. Formula (7) is monotone in d, and so are the prefix and maximum-weight bounds. The resulting cap does not depend on T and survives the outer optimization. □

For rank-one Choi operators $J_i = |U_i\rangle\rangle\langle\langle U_i|$, the oracle matroid is the vector matroid of the Choi vectors. The theorem therefore bounds list discrimination of unitary channels by the union powers of their labeled linear-dependence matroid. We make no claim that the tester-independent cap is always attainable, or that indefinite causal order is irrelevant to the actual optimum.

For clarity, equality in the effective and original-support caps are different claims. For fixed T, Theorem 4.3 characterizes equality in the cap built from S_i^T. Equality in the coarser cap built from S_i^0 additionally requires that compression cause no weighted prefix-rank loss. We never infer the latter from saturation of the former.

5.1 A robust core-and-tail certificate

Exact support can change discontinuously under a small full-rank perturbation. The following secondary result retains a low-rank union-matroid core and pays explicitly for the discarded tail. It extends the one-label robust certificate of Ref. [3] to every list budget.

Corollary 5.3 (Robust process cores)**.** *Let $0 \le L_i \le J_i$ and $R_i = J_i - L_i$. For a fixed deterministic tester normalization T, put*

$$\lambda_i = T^{1/2} L_i T^{1/2}, \qquad S_i^L = \operatorname{ran} \lambda_i,$$

and form the union-matroid cap $\beta_\ell(T, L, p)$ from the subspaces (S_i^L). Every deterministic ℓ-list tester with normalization T obeys

$$P_{\text{list}} \le \min\left\{1,\ \beta_\ell(T, L, p) + \sum_i p_i \operatorname{Tr}(T R_i)\right\}. \tag{18}$$

Consequently, for an architecture class $\mathfrak{T}$ of deterministic normalizations,

$$P_{\text{list}}^{\mathfrak{T}} \le \min\left\{1, \sup_{T \in \mathfrak{T}} \beta_\ell(T, L, p) + \sum_i p_i \sup_{T \in \mathfrak{T}} \operatorname{Tr}(T R_i)\right\}. \tag{19}$$

The first supremum may be replaced by the coarser cap computed from the uncompressed core supports $\operatorname{ran} L_i$*.*

Proof. Decompose $s_i = s_i^L + s_i^R$ using $J_i = L_i + R_i$. The compressed core obeys $0 \le \lambda_i \le \rho_i$, where ρ_i is the density operator from Theorem 5.1. Thus λ_i is bounded by the projector onto any core support sum containing it. The proof of Theorem 4.1 applies to the core contributions even though $\operatorname{Tr} \lambda_i$ need not be one, and gives $\sum_i p_i s_i^L \le \beta_\ell(T, L, p)$.

For the tail,

$$\sum_{L \ni i} T_L \le \sum_L T_L = T,$$

so positivity gives $s_i^R \leq \mathrm{Tr}(TR_i)$. Summing with the priors and capping a success probability by one proves (18). Taking the architecture supremum and bounding each tail term separately proves (19). Common congruence can only lower core support-sum ranks, proving the final statement. $\square$

Remark 5.4. The tail term in (19) deliberately contains separate suprema. It does not assert that one tester normalization simultaneously maximizes all tails. For ordinary state discrimination, $T = I$ and the penalty reduces to $\sum_i p_i \operatorname{Tr} R_i$.

6 Input-dependent channels and an exact architecture phase

The tester theorem above bounds each architecture after its deterministic normalization has been chosen. It does not normally optimize that choice or compare architectures. We now give a channel family for which both tasks close exactly, including arbitrary entangled parallel probes and arbitrary adaptive quantum memories.

Let X and Y be finite alphabets. Hypothesis i specifies a deterministic function $f_i : \mathsf{X} \to \mathsf{Y}$ and the entanglement-breaking channel

$$\Phi_i(\tau) = \sum_{x \in \mathsf{X}} \langle x|\tau|x\rangle \, |f_i(x)\rangle\langle f_i(x)|. \tag{20}$$

The same unknown channel may be called repeatedly. A parallel strategy uses q copies simultaneously and may prepare an arbitrary state entangled across the q inputs and a reference. An adaptive strategy may retain an arbitrary quantum memory and choose each later input after the earlier classical outputs. No indefinite-causal-order strategy is included in the comparison.

For a deterministic query tuple $\boldsymbol{x} = (x_1, \ldots, x_q)$, set

$$c_{\boldsymbol{x}}(i) = (f_i(x_1), \ldots, f_i(x_q)), \qquad C_z(\boldsymbol{x}) = \{i : c_{\boldsymbol{x}}(i) = z\}.$$

If $w = (w_i)_{i \in E}$ is a nonnegative weight vector, write

$$B_\ell(\boldsymbol{x}; w) = \sum_z \sum_{j=1}^{\min\{\ell, |C_z(\boldsymbol{x})|\}} w^{\downarrow}_{C_z(\boldsymbol{x}),j}, \tag{21}$$

where $w^{\downarrow}_{C,j}$ is the jth largest weight in C.

Theorem 6.1 (Exact reduction of classical channels)**.** *For the channels in* (20) *and arbitrary priors p, the optimum of every q-use parallel quantum strategy is*

$$P^{(\ell)}_{\mathrm{par}}(q) = \max_{\boldsymbol{x} \in \mathsf{X}^q} B_\ell(\boldsymbol{x}; p). \tag{22}$$

The adaptive optimum is the exact Bellman value

$$V_0(w) = \sum_{j=1}^{\min\{\ell, |\operatorname{supp} w|\}} w^{\downarrow}_j, \tag{23}$$

$$V_q(w) = \max_{x \in \mathsf{X}} \sum_{y \in \mathsf{Y}} V_{q-1}\big((w_i \mathbf{1}_{\{f_i(x)=y\}})_i\big), \qquad P^{(\ell)}_{\mathrm{ad}}(q) = V_q(p). \tag{24}$$

For fixed $\boldsymbol{x}$, the effective supports form the partition matroid

$$\bigoplus_z U_{1,|C_z(\boldsymbol{x})|}. \tag{25}$$

Its ℓ-fold union bound is exactly (21) *and is attained by reporting the ℓ largest-prior labels in each response cell.*

Proof. For deterministic basis inputs, the output word is the orthogonal classical state $|c_{\boldsymbol{x}}(i)\rangle$. Labels in one cell are identical and different cells are perfectly distinguishable. The cellwise decision just described is therefore optimal. Its support matroid is (25); taking ℓ union copies changes every class capacity from one to $\min\{\ell, |C_z|\}$, and matroid greedy optimization gives (21).

For a general parallel input on $\mathsf{X}^{\otimes q} \otimes R$, each call first measures its input in the fixed basis. Give the decoder a genie flag containing the resulting tuple $\boldsymbol{x}$. Conditional on this flag, the reference operator is independent of i and all label dependence lies in $c_{\boldsymbol{x}}(i)$. Revealing a flag cannot lower success, so the original strategy is bounded by a convex combination of the values $B_\ell(\boldsymbol{x}; p)$ and hence by their maximum. A deterministic basis tuple attains that maximum, proving (22).

The same flag may be revealed after every call of a sequential strategy. Conditional on the complete prior input–output transcript, induction shows that the retained quantum state is fixed by the strategy and independent of the label inside the surviving response cell. Thus every quantum strategy is bounded by a randomized classical query tree with the same transcript. At each node the value is affine in the randomized next query, so an extreme deterministic query is optimal. The terminal reward is the sum of the ℓ largest surviving weights, and conditioning on the next output gives (23)–(24). Conversely, every deterministic query tree is an admissible adaptive quantum strategy, so the bound is exact. □

We now choose a family on which the recursion and the parallel cell problem both have closed forms. Fix $h \geq 1$ and label $M = 2^h$ channels by the binary leaves $u \in \{0,1\}^h$. The input alphabet consists of all prefixes v with $|v| < h$, and

$$f_u(v) = \mathbf{1}_{\{v1 \text{ is a prefix of } u\}}. \tag{26}$$

Thus query v asks membership in the right child below v. The queried cylinders $A_v = \{u : f_u(v) = 1\}$ are laminar: any two are nested or disjoint.

Theorem 6.2 (Complete parallel–adaptive list phase)**.** *Let the channels* (26) *have uniform priors, let $\ell = 2^s$ with $0 \leq s \leq h$, and let $q \geq 0$. Then*

$$P^{(\ell)}_{\text{par}}(q) = \min\left\{1, \frac{\ell(q+1)}{2^h}\right\}, \qquad P^{(\ell)}_{\text{ad}}(q) = \min\left\{1, \frac{\ell 2^q}{2^h}\right\}. \tag{27}$$

Consequently the exact numbers of calls required for perfect list success are

$$q^*_{\text{par}} = 2^{h-s} - 1 = \frac{M}{\ell} - 1, \qquad q^*_{\text{ad}} = h - s = \log_2 \frac{M}{\ell}. \tag{28}$$

In particular, one-label identification needs $2^h - 1$ parallel calls but only h adaptive calls.

Table 1: Exact resource phase for the laminar channel family, with $M = 2^h$ and $\ell = 2^s$. "Cells" denotes the maximum number of classical response cells before saturation.

Architecture	cells after q calls	list success	calls for success one
parallel, no feedback	$q+1$	$\min\{1, \ell(q+1)/M\}$	$M/\ell - 1$
adaptive feed-forward	2^q	$\min\{1, \ell 2^q/M\}$	$\log_2(M/\ell)$

Proof. Any q fixed queries select q sets from a laminar family. Such sets have at most $q+1$ nonempty membership cells: insert them in an order compatible with inclusion; a new set lies inside one old cell and can split only that cell. For uniform priors, Theorem 6.1 assigns at most $\ell/2^h$ success to each cell. This proves the parallel upper bound.

Put $d = h - s$. Query the prefixes of lengths below d in breadth-first order. Each new right-child cylinder splits one current prefix block into two, and until all $2^d - 1$ queries have been used every resulting block contains at least $2^s = \ell$ leaves. After $q \le 2^d - 1$ queries there are therefore exactly $q + 1$ cells, each contributing $\ell/2^h$. At $q = 2^d - 1$ the cells are the 2^d length-d prefix blocks, each of size ℓ; later calls can be ignored. This attains the first formula.

A binary adaptive tree of depth q has at most 2^q transcripts, and each terminal list covers at most ℓ labels. Hence its uniform success is at most the second expression in (27). To attain it, start at the empty prefix and query the currently certified prefix v. Output one selects child $v1$, while output zero selects $v0$. After q rounds the transcript identifies the first q bits and the surviving cell has size 2^{h-q}. At $q = d$ this size is ℓ, proving both the second formula and (28). $\square$

The depth-two member has four channels with function tables

$$000, \qquad 010, \qquad 100, \qquad 101.$$

With two calls, Theorem 6.2 gives

$$P_{\mathrm{par}}^{(1)} = \frac{3}{4}, \qquad P_{\mathrm{par}}^{(2)} = 1, \qquad P_{\mathrm{ad}}^{(1)} = 1. \tag{29}$$

Thus one feedback bit and one extra reported label close the same parallel one-guess deficit. The example is entirely classical after the input measurement; it proves neither a quantum-memory advantage nor an indefinite-causal-order advantage. Binary search, membership-query learning, and adaptive channel advantages are classical or established mechanisms [16, 7]. The result here is the exact list-valued phase against the full parallel and adaptive quantum strategy classes specified above, together with its attained union-matroid interpretation.

7 Entanglement spectra and multitime unitary histories

The preceding family isolates feedback but is classical after dephasing. We next give an input-sensitive quantum family in which the complete Schmidt spectrum, rather than only support rank, determines the optimum. The result also separates three precisely delimited multitime wirings.

Let $\dim A = D$ and let $(U_g)_{g=1}^{D^2}$ be a complete unitary error basis, so

$$\mathrm{Tr}(U_g^* U_{g'}) = D\delta_{gg'}. \tag{30}$$

For a pure probe $\psi \in A \otimes R$, put $\rho_R = \mathrm{Tr}_A |\psi\rangle\langle\psi|$ and $\psi_g = (U_g \otimes I)\psi$.

Theorem 7.1 (Dense-coding spectrum law in process form)**.** *For uniform discrimination of the D^2 channels $\Phi_g = \mathrm{Ad}_{U_g}$ with the fixed probe ψ,*

$$P_{\mathrm{guess}}(\psi) = \frac{(\mathrm{Tr}\sqrt{\rho_R})^2}{D}. \tag{31}$$

Consequently, among probes of Schmidt rank at most $r \le D$,

$$\max_{\mathrm{SR}(\psi) \le r} P_{\mathrm{guess}}(\psi) = \frac{r}{D}, \tag{32}$$

with equality exactly for a flat nonzero Schmidt spectrum. Every ℓ-list measurement for the same fixed probe satisfies

$$P_{\mathrm{list}}^{(\ell)}(\psi) \le \min\left\{1, \frac{\ell(\mathrm{Tr}\sqrt{\rho_R})^2}{D}\right\}. \tag{33}$$

The one-guess identity (31), including optimality for an arbitrary pure entangled resource, is the approximate dense-coding result of Feng, Duan, and Ji [17]. We include the short frame/dual proof to fix conventions and to make the list and multitime consequences below self-contained; we do not claim the identity itself as new.

Proof. Hilbert–Schmidt completeness gives the depolarizing identity

$$\sum_g U_g X U_g^* = D\,\mathrm{Tr}(X) I_A.$$

Therefore, on $A \otimes \operatorname{supp} \rho_R$,

$$S := \sum_g |\psi_g\rangle\langle\psi_g| = D I_A \otimes \rho_R. \tag{34}$$

The square-root measurement $M_g = S^{-1/2}|\psi_g\rangle\langle\psi_g|S^{-1/2}$ sums to the identity on this support. Moreover

$$c := \langle\psi_g|S^{-1/2}|\psi_g\rangle = \frac{\mathrm{Tr}\sqrt{\rho_R}}{\sqrt{D}}$$

is independent of g, so its success is c^2.

For the dual minimum-error SDP, set $Y = (c/D^2)S^{1/2}$. The rank-one domination criterion gives $Y \succeq |\psi_g\rangle\langle\psi_g|/D^2$, because

$$\langle\psi_g|(cS^{1/2})^{-1}|\psi_g\rangle = 1.$$

Also $\mathrm{Tr}\, Y = (\mathrm{Tr}\sqrt{\rho_R})^2/D$, proving optimality. If the nonzero Schmidt coefficients are $\lambda_1, \ldots, \lambda_r$, Cauchy–Schwarz gives $(\sum_j \sqrt{\lambda_j})^2 \le r$, with equality exactly when they are all $1/r$.

Finally, for a list POVM (M_L) define $N_g = \ell^{-1}\sum_{L\ni g} M_L$. Then $\sum_g N_g \preceq I$; completing this sub-POVM to a one-label POVM shows $P_{\text{guess}} \ge P^{(\ell)}_{\text{list}}/\ell$. Combining with (31) and the trivial cap one proves (33). □

The formula is stable under independent time slots. A tensor product of complete unitary error bases is a complete basis on dimension $D = \prod_t d_t$, so Theorem 7.1 already allows inputs entangled across different times. In particular, without a retained reference every pure parallel input has success $1/D$, whereas a global Schmidt-rank budget r gives exact optimum $\min\{r, D\}/D$.

For a list-valued wiring comparison, let each of n time slots carry an independent uniformly labeled Weyl operator $X^{a_t}Z^{b_t}$ on $\mathbb{C}^d$. There are $M = d^{2n}$ history hypotheses. We compare:

- SER_0: one d-dimensional carrier is reused, with known interleaving unitaries but no side memory, intermediate measurement, or retained transcript;
- PAR_0: n fresh inputs are used in parallel and may be entangled with each other, but no reference is retained;
- $\mathrm{PAR}_{\mathrm{Bell}}$: every input has a retained d-dimensional Bell reference.

Theorem 7.2 (Exact multitime wiring trichotomy)**.** *For the uniform Weyl histories above,*

$$\begin{aligned} P^{(\ell)}_{\mathrm{SER}_0} &= \min\{1, \ell/d^{2n-1}\},\\ P^{(\ell)}_{\mathrm{PAR}_0} &= \min\{1, \ell/d^{n}\},\\ P^{(\ell)}_{\mathrm{PAR}_{\mathrm{Bell}}} &= 1. \end{aligned} \tag{35}$$

Before saturation, fresh parallel inputs improve on a carrier-only serial wiring by the exact factor d^{n-1}.

Table 2: Exact list values for $M = d^{2n}$ independent Weyl histories. The resource restrictions are part of the architecture definitions preceding Theorem 7.2.

Architecture	terminal support dimension	optimal list success
SER_0	d	$\min\{1, \ell/d^{2n-1}\}$
PAR_0	d^n	$\min\{1, \ell/d^n\}$
$\mathrm{PAR}_{\mathrm{Bell}}$	d^{2n} orthogonal labels	1

Proof. A uniform ensemble of M pure states in dimension q_0 obeys $P_{\mathrm{list}}^{(\ell)} \le \min\{1, \ell q_0/M\}$ by Theorem 4.1. Every SER_0 output occupies only the terminal d-dimensional carrier. With no interleaving, Weyl multiplication produces the net label $(A, B) = (\sum_t a_t, \sum_t b_t)$ modulo d; input $|0\rangle$ and a computational measurement reveal A. Each value leaves exactly d^{2n-1} histories, so reporting any ℓ of them attains the first formula.

Every PAR_0 output occupies dimension d^n, even when its inputs are entangled across slots. The product input $|0\rangle^{\otimes n}$ reveals the full shift vector $(a_1, \dots, a_n)$ and leaves exactly d^n phase histories, attaining the second formula. Local Bell inputs turn the Weyl outputs into an orthonormal product Bell basis and prove the third. □

The support-rank upper bound in (33) need not be attained for an intermediate entanglement spectrum. For example, take $D = 4$ and the rank-two probe $(|00\rangle + |11\rangle)/\sqrt{2}$ for the 16 Weyl channels. The outputs split into four orthogonal shift sectors, each containing the phase states

$$\phi_b = (|0\rangle + \mathrm{i}^b|1\rangle)/\sqrt{2}, \qquad b = 0, 1, 2, 3.$$

Adjacent-pair reward operators have largest eigenvalue $1 + 1/\sqrt{2}$ and their top eigenvectors form a tight frame; the covariant dual is $(1 + 1/\sqrt{2})I/4$. Pinching over the shift sectors therefore proves the exact fixed-probe value

$$P_{\mathrm{list}}^{(2)} = \frac{2 + \sqrt{2}}{4} < 1, \tag{36}$$

although the coarse right-hand side of (33) equals one. This is a fixed-probe statement, not a resource-optimized impossibility theorem.

Unitary-error bases, dense coding, group-covariant discrimination, and parallel treatment of independent unitary channels are established subjects [17, 18, 19, 20, 21, 6]. We claim neither those ingredients, the one-guess spectrum formula, nor the dimension bound as new. The contribution of this section is narrower: the list-cap embedding, the exact multitime wiring trichotomy, and the explicit intermediate-spectrum failure of the coarse list cap.

8 Uniform congestion, deficits, and obstruction thresholds

For equal priors, Theorem 4.2 depends only on the rank of the full ground set.

Corollary 8.1 (Exact support deficit)**.** *For $p_i = 1/M$,*

$$P_{\mathrm{list}} \le \frac{r_\ell(E)}{M} = 1 - \frac{\Delta_\ell}{M}, \qquad \Delta_\ell = \max_{C \subseteq E}\bigl(|C| - \ell d(C)\bigr). \tag{37}$$

The maximum includes $C = \varnothing$, so $\Delta_\ell \ge 0$.

Proof. Taking $A = E$ in (7),

$$r_\ell(E) = \min_C(M - |C| + \ell d(C)) = M - \max_C(|C| - \ell d(C)).$$

Divide by M. □

The integer Δ_ℓ counts the rank deficit of the union matroid, not a continuous geometric error. It gives a fail-closed certificate: if any label cluster has more than ℓ labels per available support dimension, uniform perfect-list discrimination is impossible.

Corollary 8.2 (Disappearance threshold of the matroid obstruction)**.** *The support bound permits $P_{\mathrm{list}} = 1$ exactly when*

$$|C| \le \ell d(C) \qquad \text{for all } C \subseteq E. \tag{38}$$

The smallest list budget for which this happens is

$$\ell_{\mathrm{supp}} = \max_{\varnothing \ne C \subseteq E} \left\lceil \frac{|C|}{d(C)} \right\rceil. \tag{39}$$

Equivalently, E can then be partitioned into ℓ_{supp} independent sets of the support Rado matroid.

Proof. The cap equals one iff $r_\ell(E) = M$, equivalently iff $\Delta_\ell = 0$. This is exactly (38); minimizing the integer ℓ gives (39). The final statement is Edmonds' matroid-partition criterion applied to $\mathcal{R}$, or directly the condition that E be independent in $\mathcal{R}^{\vee\ell}$. □

The adjective "support" in ℓ_{supp} is essential. It is a lower bound on the list budget required for perfect physical success, but equality need not hold; Section 10 gives an exact failure.

It is also not the strongest known feasibility obstruction. The projector test for conclusive k-state exclusion of Stratton, Hsieh, and Skrzypczyk [9], applied after Theorem 2.3 and then to each restricted label set C, requires

$$\sum_{i \in C} \Pi_i \le \ell P_C, \tag{40}$$

where Π_i projects onto S_i and P_C onto $\sum_{i \in C} S_i$. For pure states, taking the trace recovers $|C| \le \ell d(C)$. For mixed states, and even geometrically for pure states, the operator inequality can be strictly stronger. Thus (39) is precisely the disappearance threshold of the *matroid* obstruction, not a new feasibility characterization.

Proposition 8.3 (Strict improvement over total dimension)**.** *Let five distinct pure states lie in a common two-dimensional subspace and let a sixth pure state span an orthogonal line. Give the five coplanar labels prior $a = 19/100$ each and the orthogonal label prior $b = 5/100$. For list budget $\ell = 2$, the total-dimension relaxation is one, whereas the matroid-union cap is*

$$\beta_2 = 4a + b = \frac{81}{100}. \tag{41}$$

Proof. Choose, for example, $\phi_j = (e_0 + j e_1)/\sqrt{1 + j^2}$ for $j = 0, \ldots, 4$, and $\phi_5 = e_2$. The vector matroid is $U_{2,5} \oplus U_{1,1}$. Its two-fold union is $U_{4,5} \oplus U_{1,1}$, so a maximum-weight independent set retains four coplanar labels and the coloop, proving (41). The total dimension is $D = 3$ and $\ell D = 6 = M$; hence the relaxation that keeps only D and the singleton bounds gives one. No attainability claim is made for the value 81/100. □

9 Sharp families

We first show that the arbitrary-prior theorem is exactly attainable for a complete family of partition matroids.

Proposition 9.1 (Orthogonal parallel classes)**.** *Let $(e_a)_{a=1}^q$ be orthonormal. Partition E into nonempty classes E_a, and set $\rho_i = |e_a\rangle\langle e_a|$ for $i \in E_a$. For every prior and list budget ℓ,*

$$P_{\mathrm{list}}^{\mathrm{opt}} = \sum_{a=1}^{q} \sum_{i \in \mathrm{Top}_{\min(\ell, |E_a|)}(E_a)} p_i. \tag{42}$$

This equals the matroid-union cap.

Proof. The support matroid is the partition matroid with capacity one in each class; its ℓ-fold union has capacity ℓ in each class. The maximum-weight independent set therefore selects the $\min(\ell, |E_a|)$ largest priors in each class, proving the upper bound. Measure the orthogonal projectors $|e_a\rangle\langle e_a|$ and, on outcome a, report precisely those top-prior labels. This attains (42). $\square$

Repeated states make Theorem 9.1 combinatorially transparent. The next family uses only distinct, pairwise nonorthogonal pure states.

9.1 The regular-simplex transition

Let $m \geq 3$ and let unit vectors $\phi_1, \ldots, \phi_m \in \mathbb{C}^{m-1}$ satisfy

$$\langle \phi_i, \phi_j \rangle = -\frac{1}{m-1} \quad (i \neq j). \tag{43}$$

Their Gram matrix has one zero eigenvalue and eigenvalue $m/(m-1)$ with multiplicity $m-1$. Consequently

$$\sum_{i=1}^{m} \phi_i = 0, \qquad \sum_{i=1}^{m} |\phi_i\rangle\langle\phi_i| = \frac{m}{m-1} I. \tag{44}$$

Every proper subset is linearly independent, so their vector matroid is the uniform matroid $U_{m-1,m}$.

Theorem 9.2 (Exact one-to-two-list jump). *For the equiprobable regular-simplex ensemble,*

$$P_{\mathrm{opt}}^{(1)} = \frac{m-1}{m}, \qquad P_{\mathrm{opt}}^{(2)} = 1. \tag{45}$$

An optimal one-guess POVM is

$$N_i = \frac{m-1}{m} |\phi_i\rangle\langle\phi_i|. \tag{46}$$

An optimal two-list POVM is indexed by unordered pairs $i < j$:

$$\psi_{ij} = \frac{\phi_i - \phi_j}{\sqrt{2m/(m-1)}}, \qquad M_{ij} = \frac{2}{m} |\psi_{ij}\rangle\langle\psi_{ij}|, \tag{47}$$

and outcome (i, j) reports $\{i, j\}$.

Proof. For one guess, the matroid cap is $r_1(E)/m = (m-1)/m$. Equations (44) and (46) show that (N_i) is a POVM, and its conditional correct probability is $(m-1)/m$ for every label. Hence the cap is attained.

For two guesses, first note that

$$\sum_{i<j} |\phi_i - \phi_j\rangle\langle\phi_i - \phi_j| = m \sum_i |\phi_i\rangle\langle\phi_i| - |\sum_i \phi_i\rangle\langle\sum_i \phi_i| = \frac{m^2}{m-1} I. \tag{48}$$

Since $\|\phi_i - \phi_j\|^2 = 2m/(m-1)$, substituting (47) into (48) gives $\sum_{i<j} M_{ij} = I$.

If $k \notin \{i, j\}$, then (43) gives

$$\langle \phi_i - \phi_j, \phi_k \rangle = 0.$$

Thus an outcome whose reported pair omits k has zero probability on state ϕ_k. Every possible outcome contains the true label, and the two-list success is one. No probability can exceed one, completing the proof. $\square$

The pair differences in (47) form the normalized root frame of type A_{m-1}. For $m = 3$, the construction is the usual trine exclusion measurement. Antidistinguishability and conclusive exclusion are established subjects [1]; novelty is not claimed for that isolated fact. Its role here is to prove exact attainment of the first nontrivial union step for an all-distinct family of arbitrary size.

10 A full union matroid is not sufficient

The condition $r_\ell(E) = M$ says that the support inequalities alone do not forbid perfect list success. It does not construct a POVM. The following exact example prevents the combinatorial theorem from being misread as a geometric characterization.

Proposition 10.1 (Hemisphere obstruction)**.** *Consider the three pure qubit states*

$$\phi_1 = \begin{pmatrix}1\\0\end{pmatrix}, \qquad \phi_2 = \frac{1}{\sqrt{5}}\begin{pmatrix}2\\1\end{pmatrix}, \qquad \phi_3 = \frac{1}{\sqrt{5}}\begin{pmatrix}2\\i\end{pmatrix}. \tag{49}$$

Their support matroid is $U_{2,3}$, so $r_{\mathcal{R}^{\vee 2}}(E) = 3$ and the two-list support cap equals one. Nevertheless no perfect two-list measurement exists. For uniform priors its exact optimum is

$$P^{\mathrm{opt}}_{\mathrm{list}} = \frac{2 + 1/\sqrt{3}}{3} < 1. \tag{50}$$

Proof. The squared pairwise overlaps are

$$|\langle\phi_1,\phi_2\rangle|^2 = \frac{4}{5}, \quad |\langle\phi_1,\phi_3\rangle|^2 = \frac{4}{5}, \quad |\langle\phi_2,\phi_3\rangle|^2 = \frac{17}{25}. \tag{51}$$

Every pair is therefore linearly independent. The matroid is $U_{2,3}$, whose two-fold union is the free matroid on three labels.

Suppose a perfect two-list measurement existed. Every outcome list omits at least one of the three labels. Assign each outcome to one omitted label and sum the corresponding POVM effects, obtaining effects F_1, F_2, F_3 with $\sum_i F_i = I$. Perfect inclusion implies that every outcome omitting i has zero probability on ϕ_i, so

$$F_i\phi_i = 0. \tag{52}$$

Thus (F_i) would antidistinguish the three states.

Let $n_i \in \mathbb{R}^3$ be their Bloch vectors. A positive qubit operator annihilating ϕ_i has the form $F_i = a_i|\phi_i^\perp\rangle\langle\phi_i^\perp|$ with $a_i \geq 0$. Completeness requires

$$\sum_i a_i = 2, \qquad \sum_i a_i n_i = 0. \tag{53}$$

For (49), the z coordinates of the Bloch vectors are

$$(n_1)_z = 1, \qquad (n_2)_z = (n_3)_z = \frac{3}{5}. \tag{54}$$

They lie in a strict open hemisphere. The second equation in (53) cannot hold with nonnegative a_i whose sum is two. This contradicts (52) and proves nonperfectness.

It remains to prove the exact value (50). For three labels, minimum list failure equals minimum exclusion error. An exclusion POVM (F_i) yields the list $E \setminus \{i\}$ on outcome i. Conversely, from any two-list POVM assign each outcome to one label omitted by its list; the resulting exclusion error is no larger than the original list failure. Taking infima in the two directions proves equality.

The uniform exclusion SDP and its dual are

$$e_* = \min_{F_i \geq 0,\ \sum_i F_i = I} \frac{1}{3}\sum_i \mathrm{Tr}(F_i\rho_i) = \max_{Y \leq \rho_i/3} \mathrm{Tr}\, Y. \tag{55}$$

Put

$$u = \left(\frac{1}{3}, \frac{1}{3}, \frac{2}{3}\right), \qquad R = \frac{1}{\sqrt{3}}.$$

The Bloch vectors are

$$n_1 = (0,0,1), \quad n_2 = (4/5,0,3/5), \quad n_3 = (0,4/5,3/5),$$

and satisfy $|u - n_i| = R$ and $u \cdot n_i = 2/3$. The dual operator

$$Y = \frac{1}{6}\big[(1-R)I + u\cdot\sigma\big] \tag{56}$$

is feasible because

$$\frac{1}{3}\rho_i - Y = \frac{1}{6}\big[RI + (n_i - u)\cdot\sigma\big] \geq 0.$$

Its trace is $(1-R)/3$.

For primal attainment, define

$$m_i = \frac{u - n_i}{R}, \qquad (w_1, w_2, w_3) = \left(\frac{1}{3}, \frac{5}{6}, \frac{5}{6}\right), \qquad F_i = \frac{w_i}{2}(I + m_i\cdot\sigma).$$

Here $|m_i| = 1$, $\sum_i w_i = 2$, and $\sum_i w_i n_i = 2u$, hence $\sum_i F_i = I$. Moreover $m_i \cdot n_i = -R$, and therefore

$$\frac{1}{3}\sum_i \mathrm{Tr}(F_i\rho_i) = \frac{1}{3}\sum_i \frac{w_i}{2}(1-R) = \frac{1-R}{3}.$$

Primal and dual values coincide. Thus $e_* = (1 - 1/\sqrt{3})/3$ and $P_{\mathrm{list}}^{\mathrm{opt}} = 1 - e_*$, proving (50). □

The example also shows why perturbing a degenerate support arrangement can change attainability without changing the matroid. Support ranks are piecewise constant; antidistinguishability is controlled by a convex geometric condition inside that rank stratum.

11 Computation and exact certificates

For a represented subspace family, the cap is finite and combinatorial. One may evaluate $d(C)$ by exact matrix rank and use (7) directly for small M. For larger instances, standard matroid-union and submodular-minimization algorithms apply; no quantum SDP is needed to certify the cap. Cunningham gives improved algorithms for matroid partition and intersection [22]. This computational observation concerns the upper bound, not construction of an optimal POVM.

The accompanying verifiers use the Python standard library and exact rational or algebraic arithmetic. They perform the following fail-closed checks:

1. exhaustive comparison of (7) with a brute-force coloring definition on 240 small subspace instances;

2. all subset inequalities for an exact list-measurement fixture;

3. the one-guess and edge-root POVMs for regular simplices $m = 3, \ldots, 8$ through rational Gram identities;

4. exhaustive laminar-channel phase values for 70 small (h, s, q) instances, including the partition-matroid cell formula;

5. complete-unitary-basis frame spectra, Schmidt-rank caps, Weyl wiring values, and the rank-two fixed-probe list certificate;

6. the exact overlaps, free union matroid, strict-hemisphere obstruction, and primal–dual optimum of Theorem 10.1;

7. the strict-flat value 81/100 versus the total-dimension value one;

8. monotonicity of support-sum ranks under a common process compression.

The finite tests are not a proof of the general theorem. They guard the rank formula, normalizations, and fixtures against transcription and convention errors. The general proofs are Theorems 3.2, 4.1, 5.2 and 6.1 and Theorems 6.2, 7.1 and 7.2.

12 Related work and priority boundary

Quantum minimum-error discrimination is classically formulated through the Holevo–Yuen–Kennedy–Lax optimality conditions [23, 24] and is reviewed in standard texts and surveys [25, 26]. State exclusion reverses the decision: an outcome rules out one or more hypotheses. Bandyopadhyay et al. give an SDP, optimality conditions, and variants including unambiguous and worst-case exclusion [1]. Experimental elimination of multiple states is demonstrated by Webb et al. [2]. Stratton, Hsieh, and Skrzypczyk derive projector and rank conditions for weak and strong k-state exclusion [9]. Johnston, Russo, and Sikora characterize pure-state antidistinguishability through Gram-matrix incoherence and give tight overlap bounds [27]. Recent work studies group-generated exclusion [28], global-versus-LOCC higher-order exclusion [11], and zero-error list decoding of classical–quantum channels, including the trine at list size two [10]. By Theorem 2.3, our list success is the complementary set-valued formulation with $k = M - \ell$. Therefore neither list decoding, the simplex/trine measurement, state exclusion, nor the nonsufficiency of rank conditions is presented as new.

Rado's theorem on independent representatives is classical [12]. Edmonds established the minimum-partition criterion and matroid union [13, 14], as well as the rank-inequality description of matroid polyhedra [15]. Our use of these results is direct and fully credited. The potentially new point is narrower: the all-subset Bayesian inequality that places the entire inclusion-probability vector in an exact union-matroid polytope, together with arbitrary priors, soft rewards, the robust core-tail version, and canonical process-tester compression.

Quantum testers and combs provide a common language for discrimination of multi-time processes [4]. Canonical inverse-square-root normalization is standard [5]; general and indefinite-order strategies have been compared in concrete unitary discrimination problems [8, 6], and restricted tester classes have a general convex formulation [29]. Entanglement-breaking channels can already exhibit finite-query adaptive advantages [7]; asymptotic comparisons obey different laws [30]. Theorem 5.1 recalls the normalization only to make the list theorem convention-safe. The input-dependent application does not claim adaptivity, binary search, or membership queries as new. Its specific contribution is the exact all-depth list phase and the proof that arbitrary quantum parallel probes and memories reduce to the stated cell and decision-tree optimizations for this family.

Complete unitary-error bases, dense coding, and symmetric operation discrimination are likewise established [17, 18, 19, 21]. In particular, Feng, Duan, and Ji already derive and attain the approximate dense-coding success law that becomes (31) in our notation. Independent unitary histories admit strong parallel reductions [20]. Accordingly, Theorems 7.1 and 7.2 do not claim the twirl, dense-coding protocol, spectrum formula, or support-dimension converse as new. The narrower addition is the process-list translation, its Schmidt-rank equality audit, the list-wiring trichotomy, and the explicit intermediate-rank failure of the coarse list cap.

The one-label Rado-matroid support bound was developed separately in Ref. [3]. The present paper cites that work rather than repackaging it. Its distinct contribution is the operational list hierarchy, the closed union-rank law, soft rewards, obstruction threshold, exact attainment/insufficiency pair, and the two architecture-sensitive channel theorems above. Targeted searches did not locate this combined theorem package. Absence from a search is not proof of priority; no claim of being first is made.

13 Limitations and conclusions

Several boundaries are essential.

First, the theorem is an upper bound. The inclusion vector belongs to a matroid polytope, but not every point of that polytope is generated by a POVM. Theorem 10.1 shows that even a free union matroid need not permit perfect list discrimination.

Second, the theorem uses exact support. An arbitrarily small full-rank noise component can enlarge every support and weaken the raw cap discontinuously. Theorem 5.3 mitigates this with chosen positive cores and an explicit tail penalty, but it does not canonically choose those cores or guarantee that the resulting certificate improves an SDP bound.

Third, the tester-independent process cap uses the uncompressed Choi support. A fixed tester normalization can lower the Rado-union rank and give a stronger certificate. Conversely, the cap does not determine which causal architecture attains the actual optimum. The exact architecture formulas proved here rely on either complete dephasing with a common input measurement or a complete unitary-error frame; they do not extend automatically to noisy, incomplete, or nonuniform channel families.

Fourth, the adaptive channel comparison excludes indefinite causal order, coherent phase-oracle access, inverse calls, and controlled bypass. The SER_0 wiring excludes side memory, intermediate measurements, and retained transcripts. These resource definitions are part of the theorems, not incidental implementation details.

Fifth, the soft-list extension assumes rewards in $[0, 1]$ and total reward budget at most ℓ for each outcome. General cost matrices with negative entries or larger column sums are outside the theorem.

Within these limits, the result gives a closed answer to a precise question: what can support dependence alone certify about quantum list decisions? The answer is the independence polytope of a matroid union. It yields arbitrary prior bounds, an integer congestion deficit, an exact obstruction-disappearance threshold, and a process-tester version under the stated Choi interface. The simplex and hemisphere examples show both sides of the boundary: the cap can be exactly attained by an all-distinct nonorthogonal family, but support combinatorics alone cannot replace quantum geometry. The channel theorems then identify two settings in which architecture optimization does close exactly: feedback doubles the number of resolved laminar cells at every call, while complete unitary frames convert the Schmidt spectrum and wiring dimension into exact success laws.

Data and code availability

The accompanying source package contains the LaTeX manuscript, bibliography, claim ledger, priority audit, research memo, and four standard-library Python verifiers. They perform exact rational or algebraic checks and require no network access or external data.

AI-assistance disclosure

The author used AI assistance for literature triage, algebraic cross-checks, code generation, and editorial revision. The mathematical claims, conventions, and final responsibility remain with the author.

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A A direct derivation for deterministic lists

For completeness, specialize Theorem 4.1 to a deterministic list POVM. For $A \subseteq E$,

$$\begin{aligned}
s(A) &= \sum_{i \in A} \sum_{L \ni i} \operatorname{Tr}(M_L \rho_i) \\
&= \sum_{L} \operatorname{Tr}\left(M_L \sum_{i \in A \cap L} \rho_i \right) \\
&\leq \sum_{L} |A \cap L| \operatorname{Tr}(M_L P_A) \\
&\leq \ell \operatorname{Tr}\left(P_A \sum_{L} M_L \right) = \ell d(A).
\end{aligned}$$

For $C \subseteq A$, split $A = C \,\dot{\cup}\, (A \setminus C)$ and use $s_i \leq 1$ on the second part. This gives

$$s(A) \leq \ell d(C) + |A \setminus C|.$$

The minimization over C is not an optional relaxation: by Theorem 3.2, it is exactly the rank of $\mathcal{R}^{\vee \ell}$.