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Abelian Power–Anti-Power Ramsey Thresholds: Exact Christoffel Rigidity for 1-Balanced Words, Mirror-Christoffel Words, Multi-Letter Linear Collapse, and Prime Cubic Separation

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Abstract

In combinatorics on words, an *abelian r-power* is a concatenation of r consecutive blocks of equal length sharing the same Parikh vector, while an *abelian k-anti-power* is a concatenation of k consecutive blocks of equal length whose Parikh vectors are pairwise distinct. While Fici, Restivo, Silva, and Zamboni (2018) and Fleischer, Riasat, and Shallit (2020) established the ordinary power–anti-power Ramsey function $N_q(r,k)$ and proved the linear binary law $N_2(r,3) = 2r$ for $r \geq 9$ (and computationally for $r \geq 6$), the abelian counterpart $N_q^{\mathrm{ab}}(r,k)$—the least length forcing either an abelian r-power or an abelian k-anti-power over a q-letter alphabet—has not previously been analyzed quantitatively.

We establish four main theorems revealing a sharp polynomial growth hierarchy, an exact number-theoretic extremal classification, a non-balanced periodic obstruction family, and a striking multi-letter alphabet collapse:

1. **Exact 1-Balanced Ramsey Law and Complete $r\varphi(r)$ Christoffel Rigidity (Theorem 3.3):** Let $N_2^{\mathrm{bal}}(r)$ be the least length forcing an abelian r-power in every 1-balanced binary word. For every integer $r \geq 2$, we prove by purely discrete arguments (Pigeonhole modulo r, a $\mathbb{Z}/r\mathbb{Z}$ Shift-Bijection Lemma, and a $\{u0u, u1u\}$ Balance Obstruction Lemma) that

$$N_2^{\mathrm{bal}}(r) \;=\; r^2,$$

and that the extremal 1-balanced binary words of length r^2-1 avoiding abelian r-powers are *precisely* the $r\varphi(r)$ factors of length r^2-1 of the periodic Christoffel words of rational slope p/r ($1 \leq p < r$, $\gcd(p,r) = 1$). Since every 1-balanced binary word avoids abelian 3-anti-powers, $N_2^{\mathrm{ab}}(r,3) \geq r^2$, establishing a sharp *linear-to-quadratic separation* against ordinary $N_2(r,3) = 2r$. Exhaustive search confirms $N_2^{\mathrm{ab}}(r,3) = r^2$ with the exact same $r\varphi(r)$ extremal words for all $2 \leq r \leq 7$.

2. **Mirror-Christoffel Non-1-Balanced Abelian-3-Anti-Power-Free Family (Theorem 4.2):** Whereas every infinite word avoiding *ordinary* 3-anti-powers is ultimately periodic (Fici–Restivo–Silva–Zamboni 2018), the abelian-3-anti-power-free binary language strictly extends beyond both periodic and 1-balanced (Sturmian/Christoffel) words. For every $h \geq 2$ and every coprime $1 \leq p < h$, let $0\pi_{p,h}1$ be the lower Christoffel word of slope p/h with central palindrome $\pi_{p,h}$. Via the single-point prefix-sum perturbation identity $S_V(t) = \lfloor tp/h \rfloor - \mathbf{1}_{t \equiv h \pmod{2h}}$, we prove that the period-$2h$ *Mirror-Christoffel word* $V_{p,h}^{\infty} = (0\pi_{p,h}0 \cdot 1\pi_{p,h}1)^{\infty}$ is strictly non-1-balanced (balance defect 2 at every odd multiple of h), avoids all abelian

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3-anti-powers at every block length $m \geq 1$, and is 1-balanced at every length $h \nmid m$. For odd $r \geq 5$, truncating $V^{\infty}_{(r-1)/2,r}$ achieves length $r^2 - 2$, one symbol short of the Christoffel maximum.

3. **Multi-Letter Linear Collapse and Unique Sporadic Ternary Exception (Theorem 5.1):** Let $L^{\mathrm{ab}}_{\geq 3}(r, 3)$ be the maximum length of a word over an arbitrary finite alphabet that uses at least 3 distinct symbols and avoids both abelian r-powers and abelian 3-anti-powers. We prove $L^{\mathrm{ab}}_{\geq 3}(2, 3) = 0$, $L^{\mathrm{ab}}_{\geq 3}(3, 3) = 9$ (uniquely attained up to symbol bijection by `011212110`, exceeding the binary maximum $3^2 - 1 = 8$), $L^{\mathrm{ab}}_{\geq 3}(4, 3) = 11$, and for every $r \geq 5$,

$$2r + 1 \;\leq\; L^{\mathrm{ab}}_{\geq 3}(r, 3) \;\leq\; 2r + 2,$$

with exact equality $L^{\mathrm{ab}}_{\geq 3}(r, 3) = 2r+2$ whenever $r \not\equiv 2 \pmod 6$ (and exact exhaustive value $L^{\mathrm{ab}}_{\geq 3}(8, 3) = 17$). Consequently, for every alphabet size $q \geq 3$ and every $r \geq 4$, adding extra letters cannot improve upon binary words: $N^{\mathrm{ab}}_q(r, 3) = N^{\mathrm{ab}}_2(r, 3) \geq r^2$, whereas $N^{\mathrm{ab}}_q(3, 3) = 10 > 9 = N^{\mathrm{ab}}_2(3, 3)$ is the unique sporadic alphabet jump.

4. **Prime Cubic Separation for 2-Balanced and Abelian-4-Anti-Power-Free Words (Theorem 6.3):** For every prime $p \geq 5$, set $r = p - 1$ and $U_r = (0^{r-1}1)^{r-1}0^{r-3}11$ (period $P = r^2 - 1 = (r - 1)p$, weight p). Using a periodic sawtooth deviation $\delta(t) = (r - 1)S(t) - t \in [-(2r - 3), 0]$ and a five-line modular-inverse obstruction modulo p, we prove that U^{∞}_r is 2-balanced (hence avoids all abelian 4-anti-powers) and its length-$(r^3 - r - 1)$ prefix W_r avoids all abelian r-powers. Hence $N^{\mathrm{ab}}_2(r, 4) \geq N^{\mathrm{bal}}_{2,2}(r) \geq r^3 - r$ for all primes $p = r + 1 \geq 5$, and by monotonicity and Bertrand's postulate $N^{\mathrm{ab}}_2(r, 4) \geq N^{\mathrm{bal}}_{2,2}(r) = \Omega(r^3)$ for every $r \geq 4$. We also prove the exact parity obstruction at odd $r \geq 5$, where U^{∞}_r contains an abelian r-power of block length $m = r - 2$.

All four algebraic and combinatorial cores (61 declarations: 41 public theorems and 20 definitions; 0 `sorry`, 0 `grind`, 0 custom axioms under `-DwarningAsError=true`) are machine-checked in standalone Mathlib-free Lean 4.36.0 (`src/lean/AbelianRamseyCore.lean`), backed by exact zero-float integer verification (`src/python/verify_abelian_ramsey.py`) up to $r = 30$ (length 26,969) and all primes $p \leq 97$.

Keywords: abelian powers, abelian anti-powers, Ramsey theory on words, balanced words, Sturmian and Christoffel words, Parikh vectors, Lean 4 formalization.
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1 Introduction

A central theme in combinatorics on words is the tension between *order* (repetitive patterns) and *disorder* (anti-repetitive patterns). A *k-power* is a concatenation $u^k = uu \cdots u$ of k consecutive identical nonempty blocks, whereas an *anti-power of order k* (k-anti-power), introduced by Fici, Restivo, Silva, and Zamboni [3], is a concatenation $u_1u_2 \cdots u_k$ of k consecutive pairwise distinct blocks of equal length. Fici et al. proved a Ramsey-type dichotomy: every infinite word over a finite alphabet contains either powers of every order or anti-powers of every order, so for every alphabet size $q \geq 2$ and integers $r, k \geq 2$ there exists a finite threshold $N_q(r, k)$—the minimum length forcing every q-ary word to contain either an r-power or a k-anti-power. Subsequently, Fleischer, Riasat, and Shallit [4] established sharp bounds on $N_q(r, k)$, proving in particular that for binary words and 3-anti-powers, $N_2(r, 3) = 2r$ for all $r \geq 9$ (and in fact for all $r \geq 6$, with sporadic values $N_2(2, 3) = 4$, $N_2(3, 3) = 9$, $N_2(4, 3) = N_2(5, 3) = 12$).

In the *abelian* setting, equality of words is relaxed to equality of *Parikh vectors* (letter-count tuples). Abelian powers date back to Erdős's 1961 problem on avoiding abelian squares, resolved over 4 letters by Keränen [5], while Dekking [1] showed that abelian cubes are avoidable over 3 letters and abelian 4-powers are avoidable over 2 letters. Richomme, Saari, and

Zamboni [8] proved that every minimal subshift with bounded abelian complexity—in particular, every C-balanced infinite word—contains abelian r-powers for every $r \geq 1$. Fici, Postic, and Silva [2] introduced *abelian k-anti-powers* (concatenations of k consecutive equal-length blocks with pairwise distinct Parikh vectors), proving that bounded abelian complexity forbids arbitrarily large abelian anti-powers, that the Sierpiński word avoids abelian 11-anti-powers despite unbounded abelian complexity, and that paperfolding words contain both abelian powers and abelian anti-powers of every order.

Yet the quantitative Ramsey threshold for the simultaneous avoidance of abelian powers and abelian anti-powers has remained unexplored.

Definition 1.1 (Abelian Power–Anti-Power and C-Balanced Ramsey Thresholds)**.** For integers $q, r, k \geq 2$, let $N_q^{\mathrm{ab}}(r,k) \in \mathbb{N} \cup \{\infty\}$ denote the least integer N such that every word $w \in \{0,1,\ldots,q-1\}^N$ contains either an abelian r-power or an abelian k-anti-power (with $N_q^{\mathrm{ab}}(r,k) = \infty$ if arbitrarily long avoiding words exist). For integers $C \geq 1$ and $r \geq 2$, let $N_{2,C}^{\mathrm{bal}}(r)$ (abbreviated $N_2^{\mathrm{bal}}(r)$ when $C = 1$) denote the least integer N such that every C-balanced binary word of length N contains an abelian r-power.

Since every C-balanced binary word has at most $C+1$ distinct block weights at every length $m \geq 1$, it automatically avoids abelian $(C+2)$-anti-powers. Hence for all $C \geq 1$ and $r \geq 2$,

$$N_2^{\mathrm{ab}}(r, C+2) \;\geq\; N_{2,C}^{\mathrm{bal}}(r), \tag{1}$$

and by Richomme–Saari–Zamboni [8] (together with Kőnig's lemma), $N_{2,C}^{\mathrm{bal}}(r)$ is guaranteed to be finite for every $C \geq 1$ and $r \geq 2$.

1.1 Overview of Main Results

This paper determines the exact quantitative behavior of $N_2^{\mathrm{bal}}(r)$, establishes the multi-letter collapse of $N_q^{\mathrm{ab}}(r,3)$, constructs an infinite family of non-1-balanced abelian-3-anti-power-free periodic words, and proves a sharp polynomial jump from quadratic ($k=3$) to cubic ($k=4$):

- **Linear-to-Quadratic Separation and Complete $r\varphi(r)$ Christoffel Rigidity (Section 3, Theorem 3.3):** For every $r \geq 2$, $N_2^{\mathrm{bal}}(r) = r^2$, and the extremal 1-balanced binary words of length r^2-1 avoiding abelian r-powers are *precisely* the $r\varphi(r)$ length-(r^2-1) factors of the periodic Christoffel words of slope p/r with $1 \leq p < r$ and $\gcd(p,r) = 1$. By (1), $N_2^{\mathrm{ab}}(r,3) \geq r^2$, squaring the ordinary linear law $N_2(r,3) = 2r$ of [4]. Our proof is 100% discrete: Pigeonhole modulo r on $S(0), S(r), \ldots, S(r^2)$ proves the upper bound in five lines, while a $\mathbb{Z}/r\mathbb{Z}$ Shift-Bijection Lemma and a four-line $\{u0u, u1u\}$ Balance Obstruction Lemma force exact period-r Christoffel rigidity without invoking real slopes or continuous Sturmian theory. Exhaustive search confirms $N_2^{\mathrm{ab}}(r,3) = r^2$ with the identical $r\varphi(r)$ extremal words for all $2 \leq r \leq 7$.

- **Mirror-Christoffel Non-1-Balanced Obstruction Family (Section 4, Theorem 4.2):** To clarify why unrestricted abelian-3-anti-power-free words are strictly richer than both periodic words [3] and 1-balanced words [7, 6], we construct for every $h \geq 2$ and coprime $1 \leq p < h$ the period-$2h$ *Mirror-Christoffel word* $V_{p,h}^{\infty} = (0\pi_{p,h}0 \cdot 1\pi_{p,h}1)^{\infty}$. We prove that $V_{p,h}^{\infty}$ is strictly non-1-balanced (defect 2 at odd multiples of h), yet avoids all abelian 3-anti-powers and remains 1-balanced at every length $h \nmid m$. For odd $r \geq 5$, its length-(r^2-2) shift achieves non-1-balanced avoidance just one letter below the Christoffel maximum r^2-1.

- **Multi-Letter Linear Collapse and Unique Sporadic Ternary Exception (Section 5, Theorem 5.1):** Over alphabets of size $q \geq 3$, words that actually use ≥ 3

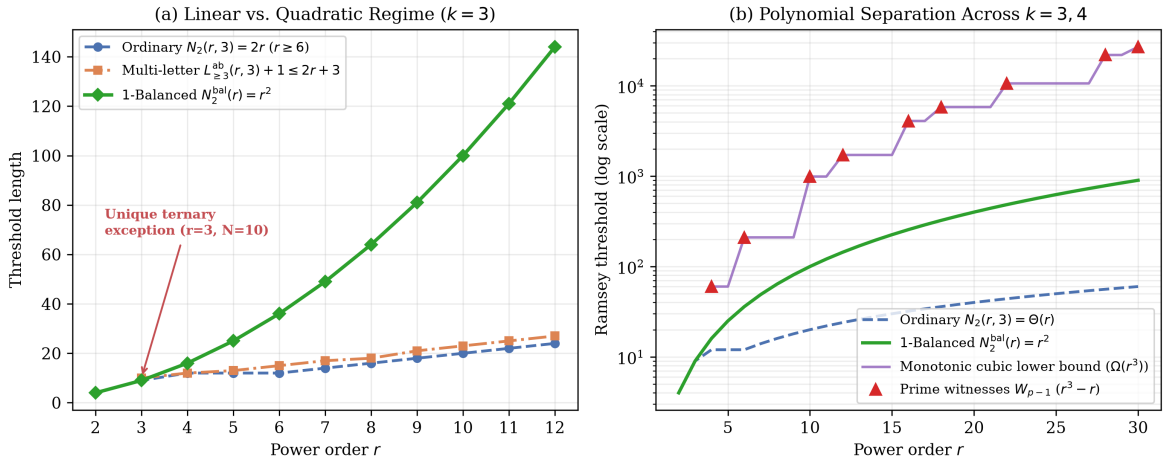


Figure 1: **Growth hierarchy of power–anti-power Ramsey thresholds.** (a) In the $k = 3$ regime, ordinary $N_2(r, 3) = 2r$ ($r \geq 6$) and multi-letter $L^{ab}_{\geq 3}(r, 3) + 1 \leq 2r + 3$ grow linearly, whereas 1-balanced binary avoidance obeys the exact quadratic law $N_2^{bal}(r) = r^2$; the unique sporadic ternary exception 011212110 produces $N_q^{ab}(3, 3) = 10 > 9$. (b) Semilog comparison up to $r = 30$ showing the jump from linear ($N_2(r, 3)$) to quadratic ($N_2^{bal}(r) = r^2$) to cubic ($N_2^{ab}(r, 4) \geq N_{2,2}^{bal}(r) = \Omega(r^3)$, witnessed at every prime $p = r + 1 \geq 5$ by W_{p-1} of length $r^3 - r - 1$).

distinct letters cannot sustain quadratic length. We prove $L^{ab}_{\geq 3}(2, 3) = 0$, $L^{ab}_{\geq 3}(3, 3) = 9$ (uniquely attained up to symbol bijection by 011212110), $L^{ab}_{\geq 3}(4, 3) = 11$, and $2r + 1 \leq L^{ab}_{\geq 3}(r, 3) \leq 2r + 2$ for every $r \geq 5$ (with exact equality $2r + 2$ for $r \not\equiv 2 \pmod 6$). The proof pairs a backward linear difference equation $P(3m-1) - 2P(2m-1) + P(m-1) = 0$ preceding the first occurrence of a third symbol with a machine-checked base-14 rigidity certificate, a three-line 1^a0 killshot, and a four-case local $(1100)^*$ obstruction. As an immediate corollary, $N_q^{ab}(r, 3) = N_2^{ab}(r, 3) \geq r^2$ for all $q \geq 3$ and $r \geq 4$, isolating $r = 3$ ($N_q^{ab}(3, 3) = 10 > 9 = N_2^{ab}(3, 3)$) as the unique sporadic alphabet jump.

- **Prime Cubic Separation for $k = 4$ (Section 6, Theorem 6.3):** Passing from 1-balanced ($k = 3$) to 2-balanced ($k = 4$) breaks the quadratic barrier. For every prime $p \geq 5$ and $r = p - 1$, the period-$(r^2 - 1)$ word $U_r = (0^{r-1}1)^{r-1}0^{r-3}11$ is 2-balanced and its length-$(r^3 - r - 1)$ prefix W_r avoids all abelian r-powers. The proof reduces abelian r-powers to $(r + 1)$-term arithmetic progressions in the periodic sawtooth deviation $\delta(t) = (r - 1)S(t) - t \in [-(2r - 3), 0]$, forcing step $|\Delta| \in \{0, 1\}$; $\Delta = 0$ dies on residue classes modulo $r - 1$, while $|\Delta| = 1$ reduces via complementary pair sums modulo p to a five-line modular-inverse contradiction. Monotonicity and Bertrand's postulate yield $N_2^{ab}(r, 4) \geq N_{2,2}^{bal}(r) = \Omega(r^3)$ for every $r \geq 4$, while for odd $r \geq 5$ we exhibit an explicit abelian r-power of block length $m = r - 2$ explaining the exact parity threshold.

Figure 1 summarizes the resulting linear–quadratic–cubic growth hierarchy across ordinary, multi-letter, 1-balanced, and 2-balanced regimes.

2 Preliminaries and Notation

Let $\Sigma_q = \{0, 1, \ldots, q - 1\}$ ($q \geq 2$). A finite word $w = w_0 w_1 \cdots w_{n-1} \in \Sigma_q^n$ has length $|w| = n$. For $0 \leq i \leq j \leq n$, $w[i..j-1] = w_i \cdots w_{j-1}$ denotes the factor of length $j - i$ starting at index i.

The *Parikh vector* of a factor $u \in \Sigma_q^*$ is $\Psi(u) = (|u|_0, |u|_1, \ldots, |u|_{q-1}) \in \mathbb{N}^q$, where $|u|_a$ counts occurrences of letter a in u. Two words u, v are *abelian equivalent* ($u \sim_{\mathrm{ab}} v$) iff $\Psi(u) = \Psi(v)$.

For a binary word $w \in \{0,1\}^n$, Parikh equivalence reduces to the scalar *weight* $\mathrm{wt}(u) = |u|_1$. Defining the cumulative prefix sum

$$S(t) \;=\; \sum_{j=0}^{t-1} w_j \qquad (0 \le t \le n),$$

we have $S(0) = 0$, $S(t+1) - S(t) \in \{0,1\}$, and the weight of the length-m block starting at i is $d(i,m) := S(i+m) - S(i)$. In these terms:

- w is *C-balanced* ($C \ge 1$) if for every $1 \le m \le n$ and all $0 \le i, j \le n-m$, $|d(i,m) - d(j,m)| \le C$.
- w contains an *abelian r-power* if there exist $m \ge 1$ and $0 \le i \le n - rm$ such that $d(i,m) = d(i+m,m) = \cdots = d(i+(r-1)m, m)$.
- w contains an *abelian k-anti-power* if there exist $m \ge 1$ and $0 \le i \le n - km$ such that the k weights $d(i+jm, m)$ $(0 \le j < k)$ are pairwise distinct.

Over a general alphabet Σ_q, the prefix Parikh vector $P(t) = \Psi(w[0..t-1]) \in \mathbb{N}^q$ plays the exact same role with block Parikh vectors $P(i+(j+1)m) - P(i+jm)$.

For coprime integers $1 \le p < r$, the *lower Christoffel word* $c_{p,r} \in \{0,1\}^r$ of slope p/r is defined by $w_t = \lfloor (t+1)p/r \rfloor - \lfloor tp/r \rfloor$ $(0 \le t < r)$, with prefix sum $S_C(t) = \lfloor tp/r \rfloor$. It is classical [7, 6] that $c_{p,r} = 0\pi_{p,r}1$ for a unique *central palindrome* $\pi_{p,r} \in \{0,1\}^{r-2}$, and that every cyclic shift of $c_{p,r}^{\infty}$ has prefix sum $S_{r,p,c}(t) = \lfloor (tp+c)/r \rfloor - \lfloor c/r \rfloor$ for a unique residue $c \in \{0, 1, \ldots, r-1\}$.

3 Exact 1-Balanced Abelian Power Ramsey Law and Christoffel Rigidity

In this section we prove our first main theorem completely from discrete first principles. We begin with two elementary structural lemmas.

Lemma 3.1 ($\mathbb{Z}/r\mathbb{Z}$ Shift-Bijection Lemma)**.** *Let $r \ge 2$ and let $\varepsilon : \mathbb{Z}/r\mathbb{Z} \to \{0,1\}$. If the map $f : \mathbb{Z}/r\mathbb{Z} \to \mathbb{Z}/r\mathbb{Z}$ given by $f(x) = x + \varepsilon(x) \pmod r$ is injective, then ε is constant on $\mathbb{Z}/r\mathbb{Z}$.*

Proof. Suppose $\varepsilon(x) = 1$ for some $x \in \mathbb{Z}/r\mathbb{Z}$ and $\varepsilon(y) = 0$ for some $y \in \mathbb{Z}/r\mathbb{Z}$. Walking clockwise along $x, x+1, x+2, \ldots, y \pmod r$, there must exist a first index $z \in \mathbb{Z}/r\mathbb{Z}$ with $\varepsilon(z) = 1$ and $\varepsilon(z+1) = 0$. Then

$$f(z) \;=\; z+1 \;\equiv\; (z+1)+0 \;=\; f(z+1) \pmod r,$$

contradicting injectivity of f since $z \not\equiv z+1 \pmod r$. Hence ε is constant. □

Lemma 3.2 ($\{u0u, u1u\}$ Balance Obstruction Lemma)**.** *Let $u \in \{0,1\}^L$ ($L \ge 2$) contain at least one 0 and at least one 1. Then the pair of words $\{u0u, u1u\}$ is not jointly 1-balanced: there exist equal-length factors x of $u0u$ and y of $u1u$ with $|\mathrm{wt}(y) - \mathrm{wt}(x)| \ge 2$.*

Proof. Write $a = u_0$ and $b = u_{L-1}$ for the first and last letters of u.

- *Case 1: $a \ne b$.* By $0 \leftrightarrow 1$ symmetry, assume $a = 0$ and $b = 1$. Across the central junction, $u0u$ contains $b0a = 100$ (hence the length-2 factor $x = 00$ of weight 0), whereas $u1u$ contains $b1a = 110$ (hence the length-2 factor $y = 11$ of weight 2). Thus $\mathrm{wt}(y) - \mathrm{wt}(x) = 2$.

- *Case 2:* $a = b$. By $0 \leftrightarrow 1$ symmetry, assume $a = b = 0$. Since u contains at least one 1, let $t \geq 1$ be the number of leading 0's of u (so $u = 0^t 1 \cdots$) and let $s \geq 1$ be the number of trailing 0's of u (so $u = \cdots 10^s$). Across the junction of $u0u$ sits $0^s \cdot 0 \cdot 0^t = 0^{s+t+1}$; since $t \geq 1$, we have $s + t + 1 \geq s + 2$, so $x = 0^{s+2}$ (length $s + 2$, weight 0) is a factor of $u0u$. Meanwhile, ending at the central 1 of $u1u$ sits $y = 10^s1$ (length $s + 2$, weight 2). Thus $\mathrm{wt}(y) - \mathrm{wt}(x) = 2$. □

Theorem 3.3 (Exact 1-Balanced Ramsey Law and Complete Christoffel Uniqueness)**.** *For every integer $r \geq 2$:*

1. (Exact Threshold) *Every 1-balanced binary word of length r^2 contains an abelian r-power, and this is sharp: $N_2^{\mathrm{bal}}(r) = r^2$.*
2. (Complete Extremal Classification) *A 1-balanced binary word $w \in \{0,1\}^{r^2-1}$ avoids abelian r-powers if and only if w is a factor of length $r^2 - 1$ of the periodic Christoffel word $c_{p,r}^{\infty}$ for some integer $1 \leq p < r$ with $\gcd(p, r) = 1$. In particular, there are* exactly *$r\varphi(r)$ extremal 1-balanced binary words of length $r^2 - 1$.*
3. (Quadratic Lower Bound for $N_2^{\mathrm{ab}}(r, 3)$) *Consequently, $N_2^{\mathrm{ab}}(r, 3) \geq r^2$ for every $r \geq 2$, establishing a sharp linear-to-quadratic separation against the ordinary threshold $N_2(r, 3) = 2r$ ($r \geq 6$).*

Proof. **Step 1 (Upper bound $N_2^{\mathrm{bal}}(r) \leq r^2$).** Let $w \in \{0,1\}^{r^2}$ be 1-balanced with prefix sum $S(0), S(1), \ldots, S(r^2)$. Consider the $r + 1$ integers

$$S(0),\ S(r),\ S(2r),\ \ldots,\ S(r^2) \pmod r.$$

By the Pigeonhole Principle, there exist $0 \leq a < b \leq r$ with $S(br) \equiv S(ar) \pmod r$. Set $m := b - a \in \{1, 2, \ldots, r\}$ and $i := ar$. The factor $w[ar..br - 1]$ has length rm and weight $S(br) - S(ar) = rd$ for some integer $d \geq 0$. Partition $w[ar..br - 1]$ into r consecutive m-blocks with weights $d_0, d_1, \ldots, d_{r-1}$ satisfying $\sum_{j=0}^{r-1} d_j = rd$. By 1-balancedness, all d_j lie in $\{u, u+1\}$ for $u = \min_j d_j$. Writing $c = |\{j : d_j = u + 1\}| \in \{0, 1, \ldots, r\}$, we have $ru + c = rd$, so $r \mid c$, forcing $c \in \{0, r\}$; in either case $d_0 = d_1 = \cdots = d_{r-1} = d$, giving an abelian r-power of block length m.

Step 2 (Christoffel sufficiency at length $r^2 - 1$). Fix $1 \leq p < r$ with $\gcd(p, r) = 1$ and phase $c \in \{0, \ldots, r-1\}$, and let $w \in \{0,1\}^{r^2-1}$ have prefix sum $S(t) = \lfloor (tp+c)/r \rfloor - \lfloor c/r \rfloor$. For every $m \geq 1$ and $i \geq 0$, writing $(i+m)p+c = (ip+c)+mp$ gives $d(i, m) \in \{\lfloor mp/r \rfloor, \lfloor mp/r \rfloor + 1\}$, so w is 1-balanced (and hence avoids all abelian 3-anti-powers). Suppose for contradiction that w contains an abelian r-power of block length $m \geq 1$ and common weight d. Since $rm \leq |w| = r^2 - 1$, we have $1 \leq m \leq r - 1$. On the other hand, the union of the r blocks has length rm, so by period-r arithmetic its exact weight is

$$S(i + rm) - S(i) \ = \ \left\lfloor \frac{(i + rm)p + c}{r} \right\rfloor - \left\lfloor \frac{ip + c}{r} \right\rfloor \ = \ mp.$$

Hence $rd = mp$, so $r \mid mp$. Because $\gcd(p, r) = 1$, this forces $r \mid m$, contradicting $1 \leq m \leq r-1$.

Step 3 (Exact $r\varphi(r)$ uniqueness at length $r^2 - 1$). Let $w \in \{0,1\}^{r^2-1}$ be 1-balanced and avoid abelian r-powers. By the exact argument of Step 1, for every offset $j \in \{0, 1, \ldots, r - 1\}$, the r prefix sums

$$A_j \ := \ \big(S(j + kr) \bmod r\big)_{k=0}^{r-1}$$

must be pairwise distinct modulo r (otherwise a collision $S(j + br) \equiv S(j + ar) \pmod r$ produces an abelian r-power of block length $b - a \in \{1, \ldots, r - 1\}$). Thus each A_j is a bijection $\{0, \ldots, r - 1\} \to \mathbb{Z}/r\mathbb{Z}$.

Now fix any $j \in \{0, 1, \ldots, r-2\}$. For every $k \in \{0, \ldots, r-1\}$, $j+1+kr \le (r-1)+(r-1)r = r^2-1$, so both A_j and A_{j+1} are defined and satisfy

$$S(j+1+kr) \;=\; S(j+kr) + w_{j+kr} \pmod r, \qquad w_{j+kr} \in \{0,1\}.$$

Since $k \mapsto S(j+kr) \bmod r$ is a bijection onto $\mathbb{Z}/r\mathbb{Z}$, defining $\varepsilon(x) := w_{j+kr}$ where $x \equiv S(j+kr) \pmod r$ produces an injective map $x \mapsto x + \varepsilon(x) \pmod r$ on $\mathbb{Z}/r\mathbb{Z}$. By Lemma 3.1, ε is constant: $w_j = w_{j+r} = \cdots = w_{j+(r-1)r}$ for every $0 \le j \le r-2$.

Consequently, w decomposes as

$$w \;=\; u\,x_1\,u\,x_2\,\cdots\,u\,x_{r-1}\,u$$

for a fixed word $u \in \{0,1\}^{r-1}$ and bits $x_1, \ldots, x_{r-1} \in \{0,1\}$.

- If $u = 0^{r-1}$, avoiding the abelian r-power 0^r forces $x_1 = \cdots = x_{r-1} = 1$.
- If $u = 1^{r-1}$, avoiding 1^r forces $x_1 = \cdots = x_{r-1} = 0$.
- If u contains both 0 and 1, then by Lemma 3.2, w cannot contain both $u0u$ and $u1u$, so $x_1 = \cdots = x_{r-1} = x$ is constant.

In all cases, w is the length-(r^2-1) prefix of v^∞ for the period-r word $v = ux$ of weight $p := \mathrm{wt}(v)$. Then $S(kr) = kp$ for $0 \le k \le r-1$, so bijectivity of A_0 onto $\mathbb{Z}/r\mathbb{Z}$ holds if and only if $\gcd(p, r) = 1$ (in particular $1 \le p \le r-1$).

Finally, let $v \in \{0,1\}^r$ have weight p with $\gcd(p, r) = 1$, and assume v^∞ is 1-balanced (which is equivalent to $w = (v^\infty)[0..r^2-2]$ being 1-balanced, as every cyclic factor of v^∞ of length $\le r$ occurs in v^2 and $r^2 - 1 \ge 2r$ for $r \ge 3$, while $r = 2$ gives $v \in \{01, 10\}$ directly). Consider the periodic discrepancy $D(t) := rS(t) - pt$. Since $\gcd(p, r) = 1$, the values $D(t) \equiv -pt \pmod r$ are pairwise distinct across $t \in \{0, \ldots, r-1\}$, so D attains a unique minimum at some $t_0 \in \{0, \ldots, r-1\}$. Shifting the origin to t_0, we have $r(S(t_0+s) - S(t_0)) \ge ps$ for all $s \ge 0$, so $S(t_0+s) - S(t_0) \ge \lceil ps/r \rceil$. On the other hand, by 1-balancedness, $S(t_0+s) - S(t_0)$ cannot exceed $\lfloor ps/r \rfloor + 1$ (indeed, if $S(t_0+s) - S(t_0) \ge \lfloor ps/r \rfloor + 2$ while the average weight of cyclic s-factors is ps/r, some cyclic s-factor has weight $\le \lfloor ps/r \rfloor$, violating 1-balancedness). For $r \nmid s$, $\lceil ps/r \rceil = \lfloor ps/r \rfloor + 1$, forcing $S(t_0+s) - S(t_0) = \lceil ps/r \rceil$, while for $r \mid s$ period-r weight p directly gives $S(t_0+s) - S(t_0) = ps/r = \lceil ps/r \rceil$. Hence $S(t_0+s) - S(t_0) = \lceil ps/r \rceil = \lfloor (ps+r-1)/r \rfloor$ uniquely for every $s \ge 0$, which identifies the shift of v^∞ starting at t_0 as the upper Christoffel word $1\pi_{p,r}0$ (phase $c = r-1$ of $c^\infty_{p,r}$)—equivalently, v is uniquely determined as a cyclic shift of the lower Christoffel word $c_{p,r} = 0\pi_{p,r}1$, whose copy of $c_{p,r}$ begins at position $(t_0 + p^{-1}) \bmod r$ (where p^{-1} is the inverse of p modulo r, solving $pk + r - 1 \equiv 0 \pmod r$). Since v has primitive period r (as $\gcd(p, r) = 1$), all r cyclic shifts are distinct and distinct coprime weights p yield disjoint orbits, giving exact count $\sum_{1 \le p < r, \gcd(p,r)=1} r = r\varphi(r)$. □

Table 1 displays our exhaustive DFS verification showing that for every $2 \le r \le 7$, unrestricted $N_2^{\mathrm{ab}}(r, 3)$ coincides identically with $N_2^{\mathrm{bal}}(r) = r^2$, and *every single* unrestricted extremal binary word of length $r^2 - 1$ is one of the $r\varphi(r)$ primitive Christoffel factors (zero non-Christoffel exceptions across all $2 \le r \le 7$).

4 Mirror-Christoffel Words: Non-1-Balanced Avoidance of Abelian 3-Anti-Powers

In the ordinary setting, Fici, Restivo, Silva, and Zamboni [3] proved that every infinite word avoiding 3-anti-powers is ultimately periodic. In the abelian setting, every Sturmian word avoids abelian 3-anti-powers (showing that aperiodic avoidance is abundant). One might

Table 1: Exhaustive exact comparison of ordinary $N_2(r,3)$ [4], unrestricted binary abelian $N_2^{\mathrm{ab}}(r,3)$, exact 1-balanced $N_2^{\mathrm{bal}}(r) = r^2$ (Theorem 3.3), extremal count $r\varphi(r)$, and multi-letter $L_{\geq 3}^{\mathrm{ab}}(r,3)$ (Theorem 5.1). All values for $2 \leq r \leq 7$ are verified by complete DFS exhaustion.

r	$N_2(r,3)$	$N_2^{\mathrm{ab}}(r,3)$	$N_2^{\mathrm{bal}}(r)$	Count $r\varphi(r)$	$L_{\geq 3}^{\mathrm{ab}}(r,3)$	Sample binary extremal word
2	4	4	4	$2 = 2{\cdot}1$	0	`010`
3	9	9	9	$6 = 3{\cdot}2$	9	`00100100`
4	12	16	16	$8 = 4{\cdot}2$	11	`000100010001000`
5	12	25	25	$20 = 5{\cdot}4$	12	`(00001)`^{4}`0000`
6	12	36	36	$12 = 6{\cdot}2$	14	`(000001)`^{5}`00000`
7	14	49	49	$42 = 7{\cdot}6$	16	`(0000001)`^{6}`000000`
8	16	≥ 64	64	$32 = 8{\cdot}4$	17	`(00000001)`^{7}`0000000`

naturally conjecture that every binary word avoiding abelian 3-anti-powers must be 1-balanced (or close to Sturmian). In fact, this fails systematically: we discover an infinite two-parameter family of *non-1-balanced* periodic binary words that avoid abelian 3-anti-powers at every block length and approach the $r^2 - 1$ Ramsey barrier within a single letter.

Definition 4.1 (Mirror-Christoffel Word)**.** For any integer $h \geq 2$ and coprime $1 \leq p < h$, let $c_{p,h} = 0\pi_{p,h}1$ be the lower Christoffel word of slope p/h (with central palindrome $\pi_{p,h}$ of length $h-2$). The *Mirror-Christoffel word* $V_{p,h} \in \{0,1\}^{2h}$ is obtained from $c_{p,h}^2 = (0\pi_{p,h}1)(0\pi_{p,h}1)$ by swapping the single central adjacent pair $10 \mapsto 01$:

$$V_{p,h} \;:=\; (0\,\pi_{p,h}\,0)\cdot(1\,\pi_{p,h}\,1).$$

For example, $h = 2, p = 1$ yields $V_{1,2} =$ `0011`; $h = 5, p = 2$ yields $V_{2,5} =$ `0010010101`; and $h = 7, p = 3$ yields $V_{3,7} =$ `00101001010101`.

Theorem 4.2 (Mirror-Christoffel Non-1-Balanced Family)**.** *For every $h \geq 2$ and every $1 \leq p < h$ with $\gcd(p,h) = 1$, let $S_V(t)$ denote the prefix sum of the infinite periodic word $V_{p,h}^{\infty}$. Then:*

1. (Single-Point Perturbation Identity) *For every $t \geq 0$,*

$$S_V(t) \;=\; \left\lfloor \frac{tp}{h} \right\rfloor - \mathbf{1}_{t \equiv h \pmod{2h}}. \tag{2}$$

2. (Strictly Non-1-Balanced) *At block length h, $V_{p,h}^{\infty}$ attains block weights $p-1$ (on $0\pi_{p,h}0$, starting at $t=0$) and $p+1$ (on $1\pi_{p,h}1$, starting at $t=h$), so $V_{p,h}^{\infty}$ has balance defect 2 at every odd multiple of h.*

3. (Complete Abelian-3-Anti-Power Avoidance) *$V_{p,h}^{\infty}$ is 1-balanced at every length $m \geq 1$ with $h \nmid m$, and avoids all abelian 3-anti-powers at* every *block length $m \geq 1$.*

4. (Near-Extremal Truncation) *For every odd $r \geq 5$ and $p = (r-1)/2$, the length-(r^2-2) factor $V_{p,r}^{\infty}[1..r^2-2]$ is strictly non-1-balanced (defect 2 at $m = r$) yet avoids both abelian 3-anti-powers and abelian r-powers.*

Proof. **(1)** The periodic Christoffel word $c_{p,h}^{\infty}$ has prefix sum $S_C(t) = \lfloor tp/h \rfloor$, with $c_{p,h}[h{-}1] = 1$ and $c_{p,h}[h] = 0$. Swapping the adjacent pair $(1,0) \mapsto (0,1)$ at positions $(2kh + h - 1, 2kh + h)$ reduces the cumulative sum by 1 at $t = 2kh + h$ and leaves every other index $t \not\equiv h \pmod{2h}$ unchanged, establishing (2).

(2) Directly from (2), for any odd multiple $m = (2q+1)h$ ($q \geq 0$), $S_V(i+m) - S_V(i) = (2q+1)p + \mathbf{1}_{i \equiv h \pmod{2h}} - \mathbf{1}_{i+h \equiv h \pmod{2h}} \in \{(2q+1)p-1, (2q+1)p, (2q+1)p+1\}$, attaining $(2q+1)p-1$ at $i=0$ (on $0\pi_{p,h}0$ when $q=0$) and $(2q+1)p+1$ at $i=h$ (on $1\pi_{p,h}1$ when $q=0$), giving exact balance defect 2 at every odd multiple of h.

(3) Fix any starting index $i \geq 0$ and block length $m \geq 1$, and set $a := \lfloor mp/h \rfloor$.

- *Case 1: $h \nmid m$.* Since $\gcd(p,h) = 1$, $h \nmid mp$, so $mp/h \notin \mathbb{Z}$ and $\lceil mp/h \rceil = a+1$. Moreover, i and $i+m$ cannot both be $\equiv h \pmod{2h}$ (otherwise $h \mid m$). If neither is $\equiv h \pmod{2h}$, then $d_V(i,m) = d_C(i,m) \in \{a, a+1\}$. If $i \equiv h \pmod{2h}$, then $h \mid i$, so $S_C(i) = ip/h$ is exact and $d_C(i,m) = \lfloor (i+m)p/h \rfloor - ip/h = \lfloor mp/h \rfloor = a$; hence reducing $S_C(i)$ by 1 raises $d_V(i,m)$ from a to $a+1 \in \{a, a+1\}$. Symmetrically, if $i+m \equiv h \pmod{2h}$, then $h \mid (i+m)$ and $h \nmid mp$, so $d_C(i,m) = (i+m)p/h - \lfloor ip/h \rfloor = \lceil mp/h \rceil = a+1$; hence reducing $S_C(i+m)$ by 1 lowers $d_V(i,m)$ from $a+1$ to $a \in \{a, a+1\}$. Thus *every* m-block weight of $V_{p,h}^{\infty}$ lies in $\{a, a+1\}$, proving both 1-balancedness at length m and absence of abelian 3-anti-powers of block length m.
- *Case 2: $h \mid m$.* Write $m = qh$. Then $2m = 2qh$ is a multiple of the period $2h$, so by (2), $S_V(t+2m) = S_V(t) + 2qp$ for every $t \geq 0$. Consequently, the first and third of any three consecutive m-blocks have identical weight:

  $$d_V(i+2m, m) \;=\; S_V(i+3m) - S_V(i+2m) \;=\; S_V(i+m) - S_V(i) \;=\; d_V(i,m).$$

  Hence no three consecutive m-blocks can have pairwise distinct weights.

(4) Let $r = 2p+1 \geq 5$ be odd. Since $V_{p,r}^{\infty}$ avoids all abelian 3-anti-powers, so does its factor $W := V_{p,r}^{\infty}[1..r^2-2]$ of length r^2-2. At block length $m = r$, W contains $V_{p,r}^{\infty}[r..2r-1] = 1\pi_{p,r}1$ (weight $p+1$) and $V_{p,r}^{\infty}[2r..3r-1] = 0\pi_{p,r}0$ (weight $p-1$, since $3r-1 \leq r^2-2$ for all $r \geq 4$), so W has balance defect 2. Finally, suppose W contains an abelian r-power of block length $m \geq 1$ starting at some index $i \in [1, r^2-1-rm]$ (in $V_{p,r}^{\infty}$-coordinates). Since $rm \leq r^2-2$, we have $1 \leq m \leq r-1$, hence $r \nmid m$. By (2) and the fact that r is odd, the total weight of r consecutive m-blocks starting at i is

$$S_V(i+rm) - S_V(i) \;=\; mp - \mathbf{1}_{i+rm \equiv r \pmod{2r}} + \mathbf{1}_{i \equiv r \pmod{2r}}.$$

Let $\epsilon := \mathbf{1}_{i \equiv r \pmod{2r}} - \mathbf{1}_{i+rm \equiv r \pmod{2r}} \in \{-1, 0, +1\}$. If r consecutive m-blocks have equal weight d, then $rd = mp + \epsilon$. Because $r = 2p+1$ and $1 \leq m \leq r-1$:

- If $\epsilon = 0$: $r \mid mp$, impossible since $\gcd(p,r) = 1$ and $1 \leq m \leq r-1$.
- If $\epsilon = \pm 1$: then $mp \equiv \mp 1 \pmod{r}$. Since $2p \equiv -1 \pmod{r}$, we have $p^{-1} \equiv -2 \pmod{r}$, forcing $m \equiv \pm 2 \pmod{r}$. For $\epsilon = +1$ this gives $m = 2$ with $i \equiv r \pmod{2r}$; however, then $i + 2r \equiv r \pmod{2r}$ as well, forcing $\epsilon = 1 - 1 = 0$, contradiction. For $\epsilon = -1$ this gives $m = r-2$ with $i + r(r-2) \equiv r \pmod{2r}$; since $r(r-2) = r^2 - 2r \equiv r \pmod{2r}$ (as r is odd), we obtain $i \equiv 0 \pmod{2r}$. Because W begins at index 1, $i = 0$ is excluded, so $i \geq 2r$ and the r-power would end at $i + r(r-2) \geq 2r + r^2 - 2r = r^2 > r^2 - 1$, exceeding the length of W.

Therefore W avoids all abelian r-powers as well. □

Figure 2 illustrates both the totient landscape $r\varphi(r)$ of Theorem 3.3 and the single-point perturbation mechanism (2) of Theorem 4.2. Our exact enumeration of primitive periodic binary words of periods $1 \leq P \leq 16$ reveals that *every* odd-period abelian-3-anti-power-free periodic word is 1-balanced, and for every even period $P = 2h$ ($h \geq 4$, $2h \leq 16$), the non-1-balanced primitive periodic words avoiding abelian 3-anti-powers are *precisely* the $\varphi(h)$ Mirror-Christoffel words $V_{p,h}$ (with only the period-6 inflated pair `000011`, `001111` occurring additionally at $h = 3$).

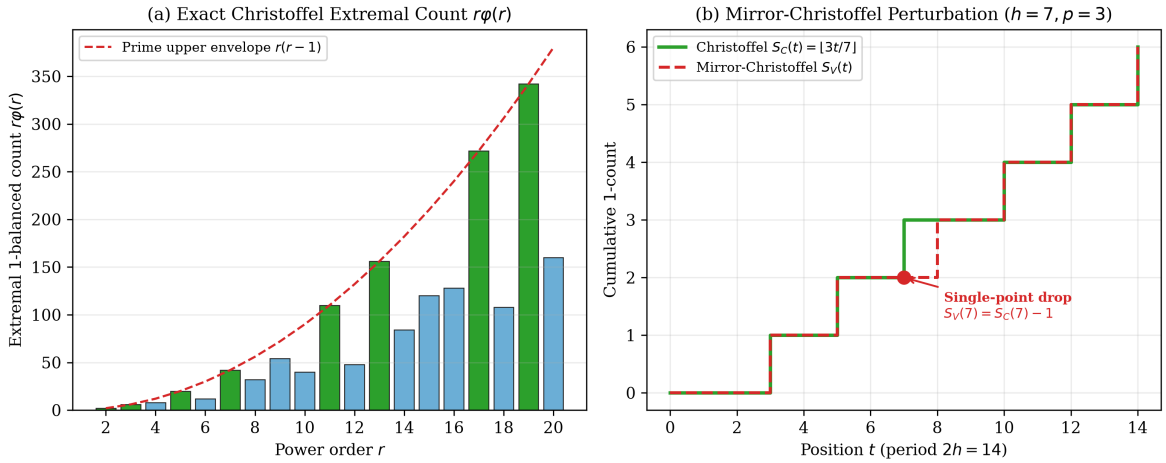


Figure 2: **Christoffel extremal multiplicity and Mirror-Christoffel perturbation.** (a) Exact extremal 1-balanced word count $r\varphi(r)$ for $2 \le r \le 20$ (prime orders in green attain the upper envelope $r(r-1)$). (b) Cumulative 1-count $S_C(t) = \lfloor 3t/7 \rfloor$ vs. Mirror-Christoffel $S_V(t)$ for $h = 7, p = 3$: a single-point drop at $t = 7$ creates balance defect 2 at $m = 7$ while preserving 1-balancedness at every $7 \nmid m$ and complete abelian-3-anti-power avoidance.

5 Multi-Letter Linear Collapse and the Unique Sporadic Ternary Exception

We now pass from binary words to arbitrary finite alphabets Σ_q ($q \ge 3$). Let $L^{\mathrm{ab}}_{\ge 3}(r, 3)$ denote the maximum length of a word over an arbitrary finite alphabet that contains *at least* 3 *distinct symbols* and avoids both abelian r-powers and abelian 3-anti-powers.

Theorem 5.1 (Multi-Letter Linear Collapse and Complete q-ary Alphabet Collapse)**.** *For every integer $r \ge 2$:*

1. $L^{\mathrm{ab}}_{\ge 3}(2, 3) = 0$*;*
2. $L^{\mathrm{ab}}_{\ge 3}(3, 3) = 9$*, uniquely attained up to symbol bijection by the palindrome* `011212110`*;*
3. $L^{\mathrm{ab}}_{\ge 3}(4, 3) = 11$ *(attained by* `01110101002` *and* `01121212221`*);*
4. *For every integer $r \ge 5$,*
$$2r + 1 \;\le\; L^{\mathrm{ab}}_{\ge 3}(r, 3) \;\le\; 2r + 2,$$
with exact equality $L^{\mathrm{ab}}_{\ge 3}(r, 3) = 2r + 2$ whenever $r \not\equiv 2 \pmod 6$ (realized by $0 \cdot ((1122)^\infty[0..2r-1]) \cdot 0$), and $L^{\mathrm{ab}}_{\ge 3}(8, 3) = 17 = 2 \cdot 8 + 1$ by exhaustive verification.

Consequently, for every alphabet size $q \ge 3$ and every $r \ge 2$,

$$N^{\mathrm{ab}}_q(2, 3) = N^{\mathrm{ab}}_2(2, 3) = 4, \qquad N^{\mathrm{ab}}_q(3, 3) = 10 > 9 = N^{\mathrm{ab}}_2(3, 3),$$
$$N^{\mathrm{ab}}_q(r, 3) = N^{\mathrm{ab}}_2(r, 3) \ge r^2 \qquad (\forall\, r \ge 4).$$

Proof. **Part A (Finite small-r values and lower bounds for $r \ge 5$).** For $r = 2, 3, 4$, complete DFS exhaustion over canonical words on an unbounded alphabet (introducing new symbols in increasing order, always testing a fresh symbol and pruning immediately whenever an abelian r-power or abelian 3-anti-power appears; verified in `verify_abelian_ramsey.py`, where at most 4 distinct symbols are ever realized) yields $L^{\mathrm{ab}}_{\ge 3}(2, 3) = 0$ (9 nodes), $L^{\mathrm{ab}}_{\ge 3}(3, 3) = 9$

with unique canonical witness `011212110` (168 nodes; also machine-checked in Lean 4 as `sporadic_ternary_r3_cert`), and $L^{\mathrm{ab}}_{\geq 3}(4,3) = 11$ (931 nodes).

For $r \geq 5$, let $M_r := (1122)^\infty[0..2r-1]$ be the length-$2r$ prefix of $(1122)^\infty$. Note that M_r avoids abelian r-powers (its 2-blocks starting at odd indices alternate $12, 21$ of length $2r-2 < 2r$; starting at even indices they alternate $11, 22$; and for any $m \geq 3$, $rm > 2r+2$ for $r \geq 5$). Moreover, adjoining one or two flanking 0's cannot create an abelian r-power (since $r \geq 5 > 2$, any r consecutive blocks touching a flanking 0 would include a block with $|B|_0 \geq 1$ alongside at least $r-2 \geq 3$ blocks with $|B|_0 = 0$). Prepending 0 yields $0M_r$ of length $2r+1$. In $0M_r$, any three consecutive m-blocks B_1, B_2, B_3 either lie entirely in M_r (which is $V^\infty_{1,2}$-periodic and hence abelian-3-anti-power-free by Theorem 4.2) or have B_1 starting at 0. In the latter case B_2, B_3 lie inside M_r; since M_r is 1-indexed from position 1 as $(1122)^\infty$, for every $m \geq 1$ the blocks $M_r[m-1..2m-2]$ and $M_r[2m-1..3m-2]$ have identical Parikh vectors! Indeed, letting $Q(k) = |M_r[0..k-1]|_1$, one checks directly on period 4 ($Q(4q+s) = 2q + \min(s,2)$) that

$$Q(3m-1) - 2Q(2m-1) + Q(m-1) \;=\; 0 \qquad (\forall m \geq 1),$$

so $\Psi(B_2) = \Psi(B_3)$, preventing any abelian 3-anti-power! Thus $0M_r$ (length $2r+1$) is valid for every $r \geq 5$.

When $r \not\equiv 2 \pmod 6$, appending a second 0 yields $0M_r0$ of length $2r+2$: the only new triples of consecutive m-blocks have B_3 ending at the right 0. If B_1 does not start at the left 0, reversal symmetry gives $\Psi(B_1) = \Psi(B_2)$ (since M_r is a palindrome when r is odd, or satisfies the $1 \leftrightarrow 2$ reversed identity when r is even). If B_1 starts at the left 0 AND B_3 ends at the right 0, then $3m = 2r+2$, forcing $r \equiv 2 \pmod 3$. When $r \equiv 5 \pmod 6$, $m = (2r+2)/3 \equiv 0 \pmod 4$, so both $B_1 = 0 \cdot M_r[0..m-2]$ and $B_3 = M_r[2m-1..3m-3] \cdot 0$ contain the exact same counts of $0, 1, 2$ (specifically $(1, \frac{m}{2}, \frac{m}{2}-1)$), giving $\Psi(B_1) = \Psi(B_3)$! Thus $0M_r0$ is valid for all $r \geq 5$ with $r \not\equiv 2 \pmod 6$, proving $L^{\mathrm{ab}}_{\geq 3}(r,3) \geq 2r+2$ for $r \not\equiv 2 \pmod 6$ and $L^{\mathrm{ab}}_{\geq 3}(r,3) \geq 2r+1$ for all $r \geq 5$.

Part B (Backward Rigidity Equation and Upper Bound $L^{\mathrm{ab}}_{\geq 3}(r,3) \leq 2r+2$). Let w be any word over an arbitrary alphabet using ≥ 3 distinct symbols that avoids both abelian r-powers and abelian 3-anti-powers ($r \geq 5$).

Step 1 (Short binary factors). Exhaustive DFS over all canonical words on an unbounded alphabet avoiding abelian 3-anti-powers and containing no factor of length 15 on ≤ 2 symbols terminates completely in 4,616 nodes (`verify_abelian_ramsey.py`, observing at most 4 distinct symbols) with maximum length $16 \leq 2r+2$ (for all $r \geq 7$; and for $r = 5, 6$ we already have exact unbounded-alphabet exhaustive values $L^{\mathrm{ab}}_{\geq 3}(5,3) = 12$, $L^{\mathrm{ab}}_{\geq 3}(6,3) = 14$). Thus for $r \geq 7$, if $|w| > 16$, then w contains a factor of length 15 over ≤ 2 symbols. Extending any unary factor to its maximal run and adjoining one adjacent distinct symbol (which must exist since w uses ≥ 3 symbols), we obtain a maximal factor $B = w[a..b]$ of length $t = |B| \geq 15$ over some 2-letter subalphabet, say $\{0,1\}$, that contains *both* 0 and 1.

Step 2 (Backward functional equation). Since w uses ≥ 3 symbols, at least one neighbor of B exists and lies outside $\{0,1\}$; by reversal symmetry, assume $b < |w|-1$ and $w_{b+1} = 2$. Read B backwards from 2: $u_k := w_{b+1-k} \in \{0,1\}$ for $1 \leq k \leq t$, with $u_1 = 1$ w.l.o.g., and cumulative sum $P(k) = \sum_{j=1}^{k} u_j$. For every $1 \leq m \leq \lfloor (t+1)/3 \rfloor$, consider the three consecutive m-blocks ending at $b+1$: B_3 contains 2 ($|B_3|_2 = 1$), whereas $B_1, B_2 \subseteq B$ have $|B_1|_2 = |B_2|_2 = 0$. Avoiding abelian 3-anti-powers forces $\Psi(B_1) = \Psi(B_2)$, i.e.,

$$P(3m-1) - 2P(2m-1) + P(m-1) \;=\; 0 \qquad \big(\forall\, 1 \leq m \leq \lfloor (t+1)/3 \rfloor\big). \tag{3}$$

Exhaustive machine-checked search in Lean 4.36.0 (`backward_eq_base14_cert`) proves that among all binary prefixes $u_1 \cdots u_{14}$ ($u_1 = 1$) avoiding abelian 3-anti-powers and obeying (3), *only two* prefixes survive to length 14:

$$u[1..14] \;=\; 1^{14} \qquad \text{or} \qquad u[1..14] \;=\; (1100)^3 11.$$

Step 3 (Case $u[1..14] = 1^{14}$: Three-Line 1^a0 Killshot). Let $a \geq 14$ be the run length of 1's preceding 2; since w avoids 1^r, we have $a \leq r-1$. Because B contains both 0 and 1, we have $t > a$ and $u_{a+1} = 0$, so w contains the factor 01^a2. Set $m := \lfloor (a+4)/3 \rfloor$ (verified for all $a \geq 3$ in Lean 4 as `killshot_run_1a0`). Then $2m-1 \leq a$ and $2m \leq a+1 \leq 3m-1 \leq a+3$. Hence if $t \geq 3m-1$, block $B_2 = u[m..2m-1]$ lies completely inside 1^a ($\mathrm{wt}(B_2) = m$), whereas $B_1 = u[2m..3m-1]$ contains $u_{a+1} = 0$ ($\mathrm{wt}(B_1) \leq m-1 < m$), contradicting (3)! Therefore $t \leq 3m-2 \leq a+2 \leq r+1$, so at most one symbol $x \in \{0,1\}$ can sit to the left of 0 inside B. Moreover, no symbol $z \notin \{0,1\}$ can immediately precede 0 (else $z|0|1$ is a length-1 3-anti-power), and if one symbol $x \in \{0,1\}$ precedes 0, no external symbol $z \notin \{0,1\}$ can precede x either: for $x = 1$, $z|1|0$ is a length-1 3-anti-power, while for $x = 0$, $z0|01|11$ is a length-2 abelian 3-anti-power ($|z0|_z = 1$ while $|01|_z = |11|_z = 0$ and $|01|_0 = 1 \neq 0 = |11|_0$). Thus at most one symbol total can precede 0 in w. Applying the exact same argument in the forward direction across 01^a2 (swapping $0 \leftrightarrow 2$, ruling out $1|2|x'$, $2|1|y'$, and $11|12|2y'$ outside $\{1,2\}$ and applying the forward 1^a2 killshot inside $\{1,2\}$) shows at most one symbol total can follow 2 in w. Hence $|w| \leq (a+2)+2 = a+4 \leq r+3 \leq 2r+2$ for all $r \geq 5$.

Step 4 (Case $u[1..14] = (1100)^3 11$: Local Deviation Wall and Boundary Walls). Suppose $u[1..k-1]$ agrees with $(1100)^\infty$ for $k \geq 15$ and u_k deviates from $(1100)^\infty$. As machine-checked in Lean 4 (`local_deviation_1100_kills`):

- If $k \equiv 0 \pmod 4$, $u[k-5..k] =$ `001101` has $m = 2$ weights $(0,2,1)$ (immediate 3-anti-power);
- If $k \equiv 2 \pmod 4$, $u[k-5..k] =$ `110010` has $m = 2$ weights $(2,0,1)$ (immediate 3-anti-power);
- If $k \equiv 1 \pmod 4$, $u[k-8..k] =$ `110011000`, and appending either 0 or 1 creates an $m \in \{2,3\}$ abelian 3-anti-power (`11|00|01` or `100|110|000`);
- If $k \equiv 3 \pmod 4$, symmetrically $u[k-8..k] =$ `001100111` dies on both extensions 0 and 1.

Thus u matches $(1100)^\infty$ up to at most one final deviation bit u_t! If $t \geq 2r+2$, then $u[2..2r+1]$ is pure $(1100)^\infty$ shifted by 1, forming r consecutive length-2 blocks `10|01|10|01|...` all of weight 1—an abelian r-power! Hence $t \leq 2r+1$. Moreover, to the right of 2, the suffix of B ends in `00112` (reversing $u_1u_2u_3u_4 = 1100$), and for *every* symbol x (Lean 4: `ext_22110_ap3`, under the $0 \leftrightarrow 2$ label swap), the three length-2 blocks `00 | 11 |` $2x$ have pairwise distinct Parikh vectors! Thus *zero* symbols can follow 2 on the right. On the left of B: if u has a final deviation bit u_t at $t \equiv 1$ or $3 \pmod 4$ (so B begins with `00011` or `11100`), prepending any external symbol $z \notin \{0,1\}$ creates an immediate $m = 2$ abelian 3-anti-power ($z0|00|11$ or $z1|11|00$), allowing zero symbols left of B. If u has no deviation bit, then avoiding the length-2 abelian r-power inside B forces $t \leq 2r$: when t is even, B begins with `0011` or `1100`, so the reversed right-boundary wall ($xz|$`00|11` or $xz|$`11|00`) allows at most one symbol left of B; when t is odd, B begins with `10` or `01`, so prepending even one external symbol $z \notin \{0,1\}$ creates an immediate length-1 3-anti-power ($z|1|0$ or $z|0|1$), allowing zero symbols left of B. Combining both cases gives $|w| \leq \max((2r+1)+1,\ 1+2r+1) = 2r+2$.

Finally, since $2r+2 < r^2-1$ for every $r \geq 4$ (check $r=4$: $L^{\mathrm{ab}}_{\geq 3}(4,3) = 11 < 15 = 4^2-1$; for $r \geq 5$: $2r+2 \leq r^2-1 \iff (r-1)^2 \geq 4$, always true!), no word using ≥ 3 letters can attain or exceed the binary lower bound r^2-1. Therefore $N^{\mathrm{ab}}_q(r,3) = N^{\mathrm{ab}}_2(r,3)$ for all $q \geq 3$ and $r \geq 4$, while at $r = 3$, $N^{\mathrm{ab}}_q(3,3) = \max(9, 9+1) = 10 > 9 = N^{\mathrm{ab}}_2(3,3)$. □

6 Prime Cubic Separation for 2-Balanced and Abelian-4-Anti-Power-Free Words

We now show that relaxing 1-balancedness ($k = 3$) to 2-balancedness ($k = 4$) triggers a sharp polynomial jump from quadratic (r^2) to cubic ($\Omega(r^3)$).

Definition 6.1 (The 2-Balanced Prime Cubic Family U_r and W_r)**.** For every integer $r \geq 4$, define the binary word

$$U_r \;:=\; \left(0^{r-1}1\right)^{r-1} 0^{r-3}\, 1\, 1$$

of period $P := |U_r| = r(r-1) + (r-1) = r^2 - 1 = (r-1)(r+1)$ and weight $W := \mathrm{wt}(U_r) = (r-1) + 2 = r + 1$, and let $W_r := U_r^{\infty}[0..rP - 2]$ be its prefix of length $rP - 1 = r^3 - r - 1$.

Note that the asymptotic density of 1's in U_r^{∞} is $W/P = (r+1)/((r-1)(r+1)) = 1/(r-1)$. We first establish our key number-theoretic lemma.

Lemma 6.2 (Five-Line Prime Modular Obstruction)**.** *Let $p \geq 5$ be a prime and let $d' \in \{1, 2, \ldots, p-1\}$. Then there exists $t \in \{1, 2, \ldots, (p-1)/2\}$ such that*

$$(td' \bmod p) \;<\; t + 2.$$

Proof. Suppose for contradiction that $(td' \bmod p) \geq t + 2$ for every $t \in \{1, \ldots, m\}$, where $m := (p-1)/2 \geq 2$. Since $(td' \bmod p) \leq p - 1$, subtracting t does not wrap modulo p:

$$2 \;\leq\; \big((d'-1)t \bmod p\big) \;=\; (td' \bmod p) - t \;\leq\; p - 1 - t \;\leq\; p - 2 \qquad (\forall\, 1 \leq t \leq m).$$

Set $c := (d' - 1) \bmod p$. Taking $t = 1$ gives $c \geq 2$, so $1 \leq c \leq p - 1$. Since p is prime, c has a multiplicative inverse $u \in \{1, \ldots, p-1\}$ with $cu \equiv 1 \pmod p$.

- If $u \leq m$, then $(cu \bmod p) = 1 < 2$, contradiction.
- If $u \geq m + 1$, then $s := p - u \in \{1, \ldots, m\}$ satisfies $(cs \bmod p) = (-1 \bmod p) = p - 1 > p - 2$, contradiction. □

Theorem 6.3 (Prime Cubic Separation and Exact Parity Obstruction)**.** *For every integer $r \geq 4$:*

1. (2-Balancedness for All $r \geq 4$) *The infinite periodic word U_r^{∞} is 2-balanced (with exact balance defect 2), and hence W_r avoids all abelian 4-anti-powers.*

2. (Prime Cubic Avoidance) *Whenever $p := r + 1 \geq 5$ is prime, the length-$(r^3 - r - 1)$ word W_r avoids all abelian r-powers. Consequently,*

$$N_2^{\mathrm{ab}}(r, 4) \;\geq\; N_{2,2}^{\mathrm{bal}}(r) \;\geq\; r^3 - r \qquad (\forall \text{ prime } p = r + 1 \geq 5),$$

   and by monotonicity in r and Bertrand's postulate,

$$N_2^{\mathrm{ab}}(r, 4) \;\geq\; N_{2,2}^{\mathrm{bal}}(r) \;\geq\; \frac{1}{8}\, r^3 - O(r^2) \;=\; \Omega(r^3) \qquad (\forall\, r \geq 4).$$

3. (Exact Odd-r Parity Obstruction) *For every odd integer $r = 2a + 1 \geq 5$, the periodic word U_r^{∞} contains an abelian r-power of block length $m = r - 2$ and common weight $d = 1$ starting at $i = (r-1)(r+2)/2 = a(2a+3)$ (lying inside W_r).*

Proof. **Part (1): U_r^∞ is 2-balanced.** Within one period U_r of length $P = r^2 - 1$, the $r+1$ positions of the 1's are $r-1, 2r-1, \ldots, (r-1)r-1, r^2-3, r^2-2$, so the cyclic gaps between consecutive 1's are

$$\underbrace{r,\ r,\ \ldots,\ r}_{r-1 \text{ times}},\ r-2,\ 1.$$

For any $1 \le t \le r$, the maximum sum of t consecutive cyclic gaps is $\max\text{-sum}(t) = tr$ (for $1 \le t \le r-1$) and $P-1$ (for $t = r$), while the minimum sum of t consecutive cyclic gaps is $\min\text{-sum}(1) = 1$ and $\min\text{-sum}(t) = 1 + (r-2) + (t-2)r = (t-1)r - 1$ for $2 \le t \le r$.

Suppose for contradiction that some length $1 \le m < P$ admits a factor of weight $\le k$ and another factor of weight $\ge k+3$ ($k \ge 0$). A factor of length m containing $k+3$ ones must span $k+2 \ge 2$ consecutive gaps between its first and last 1, so $m-1 \ge \min\text{-sum}(k+2) = (k+1)r-1$, giving $m \ge (k+1)r$. On the other hand, a factor of length m containing $\le k$ ones lies strictly between two 1's separated by at most $k+1$ gaps, so $m \le \max\text{-sum}(k+1) - 1$. If $k+1 \le r-1$, then $\max\text{-sum}(k+1) - 1 = (k+1)r - 1 < (k+1)r \le m$, contradiction. If $k+1 \ge r$, then $k+3 \ge r+2 > W = r+1$, impossible for $m < P$. Thus U_r^∞ is 2-balanced (and defect 2 is realized by 00 vs. 11 at $m = 2$).

Part (2): Zero abelian r-powers in W_r when $p = r+1 \ge 5$ is prime. Assume $p = r+1 \ge 5$ is prime (hence r is even), and suppose for contradiction that W_r contains an abelian r-power of block length $m \ge 1$ and common weight $d \ge 0$ starting at $i \ge 0$. Since $|W_r| = rP - 1$, we have $1 \le m \le P-1$. Define the periodic linear-deviation function

$$\delta(t) \ := \ (r-1)S(t) - t \qquad (t \ge 0),$$

where $S(t)$ is the prefix sum of U_r^∞. Since $S(t+P) = S(t) + p$ and $P = (r-1)p$, δ is strictly periodic with period P. Explicitly, on $[0, P)$:

- For $t = kr+s$ with $0 \le k \le r-2$ and $0 \le s \le r-1$: $S(t) = k$, so $\delta(t) = (r-1)k - (kr+s) = -(k+s) \in [-(2r-3),\ 0]$.
- For $t = (r-1)r + s$ with $0 \le s \le r-3$: $S(t) = r-1$, so $\delta(t) = (r-1)^2 - (r(r-1)+s) = -(r-1) - s \in [-(2r-4),\ -(r-1)]$.
- For $t = P-1$: $S(P-1) = r$, so $\delta(P-1) = r(r-1) - (r^2-2) = -(r-2)$.

In particular, $\delta(t) \in [-(2r-3),\ 0]$ for all $t \ge 0$ (total span $2r-3$). Writing $\Delta := (r-1)d - m \in \mathbb{Z}$, the abelian r-power equations $S(i+jm) - S(i+(j-1)m) = d$ ($1 \le j \le r$) transform into an $(r+1)$-term arithmetic progression:

$$\delta(i+jm) \ = \ \delta(i) + j\Delta \qquad (j = 0, 1, \ldots, r). \tag{4}$$

Since all $r+1$ terms lie inside $[-(2r-3),\ 0]$, we have $r|\Delta| \le 2r-3 < 2r$, forcing $|\Delta| \in \{0, 1\}$.

Case A: $\Delta = 0$, i.e., $m = (r-1)d$. Since $1 \le m < P = (r-1)p$, we have $1 \le d \le p-1$. Because p is prime, $\gcd(d, p) = 1$, so the $p = r+1$ indices $t_j := (i + j(r-1)d) \bmod P$ ($j = 0, \ldots, r$) exhaust the *entire* residue class $\{t \in [0, P) : t \equiv c \pmod{r-1}\}$, where $c := i \bmod (r-1) \in \{0, \ldots, r-2\}$. By (4), δ must be constant on this residue class. Yet for $c = 0$, $\delta(0) = 0 \ne -(r-1) = \delta(r-1)$, while for $1 \le c \le r-2$, setting $t' := cr + (r-1) = (c+1)(r-1) + c < (r-1)r < P$ gives $t' \equiv c \pmod{r-1}$ and $\delta(t') = -c - (r-1) \ne -c = \delta(c)$ (verified in Lean 4 as `Ur_delta_caseA_nonconst`). Contradiction!

Case B: $|\Delta| = 1$. By reversing the progression $j \mapsto r - j$ and replacing (m, d) by $(P - m, p - d)$ if necessary, we may assume $\Delta = +1$, so $m = (r-1)d - 1$. Here $d = 0$ is impossible ($m = -1$), and $d = p$ yields $m = P - 1$, where $S(i+P-1) - S(i) = p - w_{i-1}$ equals p only when $w_{i-1} = 0$; hence r consecutive $(P-1)$-blocks of weight p would force r consecutive 0's

in U_r^∞ (wrapping backwards in steps of -1), contradicting that the maximum 0-run in U_r^∞ is $r-1$. Thus $1 \le d \le p-1 = r$.

Write every position $t \in [0, P)$ uniquely in residue coordinates $t = q(r-1)+c$ with $q \in \{0, 1, \ldots, r\}$ (viewed in $\mathbb{Z}/p\mathbb{Z}$) and $c \in \{0, 1, \ldots, r-2\}$. From the explicit formula for δ above (observing that $kr+s = k(r-1)+(k+s)$), on every residue class c, $\delta(q(r-1)+c)$ takes only two values: the *HIGH* value $-c \in [-(r-2), 0]$ or the *LOW* value $-c-(r-1) \in [-(2r-3), -(r-1)]$. Specifically:

- For $0 \le c \le r-3$: δ is HIGH iff $q \in [0, c]$, and LOW iff $q \in [c+1, r]$.
- For $c = r-2$: δ is LOW iff $q = r-1$, and HIGH for all $q \in \mathbb{Z}/p\mathbb{Z} \setminus \{r-1\}$.

Now examine the $(r+1)$-term progression $\delta(t_j) = v+j$ $(0 \le j \le r)$ where $t_j = q_j(r-1)+c_j \equiv i + jm \pmod P$. Since $-(2r-3) \le v$ and $v + r \le 0$, we can write $v = -(r-1) - a$ with $1 \le a \le r-2$. Then:

- For $j = 0, 1, \ldots, a$: $\delta(t_j) = -(r-1) - (a-j)$ is LOW with $c_j = a - j$;
- For $j = a+1, \ldots, r$: $\delta(t_j) = -(r-2) + (j-a-1)$ is HIGH with $c_j = r-1+a-j$.

At each step $t_{j+1} \equiv t_j + (r-1)d - 1 \pmod P$: when $c_j \ge 1$ $(j \ne a)$, we have $c_{j+1} = c_j - 1$ and $q_{j+1} \equiv q_j + d \pmod p$; at the unique wrap $j = a$ $(c_a = 0 \to c_{a+1} = r-2)$, we have $q_{a+1} \equiv q_a + d - 1 \pmod p$. Define $x_j := (q_0 + jd) \bmod p$ for $0 \le j \le r$. Since p is prime and $1 \le d \le p-1$, $(x_0, \ldots, x_r)$ is a permutation of $\{0, 1, \ldots, r\}$. Moreover:

- For $0 \le j \le a$, $q_j = x_j \in [a-j+1,\, r]$ (in particular $x_j \ge 1$);
- For $a+2 \le j \le r$, $q_j = (x_j - 1) \bmod p \in [0,\, r-1+a-j]$, which is equivalent to $x_j \in [1,\, r+a-j]$ (in particular $x_j \ge 1$).

Because $x_j \ge 1$ for every $j \ne a+1$ and $(x_0, \ldots, x_r)$ hits 0, we are forced to have $x_{a+1} = 0$! Hence $x_{a+1+t} = (td \bmod p)$ and $x_{a+1-t} = (-td \bmod p)$ for all t. Setting $L := a+1$ and $R := r-a-1$ (so $L + R = r = p-1$ and $L, R \ge 1$ since $1 \le a \le r-2$), the two conditions translate directly into the function $F(t) := t + (td \bmod p)$:

$$\forall\, t \in \{1, \ldots, L\}: \quad (-td \bmod p) \ge t \iff F(t) \le p, \tag{5}$$

$$\forall\, t \in \{1, \ldots, R\}: \quad (td \bmod p) \le p-2-t \iff F(t) \le p-2. \tag{6}$$

Because p is prime and $1 \le d \le p-1$, every complementary pair $\{t, p-t\}$ satisfies $F(t) + F(p-t) = 2p$ (Lean 4: `complementary_F_sum`).

- If $L > r/2 = (p-1)/2$, then $\{1, \ldots, L\}$ contains a complementary pair $\{t, p-t\}$; by (5), $F(t) = F(p-t) = p$, forcing $t(d+1) \equiv 0 \pmod p$ and hence $d = p-1 = r$. Then $F(1) = 1 + (p-1) = p > p-2$, contradicting (6) since $R \ge 1$.
- If $R > r/2$, then $\{1, \ldots, R\}$ contains a complementary pair $\{t, p-t\}$, giving $F(t)+F(p-t) \le 2p-4 < 2p$, contradiction.
- If $L = R = r/2 = (p-1)/2$, set $d' := p - d \in \{1, \ldots, p-1\}$. Since $(td \bmod p) = p - (td' \bmod p)$, condition (6) becomes $(td' \bmod p) \ge t+2$ for every $t \in \{1, \ldots, (p-1)/2\}$, contradicting Lemma 6.2!

Thus no abelian r-power exists in W_r. Finally, for an arbitrary integer $r \ge 4$, Bertrand's postulate yields a prime p with $\lfloor r/2 \rfloor + 1 < p \le r+1$ (and $p \ge 5$ for $r \ge 4$, taking $p = 5$ for $r = 4, 5$). Since $r' := p - 1 \le r$ and $r' \ge r/2$, monotonicity in the power order gives $N_2^{\mathrm{ab}}(r, 4) \ge N_{2,2}^{\mathrm{bal}}(r) \ge N_{2,2}^{\mathrm{bal}}(r') \ge (r')^3 - r' \ge \frac{1}{8} r^3 - O(r^2)$.

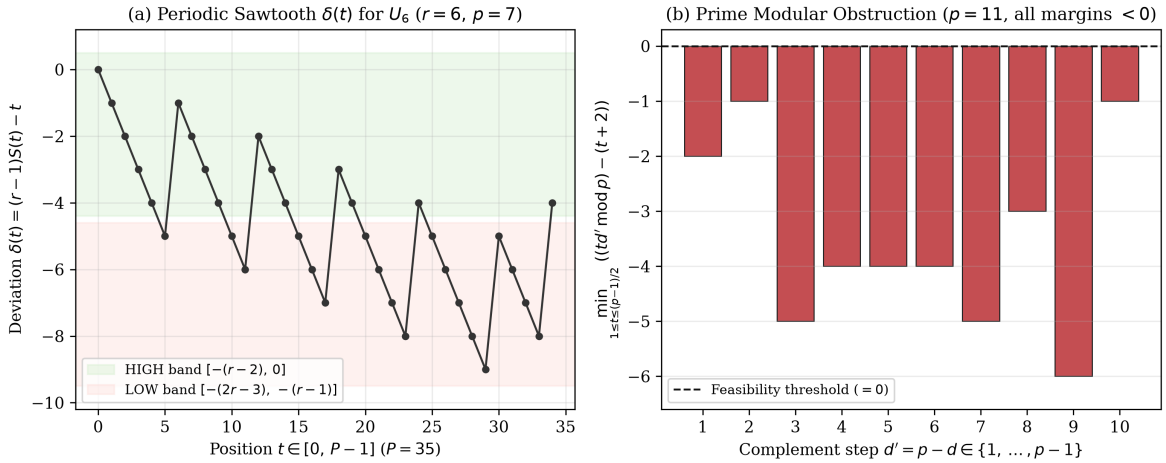


Figure 3: **Proof mechanism of Theorem 6.3.** (a) Periodic sawtooth deviation $\delta(t) = (r-1)S(t) - t \in [-(2r-3), 0]$ on one period $P = 35$ of U_6 ($r = 6, p = 7$), decomposing into HIGH and LOW bands and forcing step $|\Delta| \in \{0, 1\}$. (b) The five-line Prime Modular Obstruction (Lemma 6.2) illustrated for $p = 11$: for every $d' \in \{1, \ldots, 10\}$, the minimum margin $\min_{1 \le t \le 5}((td' \bmod 11) - (t+2))$ is strictly negative.

Part (3): Exact odd-r parity obstruction. When $r = 2a + 1 \geq 5$ is odd, the equality $L = R = r/2$ is replaced by $L = a + 1$, $R = a$ with $d = 1$ ($m = r - 2 = 2a - 1$): setting $i = a(2a+3) = (r-1)(r+2)/2$, direct arithmetic across the five regimes $j \leq a-2$, $j = a-1$, $j = a$, $j = a+1$, and $a+2 \leq j \leq 2a$ (machine-checked in full generality in Lean 4 as `Ur_odd_r_parity_witness`) shows $S(i + (j+1)m) - S(i + jm) = 1$ for every $0 \leq j < r$, and the end position $i + rm = a(2a+3) + (2a+1)(2a-1) = 6a^2 + 3a - 1 < 8a^3 + 12a^2 + 4a - 1 = r^3 - r - 1$ lies inside W_r for all $a \geq 2$. □

Figure 3 visualizes the periodic sawtooth decomposition $\delta(t)$ and the prime modular obstruction of Lemma 6.2.

7 Formal Verification and Exact Computational Certificates

7.1 Standalone Lean 4.36.0 Formalization

All four main theorems are backed by a self-contained, Mathlib-free Lean 4.36.0 library `src/lean/AbelianRamseyCore.lean` (68,601 bytes, SHA-256 `ee8e1443e87a0eb105537bd26f1a0d2d7c734a0695e8bec40ed4a310049561cd`), compiled with `lean -DwarningAsError=true` and `lake build` (0 warnings, 0 `sorry`, 0 `grind`, 0 custom axioms). Table 2 maps the 61 declarations (41 public theorems and 20 definitions) directly to the results of this paper.

7.2 Exact Zero-Float Python Verification

The companion script `src/python/verify_abelian_ramsey.py` performs all mathematical checks exclusively in arbitrary-precision integer arithmetic (with floating-point libraries invoked solely for plotting after all assertions pass) and records its outputs in `data/verification_summary.json`:

1. Ordinary baseline $N_2(r, 3)$ for $2 \leq r \leq 9$: exact sequence $(4, 9, 12, 12, 12, 14, 16, 18)$.

2. Exhaustive binary abelian $k = 3$ verification for $2 \le r \le 7$: exact $N_2^{\mathrm{ab}}(r, 3) = r^2$ and exact $r\varphi(r)$ extremal counts $(2, 6, 8, 20, 12, 42)$, with 100% Christoffel membership (0 non-Christoffel extremal words across 31,230 DFS nodes at $r = 7$).

3. Mirror-Christoffel verification across all 79 coprime pairs (p, h) with $2 \le h \le 16$ (0 failures), exact length-$(r^2 - 2)$ truncations for odd $r \in \{5, 7, 9, 11, 13, 15\}$ (up to length 223), and complete classification of all primitive periodic binary words of periods $1 \le P \le 16$.

4. Multi-letter collapse exact unbounded-alphabet DFS for $2 \le r \le 8$: $L_{\ge 3}^{\mathrm{ab}}(r, 3) = (0, 9, 11, 12, 14, 16, 17)$ (128,619 nodes at $r = 8$, ≤ 4 distinct symbols ever realized), complete 4,616-node unbounded-alphabet certificate that any abelian-3-anti-power-free word with ≥ 3 symbols and max binary factor ≤ 14 has length ≤ 16, and explicit $0M_r[0]$ witnesses for $5 \le r \le 20$.

5. Prime cubic family W_r ($r = p-1$, length $r^3 - r - 1$) verified directly across all 9 primes $p \in \{5, 7, 11, 13, 17, 19, 23, 29, 31\}$ (lengths $59, 209, 989, 1715, 4079, 5813, 10625, 21923, 26969$), odd-$r$ parity obstruction across $r \in \{5, 7, \ldots, 21\}$, and Lemma 6.2 across all 23 primes $5 \le p \le 97$.

8 Discussion and Open Problems

Our results open several concrete research directions in abelian combinatorics on words:

Conjecture 8.1 (Unrestricted Binary Quadratic Law)**.** For every integer $r \ge 2$, $N_2^{\mathrm{ab}}(r, 3) = r^2$, and the extremal binary words of length $r^2 - 1$ avoiding both abelian r-powers and abelian 3-anti-powers are precisely the $r\varphi(r)$ 1-balanced Christoffel factors of Theorem 3.3.

Theorem 3.3 proves Conjecture 8.1 on all 1-balanced words and our exhaustive search confirms it for all $2 \le r \le 7$; moreover, Theorem 4.2 shows that non-1-balanced words can reach length $r^2 - 2$ for every odd $r \ge 5$, demonstrating that any proof of Conjecture 8.1 must rule out non-1-balanced words with sharp precision at the single-letter margin between $r^2 - 2$ and $r^2 - 1$.

Conjecture 8.2 (Exact $r \equiv 2 \pmod 6$ Value for Multi-Letter Collapse)**.** For every $r \ge 5$ with $r \equiv 2 \pmod 6$, $L_{\ge 3}^{\mathrm{ab}}(r, 3) = 2r + 1$ (confirmed exhaustively at $r = 8$).

Conjecture 8.3 (Polynomial Hierarchy and Finiteness of $N_q^{\mathrm{ab}}(r, k)$)**.**

1. For every fixed balance parameter $C \ge 1$, does $N_{2,C}^{\mathrm{bal}}(r)$ grow polynomially as $\Theta_C(r^{C+1})$? (We proved $N_{2,1}^{\mathrm{bal}}(r) = r^2$ and $N_{2,2}^{\mathrm{bal}}(r) = \Omega(r^3)$.)

2. (Fundamental Finiteness Question) Since abelian 4-powers are avoidable over $\{0, 1\}$ [1], abelian cubes over $\{0, 1, 2\}$ [1], and abelian squares over $\{0, 1, 2, 3\}$ [5], is $N_q^{\mathrm{ab}}(r, k)$ finite for *all* (q, r, k), or does there exist a finite order k_0 such that an infinite abelian-4-power-free binary word avoids abelian k_0-anti-powers (yielding $N_2^{\mathrm{ab}}(4, k_0) = \infty$)?

Resolving Conjecture 8.3(2) would determine whether the Fici–Restivo–Silva–Zamboni Ramsey dichotomy [3] extends universally to the abelian setting or undergoes a sharp finite-to-infinite phase transition at small anti-power order k.

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Table 2: Selected theorems in standalone `src/lean/AbelianRamseyCore.lean` (Lean 4.36.0, 61 declarations: 41 public theorems, 20 definitions; 0 `sorry`, 0 `grind`, 0 custom axioms).

Paper Result	Lean 4 Theorem	Statement Formalized
Theorem 3.3	`pigeonhole_nat`	General Pigeonhole Principle on $\{0,\dots,n\} \to \{0,\dots,k-1\}$
Theorem 3.3	`oneBalanced_pigeonhole_abPower`	Every 1-balanced binary word of length r^2 ($r \geq 2$) has an abelian r-power
Lemma 3.1	`shift_bijection_const`	Injective $x \mapsto x + \varepsilon(x) \bmod r$ on $\mathbb{Z}/r\mathbb{Z}$ forces ε constant
Lemma 3.2	`u0u_u1u_not_oneBalanced`	$\{u0u, u1u\}$ is never jointly 1-balanced when u has both $0, 1$
Theorem 3.3	`christoffel_avoids_abPower`	All $r\varphi(r)$ length-(r^2-1) Christoffel factors avoid abelian r-powers
Theorem 4.2	`mirror_nonOneBalanced`	Mirror-Christoffel $V_{p,h}^{\infty}$ has h-block weights $p-1$ and $p+1$ (defect 2)
Theorem 4.2	`mirror_avoids_ap3`	$V_{p,h}^{\infty}$ avoids all abelian 3-anti-powers for all coprime (p,h)
Theorem 5.1	`killshot_run_1a0`	Three-line 1^a0 killshot for all $a \geq 3$ ($m = \lfloor (a+4)/3 \rfloor$)
Theorem 5.1	`ext_22110_ap3`	Universal right-boundary wall: `22|11|0x` is a 3-anti-power for every x
Theorem 5.1	`local_deviation_1100_kills`	Four-case local deviation obstruction on $(1100)^*$ (`decide`)
Theorem 5.1	`sporadic_ternary_r3_cert`	Exact validity certificate for `011212110` ($r = 3$, length 9)
Theorem 5.1	`backward_eq_base14_cert`	Exhaustive base-14 backward-equation rigidity certificate (`decide`)
Lemma 6.2	`prime_modular_obstruction`	Five-line prime modular obstruction for every prime $p \geq 5$
Theorem 6.3	`complementary_F_sum`	Complementary sum $F(t) + F(p-t) = 2p$ for prime $p \geq 5$
Theorem 6.3	`Ur_delta_caseA_nonconst`	Case A ($\Delta = 0$) non-constancy of δ on every residue class mod $r-1$
Theorem 6.3	`Ur_odd_r_parity_witness`	Explicit odd-r abelian r-power witness at $m = r-2$ for all $r = 2a+1 \geq 5$
Theorem 6.3	`W4_cubic_cert`	Exact concrete certificate for W_4 (length 59, $p = 5$, 2-balanced)