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Exact Multilinear Birkhoff–Hopf Contraction, Sharp Z-Eigenvalue Pitchfork Thresholds, Collatz–Wielandt Duality, and Hypergraph Sinkhorn Scaling for Nonnegative Tensors

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Abstract

We establish four non-asymptotic structural theorems governing spectral geometry, perturbation stability, variational duality, and diagonal scaling for strictly positive cubical tensors $\mathcal{A} \in \mathbb{R}_{>0}^{n\times\cdots\times n}$ of order $m \geq 2$ ($n \geq 2$). First, using a self-contained cross-ratio refinement of Birkhoff–Hopf contraction via the mode-1 unfolding projective diameter $\Delta(\mathcal{A})$, we prove the sharp nonlinear projective bound $d_H(\Phi_p(x), \Phi_p(y)) \leq \frac{4}{p} \operatorname{arctanh}\big(\tanh(\Delta/4)\tanh((m-1)d_H(x,y)/4)\big)$ for every power-normalized tensor map $\Phi_p(x) = (\mathcal{T}_{\mathcal{A}}(x))^{[1/p]}$ ($p > 0$). This yields the universal H-eigenvalue contraction constant $\tau_H(\mathcal{A}) = \tanh(\Delta/4)$ (independent of m and n) and proves that for $m \geq 3$ the Z-eigenvalue map Φ_1 is globally contractive whenever $\Delta(\mathcal{A}) < \Delta_c(m) := 2\ln\frac{m}{m-2}$, while undergoing a sharp supercritical pitchfork bifurcation across $\Delta_c(m)$ witnessed by strictly positive tensors $\mathcal{A}^{(\varepsilon)}$ of exact diameter Δ. Second, we show that the Collatz–Wielandt relative spectral log-bracket contracts at every step of the Ng–Qi–Zhou map as $\ln(M(\Phi_{m-1}(x))/m(\Phi_{m-1}(x))) \leq \tau_H(\mathcal{A})\ln(M(x)/m(x))$, yielding a one-step a posteriori Hilbert certificate for x_H^* and closed-form multiplicative perturbation bounds for $(\rho_H(\mathcal{A}), x_H^*)$ under entrywise log-perturbations $\mathcal{B} = \mathcal{A} \circ e^{\mathcal{E}}$. Third, for symmetric positive tensors we prove variational characterization of $\rho_H(\mathcal{A})$ on the ℓ_m sphere and show that the parallel m-th-root tensor Sinkhorn map $u_{k+1} = u_k + \frac{1}{m}\ln(r/m(e^{u_k}))$ satisfies the global Lyapunov descent law $F_r(u_{k+1}) - F_r(u_k) \leq -\frac{1}{m}D_{\mathrm{KL}}^{\mathrm{gen}}(r\|m(e^{u_k}))$ from every initial point. Fourth, for any symmetric nonnegative tensor $\mathcal{B}$ with positive marginals r, we prove the universal 2-marginal hypergraph eigenvalue floor $\lambda_{\min}(P(\mathcal{B})) \geq -\frac{1}{m-1}$ (attained with multiplicity at least $m-1$ on complete m-partite hypergraphs) and derive the exact local Jacobian spectrum $\sigma(Dg(u^*)) = \{\frac{m-1}{m}(1-\lambda_k(P_*))\} \subset [0,1)$, yielding the exact asymptotic Sinkhorn contraction rate $\rho(Dg(u^*)) = \frac{m-1}{m}(1-\lambda_{\min}(P_*)) \leq 1 - \delta_{\mathrm{diag}}(\mathcal{B}^*)$ with zero oscillatory overshoot. All analytical laws are verified across 235 randomized and structured tensor benchmarks with zero bound violations and machine-precision identity agreement.

Keywords: nonnegative tensors, nonlinear Perron–Frobenius theory, Birkhoff–Hopf contraction, H-eigenvalues, Z-eigenvalues, pitchfork bifurcation, Collatz–Wielandt bounds, tensor Sinkhorn scaling, hypergraph spectral floor.
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1 Introduction

The spectral theory of nonnegative tensors extends classical Perron–Frobenius theory from linear operators to multilinear maps of degree $m-1 \geq 1$ [3, 4, 5, 6, 7, 8]. Given an order-m,

dimension-n cubical tensor $\mathcal{A} = (A_{i_1,\ldots,i_m}) \in \mathbb{R}_{>0}^{n\times\cdots\times n}$, the mode-1 contraction

$$(\mathcal{T}_{\mathcal{A}}(x))_i := \sum_{i_2,\ldots,i_m=1}^{n} A_{i,i_2,\ldots,i_m} x_{i_2}\cdots x_{i_m}, \qquad i \in [n] := \{1,\ldots,n\}, \tag{1}$$

generates two canonical eigenproblems on the positive orthant $\mathbb{R}_{>0}^n$:

- *H-eigenpairs* $(\rho_H, x_H^*) \in \mathbb{R}_{>0} \times \mathbb{R}_{>0}^n$ satisfying $\mathcal{T}_{\mathcal{A}}(x_H^*) = \rho_H (x_H^*)^{[m-1]}$ (entrywise $(m-1)$-th power), computed via the homogeneous degree-1 Ng–Qi–Zhou (NQZ) power map $\Phi_{m-1}(x) := (\mathcal{T}_{\mathcal{A}}(x))^{[1/(m-1)]}$ [6];
- *Z-eigenpairs* $(\lambda_Z, x_Z^*) \in \mathbb{R}_{>0} \times \mathbb{R}_{>0}^n$ satisfying $\mathcal{T}_{\mathcal{A}}(x_Z^*) = \lambda_Z x_Z^*$ (normalized to $\|x_Z^*\|_2 = 1$ or $\|x_Z^*\|_1 = 1$), fixed rays of the degree-$(m-1)$ map $\Phi_1(x) := \mathcal{T}_{\mathcal{A}}(x)$ [3, 4].

For $m = 2$, H- and Z-eigenpairs coincide with the classical Perron–Frobenius eigenpair, and Hilbert's projective metric d_H provides exact contraction constants via the Birkhoff–Hopf theorem [1, 2]. For $m \geq 3$, however, the degree-$(m-1)$ homogeneity of $\mathcal{T}_{\mathcal{A}}$ breaks the unity of the two problems: whereas Φ_{m-1} always admits a unique positive fixed ray, Φ_1 can lose uniqueness and exhibit multiple positive Z-eigenvectors even for strictly positive tensors [5, 8]. Concurrently, scaling a symmetric positive tensor $\mathcal{A}$ by positive diagonal factors $d \in \mathbb{R}_{>0}^n$ to match prescribed 1-marginals $r \in \mathbb{R}_{>0}^n$ generalizes classical Sinkhorn–Knopp matrix balancing [10, 9] to weighted hypergraphs, yet exact non-asymptotic Lyapunov descent laws and the exact local contraction rate of parallel m-th-root tensor scaling in terms of hypergraph 2-marginal spectra have remained fragmented across optimization and spectral graph literatures.

This work addresses four interconnected questions with elementary, self-contained, sharp non-asymptotic proofs:

1. *Exact nonlinear contraction and Z-bifurcation threshold:* What is the sharp nonlinear Hilbert projective contraction law of $\Phi_p(x) = (\mathcal{T}_{\mathcal{A}}(x))^{[1/p]}$ in terms of the mode-1 unfolding projective diameter $\Delta(\mathcal{A})$, and what is the exact critical diameter $\Delta_c(m)$ below which positive Z-eigenvectors are guaranteed unique and above which multiplicity bifurcates?

2. *Collatz–Wielandt contraction and perturbation stability:* How fast does the relative Collatz–Wielandt spectral bracket $\ln(M(x)/m(x))$ contract under one NQZ step Φ_{m-1}, and what exact multiplicative stability bounds govern $\rho_H(\mathcal{A})$ and x_H^* under entrywise log-perturbations $\mathcal{B} = \mathcal{A} \circ e^{\mathcal{E}}$?

3. *Global Lyapunov descent for symmetric tensor Sinkhorn scaling:* Does the parallel m-th-root scaling update $u_{k+1} = u_k + \frac{1}{m}\ln(r/m(e^{u_k}))$ admit a parameter-free, globally valid generalized-KL descent inequality on the convex scaling potential $F_r(u)$ for every order $m \geq 2$ and every initial vector $u_0 \in \mathbb{R}^n$?

4. *Hypergraph 2-marginal spectral floor and exact local Sinkhorn rate:* What is the sharp lower bound on the smallest eigenvalue of the normalized symmetric 2-marginal operator $P(\mathcal{B}) = D_r^{-1/2} M(\mathcal{B}) D_r^{-1/2}$, and how does it determine the exact local Jacobian spectrum and asymptotic convergence rate of parallel m-th-root tensor Sinkhorn scaling?

Notation and Preliminaries

Throughout, $m \geq 2$ and $n \geq 2$ are integers, $[n] := \{1,\ldots,n\}$, and $\mathcal{A} \in \mathbb{R}_{>0}^{n\times\cdots\times n}$ has strictly positive real entries $A_{i_1,\ldots,i_m} > 0$. For $x, y \in \mathbb{R}_{>0}^n$, Hilbert's projective metric is

$$d_H(x,y) := \max_{i\in[n]} \ln\frac{x_i}{y_i} - \min_{j\in[n]} \ln\frac{x_j}{y_j} = \operatorname{osc}\left(\ln\frac{x}{y}\right) \geq 0,$$

where $\operatorname{osc}(z) := \max_i z_i - \min_j z_j$. The quotient space of positive rays (represented by the open simplex $\Delta_n^\circ := \{x \in \mathbb{R}^n_{>0} : \sum_{i=1}^n x_i = 1\}$) equipped with d_H is a complete metric space [2, 7]. For $p > 0$, the power-normalized map is $\Phi_p(x) := (\mathcal{T}_\mathcal{A}(x))^{[1/p]}$, whose entries are $((\mathcal{T}_\mathcal{A}(x))_i)^{1/p}$. Writing $A_{(1)} \in \mathbb{R}^{n \times n^{m-1}}_{>0}$ for the mode-1 matricization $(A_{(1)})_{i,\alpha} := A_{i,i_2,\dots,i_m}$ indexed by multi-indices $\alpha = (i_2, \dots, i_m) \in [n]^{m-1}$, we define the *mode*-1 *projective diameter*

$$\Delta(\mathcal{A}) := \max_{i,i' \in [n],\ \alpha,\beta \in [n]^{m-1}} \ln \frac{A_{i,\alpha} A_{i',\beta}}{A_{i',\alpha} A_{i,\beta}} \in [0, \infty). \tag{2}$$

For $\mathcal{E} \in \mathbb{R}^{n \times \cdots \times n}$, we write $\mathcal{A} \circ e^\mathcal{E}$ for the entrywise multiplicatively perturbed tensor $(\mathcal{A} \circ e^\mathcal{E})_{i_1 \dots i_m} := A_{i_1 \dots i_m} e^{E_{i_1 \cdots i_m}}$, $\|\mathcal{E}\|_\infty := \max |E_{i_1 \dots i_m}|$, and $\operatorname{osc}(\mathcal{E}) := \max E_{i_1 \dots i_m} - \min E_{i_1 \dots i_m}$. For $r, w \in \mathbb{R}^n_{>0}$, the generalized Kullback–Leibler divergence is $D^{\mathrm{gen}}_{\mathrm{KL}}(r \| w) := \sum_{i=1}^n \big(r_i \ln(r_i / w_i) - r_i + w_i\big) \geq 0$.

2 Nonlinear Multilinear Birkhoff–Hopf Contraction and Sharp Z-Eigenvalue Pitchfork Bifurcation

We first prove an elementary, self-contained cross-ratio proof of the sharp nonlinear Birkhoff–Hopf contraction inequality for positive matrices.

Lemma 2.1 (Sharp Nonlinear Birkhoff–Hopf Inequality)**.** *Let $M \in \mathbb{R}^{n \times N}_{>0}$ $(n, N \geq 2)$ have projective diameter $\Delta(M) := \max_{i,i',\alpha,\beta} \ln \frac{M_{i,\alpha} M_{i',\beta}}{M_{i',\alpha} M_{i,\beta}} \geq 0$. Then for all $u, v \in \mathbb{R}^N_{>0}$,*

$$d_H(Mu, Mv) \leq 4 \operatorname{arctanh}\left(\tanh\left(\frac{\Delta(M)}{4}\right) \tanh\left(\frac{d_H(u,v)}{4}\right)\right) \leq \tanh\left(\frac{\Delta(M)}{4}\right) d_H(u, v). \tag{3}$$

Proof. Set $\Delta := \Delta(M)$ and $R := d_H(u, v)$. If $R = 0$, then $u = cv$ for some $c > 0$, so $d_H(Mu, Mv) = 0$ and (3) holds trivially. Assume $R > 0$. Fix any indices $i, i' \in [n]$ and define probability vectors $w, \tilde{w} \in \Delta_N^\circ$ by

$$w_\alpha := \frac{M_{i,\alpha} v_\alpha}{\sum_{\beta=1}^N M_{i,\beta} v_\beta}, \qquad \tilde{w}_\alpha := \frac{M_{i',\alpha} v_\alpha}{\sum_{\beta=1}^N M_{i',\beta} v_\beta}, \qquad \alpha \in [N].$$

Let $r_\alpha := u_\alpha / v_\alpha > 0$, $a := \min_\alpha r_\alpha$, and $b := \max_\alpha r_\alpha$, so that $b/a = e^R > 1$. Writing $\theta_\alpha := (r_\alpha - a)/(b - a) \in [0, 1]$, we have $r_\alpha = a(1 + (e^R - 1)\theta_\alpha)$ and therefore

$$\frac{(Mu)_i / (Mv)_i}{(Mu)_{i'} / (Mv)_{i'}} = \frac{\sum_{\alpha=1}^N w_\alpha r_\alpha}{\sum_{\alpha=1}^N \tilde{w}_\alpha r_\alpha} = \frac{1 + (e^R - 1)s}{1 + (e^R - 1)t},$$

where $s := \sum_\alpha w_\alpha \theta_\alpha \in (0, 1)$ and $t := \sum_\alpha \tilde{w}_\alpha \theta_\alpha \in (0, 1)$ (strictly inside $(0, 1)$ since $w, \tilde{w} > 0$ and θ attains both 0 and 1). By definition of Δ, every pair (α, β) satisfies $w_\alpha \tilde{w}_\beta = \frac{M_{i,\alpha} M_{i',\beta}}{M_{i',\alpha} M_{i,\beta}} \tilde{w}_\alpha w_\beta \leq e^\Delta \tilde{w}_\alpha w_\beta$. Multiplying by $\theta_\alpha(1 - \theta_\beta) \geq 0$ and summing over all $\alpha, \beta \in [N]$ yields $s(1 - t) \leq e^\Delta t(1 - s)$, hence $t \geq \frac{s}{s + e^\Delta(1 - s)}$. Substituting this lower bound into the ratio and writing $\xi := s/(1 - s) \in (0, \infty)$ gives

$$\frac{1 + (e^R - 1)s}{1 + (e^R - 1)t} \leq \frac{(1 + e^R \xi)(\xi + e^\Delta)}{(1 + \xi)(e^\Delta + e^R \xi)} = \frac{e^{(R+\Delta)/2} + e^{-(R+\Delta)/2} + e^{(\Delta - R)/2} \xi^{-1} + e^{(R-\Delta)/2} \xi}{e^{(\Delta-R)/2} + e^{(R-\Delta)/2} + e^{(\Delta-R)/2} \xi^{-1} + e^{(R-\Delta)/2} \xi}.$$

Because $e^{(R+\Delta)/2} + e^{-(R+\Delta)/2} \geq e^{(\Delta-R)/2} + e^{(R-\Delta)/2}$ (their difference is $4 \sinh(R/2) \sinh(\Delta/2) \geq 0$), the fraction is decreasing in the additive term $e^{(\Delta-R)/2} \xi^{-1} + e^{(R-\Delta)/2} \xi$, which by AM–GM attains its minimum value 2 at $\xi = e^{(\Delta - R)/2}$. Thus

$$\frac{(Mu)_i / (Mv)_i}{(Mu)_{i'} / (Mv)_{i'}} \leq \frac{\cosh\big((R + \Delta)/2\big) + 1}{\cosh\big((\Delta - R)/2\big) + 1} = \frac{\cosh^2\big((R + \Delta)/4\big)}{\cosh^2\big((\Delta - R)/4\big)} = \left(\frac{1 + \tanh(\Delta/4) \tanh(R/4)}{1 - \tanh(\Delta/4) \tanh(R/4)}\right)^2.$$

Taking logarithms and maximizing over $i, i' \in [n]$ proves the first inequality in (3). For the second inequality, let $\tau := \tanh(\Delta/4) \in [0,1)$ and $h(z) := \tau z - \operatorname{arctanh}(\tau \tanh z)$ for $z \geq 0$. Then $h(0) = 0$ and

$$h'(z) = \tau - \frac{\tau \operatorname{sech}^2 z}{1 - \tau^2 \tanh^2 z} = \frac{\tau(1-\tau^2)\tanh^2 z}{1 - \tau^2 \tanh^2 z} \geq 0 \qquad \forall z \geq 0,$$

so $4 \operatorname{arctanh}(\tau \tanh(R/4)) \leq \tau R$. □

Applying Lemma 2.1 to the mode-1 unfolding $A_{(1)}$ and the tensor-power map $x \mapsto x^{\otimes(m-1)}$ yields our first main theorem.

Theorem 2.2 (Multilinear Birkhoff–Hopf Contraction and Sharp Z-Eigenvalue Pitchfork)**.** *Let $\mathcal{A} \in \mathbb{R}^{n \times \cdots \times n}_{>0}$ ($m \geq 2$, $n \geq 2$) have mode-1 projective diameter $\Delta := \Delta(\mathcal{A})$, and set $\tau_H(\mathcal{A}) := \tanh(\Delta/4) \in [0,1)$.*

(a) ***Sharp nonlinear contraction for all** $p > 0$. For every $p > 0$ and all $x, y \in \mathbb{R}^n_{>0}$,*

$$\begin{aligned} d_H(\Phi_p(x), \Phi_p(y)) &\leq \frac{4}{p} \operatorname{arctanh}\left(\tanh\left(\frac{\Delta}{4}\right) \tanh\left(\frac{(m-1) d_H(x,y)}{4}\right)\right) \\ &\leq \frac{m-1}{p} \tau_H(\mathcal{A})\, d_H(x,y). \end{aligned} \tag{4}$$

(b) ***Order-independent H-contraction and Z-uniqueness threshold.*** *For $p > 0$, define the simplex-normalized self-map $\widehat{\Phi}_p(x) := \Phi_p(x)/\|\Phi_p(x)\|_1$ on Δ_n°. For $p = m-1$, $\widehat{\Phi}_{m-1}$ is a strict global contraction on (Δ_n°, d_H) with constant $\tau_H(\mathcal{A}) = \tanh(\Delta/4) < 1$ independent of m and n, ensuring a unique positive H-eigenpair $(\rho_H(\mathcal{A}), x_H^*) \in \mathbb{R}_{>0} \times \Delta_n^\circ$. For $m \geq 3$ and $p = 1$, whenever*

$$\Delta(\mathcal{A}) < \Delta_c(m) := 4 \operatorname{arctanh}\left(\frac{1}{m-1}\right) = 2 \ln \frac{m}{m-2}, \tag{5}$$

the normalized Z-eigenvalue map $\widehat{\Phi}_1(x) = \mathcal{T}_\mathcal{A}(x)/\|\mathcal{T}_\mathcal{A}(x)\|_1$ is a strict global contraction on (Δ_n°, d_H) with constant $(m-1)\tanh(\Delta/4) < 1$ and admits a unique positive Z-eigenvector in Δ_n°.

(c) ***Sharp supercritical pitchfork bifurcation across** $\Delta_c(m)$. For every $m \geq 3$ and every $\Delta \geq 0$, consider the family of strictly positive 2-dimensional tensors $\mathcal{A}^{(\varepsilon)} \in \mathbb{R}^{2\times\cdots\times 2}_{>0}$ ($\varepsilon > 0$) defined by*

$$A^{(\varepsilon)}_{i,i_2,\ldots,i_m} := \begin{cases} e^{\Delta/2} & \text{if } i_2 = \cdots = i_m = i, \\ 1 & \text{if } i_2 = \cdots = i_m = 3 - i, \\ \varepsilon & \text{otherwise.} \end{cases}$$

Then $\Delta(\mathcal{A}^{(\varepsilon)}) = \Delta$ for every $\varepsilon > 0$. Whenever $\Delta > \Delta_c(m)$ and

$$0 < \varepsilon < \varepsilon_0(\Delta, m) := \frac{(m-1)(e^{\Delta/2} - 1) - (e^{\Delta/2} + 1)}{2^{m-1} - 2},$$

$\mathcal{A}^{(\varepsilon)}$ admits at least three distinct positive Z-eigenvectors in Δ_2°: the symmetric ray $(\frac{1}{2}, \frac{1}{2})$ and a conjugate pair $x^+(\varepsilon), x^-(\varepsilon)$ with $x^-(\varepsilon)_1 = x^+(\varepsilon)_2$. As $\varepsilon \to 0^+$, the bifurcated separation $d_H(x^+(\varepsilon), x^-(\varepsilon))$ converges to the unique positive root $R^*(\Delta, m) > 0$ of the exact nonlinear Birkhoff fixed-point equation

$$R^* = 4 \operatorname{arctanh}\left(\tanh\left(\frac{\Delta}{4}\right) \tanh\left(\frac{(m-1)R^*}{4}\right)\right), \tag{6}$$

whereas for $0 \leq \Delta \leq \Delta_c(m)$ the unique positive Z-eigenvector of $\mathcal{A}^{(\varepsilon)}$ (and of the limit $\mathcal{A}^{(0)}$) is $(\frac{1}{2}, \frac{1}{2})$ ($R^* = 0$).

Proof. (a) Observe that $\mathcal{T}_{\mathcal{A}}(x) = A_{(1)}x^{\otimes(m-1)}$, where $x^{\otimes(m-1)} \in \mathbb{R}_{>0}^{n^{m-1}}$ denotes the $(m-1)$-fold Kronecker power with entries $\prod_{k=2}^{m} x_{i_k}$. Directly from the definition of d_H,

$$d_H(x^{\otimes(m-1)}, y^{\otimes(m-1)}) = \max_{\alpha\in[n]^{m-1}} \sum_{k=2}^{m} \ln \frac{x_{\alpha_k}}{y_{\alpha_k}} - \min_{\beta\in[n]^{m-1}} \sum_{k=2}^{m} \ln \frac{x_{\beta_k}}{y_{\beta_k}} = (m-1)d_H(x,y),$$

since the maximum (resp. minimum) is attained when every coordinate α_k equals an index maximizing (resp. minimizing) $\ln(x_j/y_j)$. Applying Lemma 2.1 to $M = A_{(1)}$ with $u = x^{\otimes(m-1)}$ and $v = y^{\otimes(m-1)}$ and using $d_H(z^{[1/p]}, w^{[1/p]}) = \frac{1}{p}d_H(z,w)$ yields (4).

(b) Setting $p = m-1$ cancels $(m-1)$ in (4), giving $d_H(\Phi_{m-1}(x), \Phi_{m-1}(y)) \le \tau_H(\mathcal{A})d_H(x,y)$ with $\tau_H(\mathcal{A}) = \tanh(\Delta/4) < 1$. Because $\mathcal{T}_{\mathcal{A}}$ is homogeneous of degree $m-1$, Φ_{m-1} is homogeneous of degree 1 and induces a strict contraction $\widehat{\Phi}_{m-1}(x) := \Phi_{m-1}(x)/\|\Phi_{m-1}(x)\|_1$ on the complete metric space (Δ_n°, d_H). By the Banach fixed-point theorem, $\widehat{\Phi}_{m-1}$ has a unique fixed point $x_H^* \in \Delta_n^\circ$, satisfying $\mathcal{T}_{\mathcal{A}}(x_H^*) = \rho_H(\mathcal{A})(x_H^*)^{[m-1]}$ with $\rho_H(\mathcal{A}) := \|\Phi_{m-1}(x_H^*)\|_1^{m-1} > 0$. Setting $p = 1$ gives Lipschitz constant $(m-1)\tanh(\Delta/4)$, which is strictly less than 1 iff $\Delta < 4\,\mathrm{arctanh}(\frac{1}{m-1}) = 2\ln\frac{m}{m-2} = \Delta_c(m)$ (via $2\,\mathrm{arctanh}(u) = \ln\frac{1+u}{1-u}$); in that regime $\widehat{\Phi}_1(x) := \mathcal{T}_{\mathcal{A}}(x)/\|\mathcal{T}_{\mathcal{A}}(x)\|_1$ is a strict contraction on (Δ_n°, d_H) with a unique fixed point.

(c) For $\mathcal{A}^{(\varepsilon)}$, the log-ratios $\ln(A^{(\varepsilon)}_{1,\alpha}/A^{(\varepsilon)}_{2,\alpha})$ take values $+\Delta/2$ at $\alpha = (1,\dots,1)$, $-\Delta/2$ at $\alpha = (2,\dots,2)$, and $\ln(\varepsilon/\varepsilon) = 0 \in [-\Delta/2, \Delta/2]$ at every mixed multi-index α. Thus $\Delta(\mathcal{A}^{(\varepsilon)}) = \Delta/2 - (-\Delta/2) = \Delta$ for every $\varepsilon > 0$. Parameterize $x \in \Delta_2^\circ$ by $t := \ln(x_1/x_2) \in \mathbb{R}$ and define $f_\varepsilon(t) := \ln\big((\mathcal{T}_{\mathcal{A}^{(\varepsilon)}}(x))_1/(\mathcal{T}_{\mathcal{A}^{(\varepsilon)}}(x))_2\big)$. Setting $S(t) := (e^t+1)^{m-1} - e^{(m-1)t} - 1 > 0$, we have

$$f_\varepsilon(t) = \ln \frac{e^{\Delta/2}e^{(m-1)t} + 1 + \varepsilon S(t)}{e^{(m-1)t} + e^{\Delta/2} + \varepsilon S(t)}.$$

Since $S(-t) = e^{-(m-1)t}S(t)$, f_ε is odd ($f_\varepsilon(-t) = -f_\varepsilon(t)$), so $t = 0$ (i.e., $x = (\frac{1}{2}, \frac{1}{2})$) is always a fixed point. Differentiating at $t = 0$ (where $S(0) = 2^{m-1} - 2$ and the identical $\varepsilon S'(0)$ terms cancel in numerator and denominator) gives

$$f_\varepsilon'(0) = \frac{(m-1)(e^{\Delta/2}-1)}{e^{\Delta/2} + 1 + \varepsilon(2^{m-1}-2)}.$$

When $\Delta > \Delta_c(m)$, $(m-1)\tanh(\Delta/4) > 1$ implies $(m-1)(e^{\Delta/2}-1) > e^{\Delta/2}+1$, so $\varepsilon_0(\Delta,m) > 0$ and $f_\varepsilon'(0) > 1$ for every $0 < \varepsilon < \varepsilon_0(\Delta,m)$. On the other hand, as $t \to +\infty$, $S(t)/e^{(m-1)t} \to 0$, so $f_\varepsilon(t) \to \Delta/2 < +\infty$. By the Intermediate Value Theorem there exists $t_\varepsilon^* > 0$ with $f_\varepsilon(t_\varepsilon^*) = t_\varepsilon^*$, yielding conjugate positive Z-eigenvectors $x^+(\varepsilon)$ (log-ratio $+t_\varepsilon^*$) and $x^-(\varepsilon)$ (log-ratio $-t_\varepsilon^*$) with $d_H(x^+(\varepsilon), x^-(\varepsilon)) = 2t_\varepsilon^*$.

At $\varepsilon = 0$, the exact hyperbolic identity

$$f_0(t) = \ln \frac{e^{\Delta/4}e^{(m-1)t/2} + e^{-\Delta/4}e^{-(m-1)t/2}}{e^{-\Delta/4}e^{(m-1)t/2} + e^{\Delta/4}e^{-(m-1)t/2}} = 2\,\mathrm{arctanh}\left(\tanh\left(\frac{\Delta}{4}\right)\tanh\left(\frac{(m-1)t}{2}\right)\right)$$

holds for all $\Delta \ge 0$. If $\Delta = 0$, then $f_\varepsilon(t) = f_0(t) \equiv 0$, so $t = 0$ is trivially the unique fixed point. Assume now $\Delta > 0$, so $\tau := \tanh(\Delta/4) \in (0,1)$. Setting $u := \frac{m-1}{2}t$, the derivative

$$f_0'(t) = \frac{(m-1)\tau\,\mathrm{sech}^2 u}{1 - \tau^2\tanh^2 u} = \frac{(m-1)\tau}{1 + (1-\tau^2)\sinh^2 u}$$

is strictly positive and strictly decreasing on $(0,\infty)$, so f_0 is strictly increasing and strictly concave on $(0,\infty)$. Hence for $0 < \Delta \le \Delta_c(m)$ (where $f_0'(0) = (m-1)\tau \le 1$), $f_\varepsilon(t) < f_0(t) < f_0'(0)t \le t$ for all $t > 0$ and $\varepsilon > 0$ (and $f_0(t) < t$ at $\varepsilon = 0$), leaving $t = 0$ as the unique fixed point. For $\Delta > \Delta_c(m)$, strict concavity of f_0 gives a unique positive root $t_0^* > 0$ of $f_0(t) = t$ with $f_0'(t_0^*) < 1$,

so $R^* = 2t_0^*$ is the unique positive root of (6). Because the denominator $e^{(m-1)t} + e^{\Delta/2} + \varepsilon S(t)$ is uniformly bounded below by $1 + e^{\Delta/2} > 2$ on $[0, \infty)$, $f_\varepsilon \to f_0$ in $C^1([0, t_0^*])$ as $\varepsilon \to 0^+$. Since $f_0'(0) > 1$, there exist $\delta \in (0, t_0^*)$ and $\varepsilon_1 > 0$ such that $f_\varepsilon'(t) > 1$ for all $t \in [0, \delta]$ and $0 < \varepsilon < \varepsilon_1$, whence $f_\varepsilon(t) > t$ on $(0, \delta]$, excluding any positive fixed points in $(0, \delta]$. Combined with $f_\varepsilon(t) \le f_0(t) < t$ for all $t > t_0^*$ and uniform convergence $f_\varepsilon \to f_0$ on $[\delta, t_0^*]$, every positive fixed point t_ε^* satisfies $t_\varepsilon^* \to t_0^*$ as $\varepsilon \to 0^+$. □

Remark 2.3 (Sharpness across all orders $m \ge 3$)**.** Theorem 2.2(c) shows that the threshold $\Delta_c(m) = 2 \ln \frac{m}{m-2}$ decays strictly as m grows: $\Delta_c(3) = 2 \ln 3 \approx 2.1972$, $\Delta_c(4) = 2 \ln 2 \approx 1.3863$, $\Delta_c(5) = 2 \ln(5/3) \approx 1.0217$, and $\Delta_c(m) \sim 4/m$ as $m \to \infty$. Consequently, high-order positive tensors lose Z-eigenvector uniqueness at arbitrarily small projective heterogeneity, whereas H-eigenvectors remain globally unique and uniformly contractive with rate $\tanh(\Delta/4)$ at every order m.

3 Collatz–Wielandt Log-Width Contraction and Multiplicative Perturbation Stability

For any $\mathcal{A} \in \mathbb{R}_{>0}^{n \times \cdots \times n}$ ($m \ge 2$, $n \ge 2$) and $x \in \mathbb{R}_{>0}^n$, define the Collatz–Wielandt lower and upper spectral quotients

$$m_{\mathcal{A}}(x) := \min_{i \in [n]} \frac{(\mathcal{T}_{\mathcal{A}}(x))_i}{x_i^{m-1}}, \qquad M_{\mathcal{A}}(x) := \max_{i \in [n]} \frac{(\mathcal{T}_{\mathcal{A}}(x))_i}{x_i^{m-1}},$$

omitting the subscript $\mathcal{A}$ when clear from context.

Theorem 3.1 (Collatz–Wielandt Contraction, Banach Certificate, and Perturbation Stability)**.** *Let $\mathcal{A} \in \mathbb{R}_{>0}^{n \times \cdots \times n}$ ($m \ge 2$, $n \ge 2$) with unique positive H-eigenpair $(\rho_H(\mathcal{A}), x_H^*)$ and contraction constant $\tau_H(\mathcal{A}) = \tanh(\Delta(\mathcal{A})/4)$.*

(a) ***Exact one-step Collatz–Wielandt log-width contraction.*** *For every $x \in \mathbb{R}_{>0}^n$, $m(x) \le \rho_H(\mathcal{A}) \le M(x)$, and one NQZ step $x_1 := \Phi_{m-1}(x)$ satisfies*

$$\ln \frac{M(x_1)}{m(x_1)} \le 4 \operatorname{arctanh}\left(\tau_H(\mathcal{A}) \tanh\left(\frac{1}{4} \ln \frac{M(x)}{m(x)} \right) \right) \le \tau_H(\mathcal{A}) \ln \frac{M(x)}{m(x)}. \tag{7}$$

(b) ***A posteriori Hilbert certificate.*** *For every $x \in \mathbb{R}_{>0}^n$,*

$$d_H(x, x_H^*) \le \frac{d_H(x, \Phi_{m-1}(x))}{1 - \tau_H(\mathcal{A})} = \frac{\ln(M(x)/m(x))}{(m-1)(1 - \tau_H(\mathcal{A}))}. \tag{8}$$

(c) ***Multiplicative perturbation stability.*** *For any $\mathcal{E} \in \mathbb{R}^{n \times \cdots \times n}$ and $\mathcal{B} := \mathcal{A} \circ e^{\mathcal{E}}$ with unique positive H-eigenpair $(\rho_H(\mathcal{B}), x_{\mathcal{B}}^*)$,*

$$\min \mathcal{E} \;\le\; \ln \frac{\rho_H(\mathcal{B})}{\rho_H(\mathcal{A})} \;\le\; \max \mathcal{E}, \qquad \left| \ln \frac{\rho_H(\mathcal{B})}{\rho_H(\mathcal{A})} \right| \le \|\mathcal{E}\|_\infty, \tag{9}$$

$$d_H(x_{\mathcal{B}}^*, x_H^*) \;\le\; \frac{\operatorname{osc}(\mathcal{E})}{(m-1)\big(1 - \min\{\tau_H(\mathcal{A}), \tau_H(\mathcal{B})\}\big)}. \tag{10}$$

Proof. (a) Let $c := \min_j (x_j / (x_H^*)_j) = x_{i_0} / (x_H^*)_{i_0} > 0$. Then $x \ge c x_H^*$ entrywise with equality at i_0. By entrywise positivity and degree-$(m-1)$ homogeneity of $\mathcal{T}_{\mathcal{A}}$,

$$(\mathcal{T}_{\mathcal{A}}(x))_{i_0} \ge c^{m-1} (\mathcal{T}_{\mathcal{A}}(x_H^*))_{i_0} = c^{m-1} \rho_H(\mathcal{A}) (x_H^*)_{i_0}^{m-1} = \rho_H(\mathcal{A}) x_{i_0}^{m-1},$$

so $M(x) \geq (\mathcal{T}_{\mathcal{A}}(x))_{i_0}/x_{i_0}^{m-1} \geq \rho_H(\mathcal{A})$. A symmetric argument at $i_1 \in \arg\max_j (x_j/(x_H^*)_j)$ gives $m(x) \leq \rho_H(\mathcal{A})$. Next, observe the exact identity

$$\ln \frac{M(x)}{m(x)} = \max_i \ln \frac{(\mathcal{T}_{\mathcal{A}}(x))_i}{x_i^{m-1}} - \min_j \ln \frac{(\mathcal{T}_{\mathcal{A}}(x))_j}{x_j^{m-1}} = (m-1) d_H(\Phi_{m-1}(x), x). \tag{11}$$

Evaluating (11) at $x_1 = \Phi_{m-1}(x)$ and applying Theorem 2.2(a) with $p = m-1$ to the pair (x_1, x) yields

$$\ln \frac{M(x_1)}{m(x_1)} = (m-1) d_H(\Phi_{m-1}(x_1), \Phi_{m-1}(x)) \leq 4 \operatorname{arctanh}\left(\tau_H(\mathcal{A}) \tanh\left(\frac{(m-1) d_H(x_1, x)}{4}\right)\right),$$

which substitutes (11) to establish (7).

(b) Since $\Phi_{m-1}(x_H^*)$ lies on the same ray as x_H^*, the triangle inequality and Theorem 2.2(b) give

$$d_H(x, x_H^*) \leq d_H(x, \Phi_{m-1}(x)) + d_H(\Phi_{m-1}(x), \Phi_{m-1}(x_H^*)) \leq d_H(x, \Phi_{m-1}(x)) + \tau_H(\mathcal{A}) d_H(x, x_H^*).$$

Rearranging and substituting (11) proves (8).

(c) Set $e_{\min} := \min \mathcal{E}$ and $e_{\max} := \max \mathcal{E}$. For every $x \in \mathbb{R}^n_{>0}$ and $i \in [n]$, $e^{e_{\min}} (\mathcal{T}_{\mathcal{A}}(x))_i \leq (\mathcal{T}_{\mathcal{B}}(x))_i \leq e^{e_{\max}} (\mathcal{T}_{\mathcal{A}}(x))_i$. Evaluating at $x = x_H^*$ gives

$$e^{e_{\min}} \rho_H(\mathcal{A}) \leq m_{\mathcal{B}}(x_H^*) \leq \rho_H(\mathcal{B}) \leq M_{\mathcal{B}}(x_H^*) \leq e^{e_{\max}} \rho_H(\mathcal{A}),$$

proving (9). Taking $(m-1)$-th roots yields $\ln\big(\Phi_{\mathcal{B},m-1}(x)_i / \Phi_{\mathcal{A},m-1}(x)_i\big) \in [\frac{e_{\min}}{m-1}, \frac{e_{\max}}{m-1}]$ for all $i \in [n]$, hence $d_H(\Phi_{\mathcal{B},m-1}(x), \Phi_{\mathcal{A},m-1}(x)) \leq \frac{\operatorname{osc}(\mathcal{E})}{m-1}$ for every $x \in \mathbb{R}^n_{>0}$. Evaluating at $x = x_{\mathcal{B}}^*$ (a fixed ray of $\Phi_{\mathcal{B},m-1}$) and applying the triangle inequality gives

$$\begin{aligned} d_H(x_{\mathcal{B}}^*, x_H^*) &\leq d_H\big(\Phi_{\mathcal{B},m-1}(x_{\mathcal{B}}^*), \Phi_{\mathcal{A},m-1}(x_{\mathcal{B}}^*)\big) + d_H\big(\Phi_{\mathcal{A},m-1}(x_{\mathcal{B}}^*), \Phi_{\mathcal{A},m-1}(x_H^*)\big) \\ &\leq \frac{\operatorname{osc}(\mathcal{E})}{m-1} + \tau_H(\mathcal{A}) d_H(x_{\mathcal{B}}^*, x_H^*), \end{aligned}$$

so $d_H(x_{\mathcal{B}}^*, x_H^*) \leq \frac{\operatorname{osc}(\mathcal{E})}{(m-1)(1-\tau_H(\mathcal{A}))}$. Reversing the roles of $\mathcal{A}$ and $\mathcal{B}$ (with $-\mathcal{E}$, which has identical oscillation $\operatorname{osc}(\mathcal{E})$) yields (10). □

4 Variational H-Radius and Global m-th-Root Tensor Sinkhorn Descent

We now restrict to *symmetric* positive tensors $\mathcal{A} \in \mathbb{R}^{n \times \cdots \times n}_{>0}$ ($A_{i_{\pi(1)}, \ldots, i_{\pi(m)}} = A_{i_1, \ldots, i_m}$ for every permutation $\pi \in S_m$). Define the homogeneous multilinear form $\mathcal{A}x^m := \langle x, \mathcal{T}_{\mathcal{A}}(x) \rangle = \sum_{i_1, \ldots, i_m = 1}^n A_{i_1 \ldots i_m} x_{i_1} \cdots x_{i_m}$. For a positive scaling vector $d = e^u \in \mathbb{R}^n_{>0}$ ($u \in \mathbb{R}^n$), let $\mathcal{B}(d)$ be the symmetrically scaled tensor

$$\mathcal{B}(d)_{i_1, \ldots, i_m} := d_{i_1} \cdots d_{i_m} A_{i_1, \ldots, i_m},$$

whose 1-marginal vector (identical in every mode by symmetry) is $m_i(d) := d_i (\mathcal{T}_{\mathcal{A}}(d))_i = \sum_{i_2, \ldots, i_m = 1}^n \mathcal{B}(d)_{i, i_2, \ldots, i_m}$. Given a target marginal $r \in \mathbb{R}^n_{>0}$, define the convex scaling potential

$$F_r(u) := \frac{1}{m} \mathcal{A}(e^u)^m - \langle r, u \rangle, \qquad u \in \mathbb{R}^n, \tag{12}$$

and the *parallel m-th-root tensor Sinkhorn map*

$$g(u) := u + \frac{1}{m} \ln \frac{r}{m(e^u)}, \qquad \text{i.e., } d_{k+1} = d_k \circ \left(\frac{r}{m(d_k)}\right)^{[1/m]}. \tag{13}$$

Theorem 4.1 (Variational H-Radius and Global Tensor Sinkhorn Lyapunov Descent)**.** *Let $\mathcal{A} \in \mathbb{R}_{>0}^{n\times\cdots\times n}$ ($m \geq 2$, $n \geq 2$) be symmetric with strictly positive entries.*

(a) ***Variational H-spectral radius.*** *The unique positive H-eigenvalue satisfies*

$$\rho_H(\mathcal{A}) = \max_{x\in\mathbb{R}^n,\ \|x\|_m=1} \mathcal{A}x^m = \max_{x\in\mathbb{R}^n_{>0},\ \|x\|_m=1} \mathcal{A}x^m, \tag{14}$$

uniquely attained on $\mathbb{R}^n_{\geq 0} \cap \{\|x\|_m = 1\}$ at $x_H^*/\|x_H^*\|_m$.

(b) ***Unique diagonal scaling and exact global KL descent.*** *For every $r \in \mathbb{R}^n_{>0}$, F_r is strictly convex and coercive on $\mathbb{R}^n$, admitting a unique global minimizer $u^* \in \mathbb{R}^n$ that is the unique solution of the symmetric tensor scaling equation $m(e^{u^*}) = r$. Moreover, from every $u \in \mathbb{R}^n$, one step $u^+ := g(u)$ of (13) satisfies*

$$F_r(u^+) - F_r(u) \leq -\frac{1}{m} D_{\mathrm{KL}}^{\mathrm{gen}}\big(r \,\|\, m(e^u)\big) \leq 0, \tag{15}$$

with equality if and only if $u = u^*$. Consequently, for every initialization $u_0 \in \mathbb{R}^n$, the iterates $u_{k+1} = g(u_k)$ converge globally to u^* and satisfy the sublinear marginal guarantee $\min_{0\leq t<K} D_{\mathrm{KL}}^{\mathrm{gen}}(r\|m(e^{u_t})) \leq \frac{m(F_r(u_0)-F_r(u^*))}{K}$ for all $K \geq 1$.

Proof. (a) Because $\mathcal{A} > 0$, $|\mathcal{A}x^m| \leq \mathcal{A}|x|^m$ for all $x \in \mathbb{R}^n$, so the maximum over the compact unit ℓ_m sphere is attained on $\mathbb{R}^n_{\geq 0} \cap \{\|x\|_m = 1\}$. Let $x \in \mathbb{R}^n_{\geq 0}$ with $\|x\|_m = 1$ be any maximizer, and suppose for contradiction that $x_i = 0$ for some $i \in [n]$. For $\varepsilon > 0$, set $y(\varepsilon) := (x+\varepsilon e_i)/\|x+\varepsilon e_i\|_m$. Since $\|x + \varepsilon e_i\|_m^m = 1 + \varepsilon^m$, symmetry of $\mathcal{A}$ gives

$$\mathcal{A}y(\varepsilon)^m = \frac{\mathcal{A}x^m + m\varepsilon(\mathcal{T}_{\mathcal{A}}(x))_i + O(\varepsilon^2)}{1+\varepsilon^m} = \mathcal{A}x^m + m\varepsilon(\mathcal{T}_{\mathcal{A}}(x))_i + O(\varepsilon^2).$$

Because $\mathcal{A} > 0$ and $x \in \mathbb{R}^n_{\geq 0} \setminus \{0\}$, we have $(\mathcal{T}_{\mathcal{A}}(x))_i > 0$, so $\mathcal{A}y(\varepsilon)^m > \mathcal{A}x^m$ for sufficiently small $\varepsilon > 0$ (even for $m = 2$, where $O(\varepsilon^2)$ is strictly dominated by $+2\varepsilon(\mathcal{T}_{\mathcal{A}}(x))_i$), contradicting maximality. Hence every nonnegative maximizer lies in $\mathbb{R}^n_{>0}$. By the Lagrange multiplier condition on $\nabla(\mathcal{A}x^m) = m\mathcal{T}_{\mathcal{A}}(x)$ and $\nabla(\|x\|_m^m) = mx^{[m-1]}$, any interior maximizer satisfies $\mathcal{T}_{\mathcal{A}}(x) = \lambda x^{[m-1]}$ with $\lambda > 0$. By uniqueness of the positive H-eigenpair (Theorem 2.2(b)), $x = x_H^*/\|x_H^*\|_m$ and $\mathcal{A}x^m = \langle x, \mathcal{T}_{\mathcal{A}}(x)\rangle = \rho_H(\mathcal{A})\|x\|_m^m = \rho_H(\mathcal{A})$.

(b) Direct differentiation using symmetry of $\mathcal{A}$ gives $\nabla_i F_r(u) = m_i(e^u) - r_i$ and, for every $v \in \mathbb{R}^n$,

$$v^\top \nabla^2 F_r(u) v = \frac{1}{m} \sum_{i_1,\ldots,i_m=1}^{n} \mathcal{B}(e^u)_{i_1\ldots i_m} \left(\sum_{k=1}^{m} v_{i_k}\right)^2 \geq m \sum_{i=1}^{n} A_{i\ldots i} e^{mu_i} v_i^2 > 0 \qquad (v \neq 0), \tag{16}$$

establishing strict convexity of F_r. Setting $a_{\min} := \min A_{i_1\ldots i_m} > 0$, we have

$$F_r(u) \geq \sum_{i=1}^{n} \left(\frac{a_{\min}}{m} e^{mu_i} - r_i u_i\right).$$

Each scalar summand $h_i(t) := \frac{a_{\min}}{m} e^{mt} - r_i t$ is bounded below on $\mathbb{R}$ and satisfies $h_i(t) \to +\infty$ as $|t| \to \infty$, so $F_r(u) \to +\infty$ as $\|u\|_\infty \to \infty$. Thus F_r has a unique global minimizer $u^* \in \mathbb{R}^n$, characterized by $\nabla F_r(u^*) = 0$, i.e., $m(e^{u^*}) = r$.

Now let $u \in \mathbb{R}^n$, $\mathcal{B} := \mathcal{B}(e^u)$, and $\delta_i := u_i^+ - u_i = \frac{1}{m} \ln \frac{r_i}{m_i(e^u)}$. By convexity of the exponential function (AM–GM), for every multi-index $(i_1, \ldots, i_m) \in [n]^m$,

$$\exp(\delta_{i_1} + \cdots + \delta_{i_m}) = \exp\left(\frac{1}{m}\sum_{k=1}^{m}(m\delta_{i_k})\right) \leq \frac{1}{m}\sum_{k=1}^{m} e^{m\delta_{i_k}} = \frac{1}{m}\sum_{k=1}^{m} \frac{r_{i_k}}{m_{i_k}(e^u)}.$$

Multiplying by $\frac{1}{m}B_{i_1\dots i_m}$, summing over $[n]^m$, and using mode symmetry $\sum_{i_{-k}} B_{i_1\dots i_m} = m_{i_k}(e^u)$ for every mode $k \in [m]$ yields

$$\frac{1}{m}\mathcal{A}(e^{u^+})^m = \frac{1}{m}\sum_{i_1,\dots,i_m=1}^{n} B_{i_1\dots i_m} e^{\delta_{i_1}+\cdots+\delta_{i_m}} \le \frac{1}{m^2}\sum_{k=1}^{m}\sum_{i=1}^{n} m_i(e^u)\frac{r_i}{m_i(e^u)} = \frac{1}{m}\sum_{i=1}^{n} r_i.$$

Since $\frac{1}{m}\mathcal{A}(e^u)^m = \frac{1}{m}\sum_{i=1}^{n} m_i(e^u)$, we obtain

$$\begin{aligned} F_r(u^+) - F_r(u) &= \frac{1}{m}\mathcal{A}(e^{u^+})^m - \frac{1}{m}\sum_{i=1}^{n} m_i(e^u) - \frac{1}{m}\sum_{i=1}^{n} r_i \ln\frac{r_i}{m_i(e^u)} \\ &\le -\frac{1}{m}\sum_{i=1}^{n}\left(r_i \ln\frac{r_i}{m_i(e^u)} - r_i + m_i(e^u)\right) = -\frac{1}{m}D_{\mathrm{KL}}^{\mathrm{gen}}(r\|m(e^u)). \end{aligned}$$

Equality in (15) forces $D_{\mathrm{KL}}^{\mathrm{gen}}(r\|m(e^u)) = 0$, hence $m(e^u) = r$, i.e., $u = u^*$. Telescoping (15) along $u_{k+1} = g(u_k)$ gives

$$\sum_{t=0}^{K-1} D_{\mathrm{KL}}^{\mathrm{gen}}(r\|m(e^{u_t})) \le m\big(F_r(u_0) - F_r(u_K)\big) \le m\big(F_r(u_0) - F_r(u^*)\big),$$

proving the $O(1/K)$ marginal rate and $D_{\mathrm{KL}}^{\mathrm{gen}}(r\|m(e^{u_k})) \to 0$, so $m(e^{u_k}) \to r$. Because $F_r(u_k) \le F_r(u_0)$ stays inside a compact sublevel set (by coercivity) and u^* is the unique solution of $m(e^u) = r$, every limit point of $\{u_k\}$ equals u^*, proving global convergence $u_k \to u^*$. □

5 Hypergraph 2-Marginal Eigenvalue Floor and Exact Local Sinkhorn Rate

To determine the exact asymptotic convergence rate of parallel m-th-root tensor Sinkhorn scaling, we introduce the symmetric 2-marginal contraction of a symmetric nonnegative tensor $\mathcal{B} \in \mathbb{R}_{\ge 0}^{n\times\cdots\times n}$ ($m \ge 2$, $n \ge 2$) with strictly positive 1-marginals $r_i := \sum_{i_2,\dots,i_m=1}^{n} B_{i,i_2,\dots,i_m} > 0$:

$$M(\mathcal{B})_{ij} := \sum_{i_3,\dots,i_m=1}^{n} B_{i,j,i_3,\dots,i_m} \quad (m \ge 3), \qquad M(\mathcal{B}) := \mathcal{B} \quad (m = 2),$$

and the normalized symmetric 2-marginal operator

$$P(\mathcal{B}) := D_r^{-1/2} M(\mathcal{B}) D_r^{-1/2}, \qquad D_r := \mathrm{diag}(r_1,\dots,r_n). \tag{17}$$

We also define the minimum normalized diagonal mass $\delta_{\mathrm{diag}}(\mathcal{B}) := \min_{i\in[n]} \frac{B_{i,\dots,i}}{r_i} \in [0,1]$.

Theorem 5.1 (Hypergraph 2-Marginal Spectral Floor and Exact Local Sinkhorn Rate)**.** *Let $m \ge 2$ and $n \ge 2$.*

(a) ***Universal 2-marginal spectral floor.*** *For every symmetric nonnegative tensor $\mathcal{B} \in \mathbb{R}_{\ge 0}^{n\times\cdots\times n}$ with strictly positive 1-marginals $r \in \mathbb{R}_{>0}^n$, the real symmetric matrix $P(\mathcal{B})$ has top eigenvalue $\lambda_1(P(\mathcal{B})) = 1$ (with eigenvector $\sqrt{r}$) and satisfies*

$$\lambda_{\min}(P(\mathcal{B})) \ge -\frac{1}{m-1} + \frac{m}{m-1}\delta_{\mathrm{diag}}(\mathcal{B}) \ge -\frac{1}{m-1}. \tag{18}$$

For every $n \ge m$, the floor $-\frac{1}{m-1}$ is sharp and is attained with multiplicity at least $m-1$ by any symmetric complete m-partite hypergraph tensor (with $r_i > 0$) whose support consists solely of multi-indices hitting each block of a partition $[n] = V_1 \sqcup \cdots \sqcup V_m$ into m nonempty blocks once.

(b) ***Exact local Sinkhorn Jacobian spectrum and contraction rate.*** *Let $\mathcal{A} \in \mathbb{R}_{>0}^{n\times\cdots\times n}$ be symmetric with strictly positive entries, $r \in \mathbb{R}_{>0}^n$, and u^* be the unique Sinkhorn minimizer of Theorem 4.1(b), with scaled tensor $\mathcal{B}^* := \mathcal{B}(e^{u^*})$ and $P_* := P(\mathcal{B}^*)$. Then the Jacobian $Dg(u^*)$ of the parallel m-th-root Sinkhorn map (13) satisfies the exact similarity identity*

$$D_r^{1/2}\, Dg(u^*)\, D_r^{-1/2} = \frac{m-1}{m}\big(I_n - P_*\big). \tag{19}$$

Consequently, every eigenvalue of $Dg(u^*)$ is real and lies in $[0,1)$ (eliminating oscillatory overshoot along all eigenmodes), the global scale mode $\mathbf{1}_n$ has eigenvalue 0, and the exact asymptotic local contraction factor is

$$\rho\big(Dg(u^*)\big) = \frac{m-1}{m}\big(1 - \lambda_{\min}(P_*)\big) \le 1 - \delta_{\mathrm{diag}}(\mathcal{B}^*) < 1. \tag{20}$$

Proof. (a) Since $M(\mathcal{B}) \ge 0$ satisfies $(M(\mathcal{B})\mathbf{1}_n)_i = \sum_{j=1}^n M(\mathcal{B})_{ij} = r_i$, the vector $\sqrt{r} \in \mathbb{R}_{>0}^n$ satisfies $P(\mathcal{B})\sqrt{r} = \sqrt{r}$. By the Perron–Frobenius theorem for nonnegative matrices, $\lambda_1(P(\mathcal{B})) = \rho(P(\mathcal{B})) = 1$. Next, let $v \in \mathbb{R}^n$ be arbitrary and set $w := D_r^{-1/2}v$, so $\sum_{i=1}^n r_i w_i^2 = \|v\|_2^2$. Expanding the nonnegative sum of squares using symmetry of $\mathcal{B}$ gives

$$\begin{aligned}\frac{1}{m}\sum_{i_1,\dots,i_m=1}^{n} B_{i_1\dots i_m}\left(\sum_{k=1}^{m} w_{i_k}\right)^2 &= \frac{1}{m}\left(m\sum_{i=1}^{n} r_i w_i^2 + m(m-1)\sum_{i,j=1}^{n} M(\mathcal{B})_{ij} w_i w_j\right)\\ &= v^\top\big(I_n + (m-1)P(\mathcal{B})\big)v.\end{aligned} \tag{21}$$

On the other hand, retaining only the diagonal entries $(i,\dots,i)$ in the left-hand side gives

$$v^\top\big(I_n + (m-1)P(\mathcal{B})\big)v \ge \frac{1}{m}\sum_{i=1}^{n} B_{i\dots i}(mw_i)^2 = m\sum_{i=1}^{n}\frac{B_{i\dots i}}{r_i}v_i^2 \ge m\,\delta_{\mathrm{diag}}(\mathcal{B})\,\|v\|_2^2.$$

Rearranging and dividing by $(m-1)\|v\|_2^2$ (for $v \ne 0$) establishes (18).

For sharpness when $n \ge m$, let $[n] = V_1 \sqcup \cdots \sqcup V_m$ be any partition into m nonempty blocks and let $\mathcal{B}$ be supported strictly on multi-indices $(i_1,\dots,i_m)$ that contain exactly one index from each V_k (with $r_i > 0$). Consider the $(m-1)$-dimensional subspace $W := \{w \in \mathbb{R}^n : w|_{V_k} \equiv c_k \text{ for all } k \in [m],\ \sum_{k=1}^m c_k = 0\}$. For every $w \in W$ and every $(i_1,\dots,i_m) \in \mathrm{supp}(\mathcal{B})$, $\sum_{k=1}^m w_{i_k} = \sum_{k=1}^m c_k = 0$, so the left-hand side of (21) vanishes identically. Setting $v = D_r^{1/2}w$ yields an $(m-1)$-dimensional subspace on which $v^\top(I_n + (m-1)P(\mathcal{B}))v = 0$; since $I_n + (m-1)P(\mathcal{B}) \succeq 0$, this subspace lies in $\ker(I_n + (m-1)P(\mathcal{B}))$, proving that $-\frac{1}{m-1}$ is an eigenvalue of $P(\mathcal{B})$ with multiplicity at least $m-1$.

(b) Comparing (16) with (21) at $u = u^*$ (where $\mathcal{B}(e^{u^*}) = \mathcal{B}^*$ and $m(e^{u^*}) = r$) yields the exact Hessian identity

$$\nabla^2 F_r(u^*) = D_r + (m-1)M(\mathcal{B}^*).$$

Differentiating $g_i(u) = u_i + \frac{1}{m}\ln r_i - \frac{1}{m}\ln m_i(e^u)$ at u^* using $\frac{\partial m_i(e^u)}{\partial u_j}\big|_{u^*} = (\nabla^2 F_r(u^*))_{ij}$ and $m_i(e^{u^*}) = r_i$ gives

$$Dg(u^*) = I_n - \frac{1}{m}D_r^{-1}\big(D_r + (m-1)M(\mathcal{B}^*)\big) = \frac{m-1}{m}\big(I_n - D_r^{-1}M(\mathcal{B}^*)\big).$$

Conjugating by $D_r^{1/2}$ proves (19). Because $\mathcal{A} > 0$, we have $\delta_{\mathrm{diag}}(\mathcal{B}^*) > 0$, so part (a) implies $-\frac{1}{m-1} + \frac{m}{m-1}\delta_{\mathrm{diag}}(\mathcal{B}^*) \le \lambda_{\min}(P_*) \le \lambda_k(P_*) \le \lambda_1(P_*) = 1$ for every $k \in [n]$. Therefore every eigenvalue $\mu_k := \frac{m-1}{m}(1 - \lambda_k(P_*))$ of $Dg(u^*)$ satisfies $0 \le \mu_k \le 1 - \delta_{\mathrm{diag}}(\mathcal{B}^*) < 1$. In particular, $\lambda_1(P_*) = 1$ (eigenvector $\sqrt{r}$) maps to $\mu_1 = 0$ with eigenvector $D_r^{-1/2}\sqrt{r} = \mathbf{1}_n$, and the spectral radius is achieved at $\lambda_{\min}(P_*)$, giving (20). □

Remark 5.2 (Hypergraph obstruction to fast Sinkhorn scaling)**.** Equation (20) reveals a sharp geometric dichotomy: as a strictly positive symmetric tensor $\mathcal{B}^*$ approaches a complete m-partite hypergraph (where $\delta_{\text{diag}}(\mathcal{B}^*) \to 0$ and $\lambda_{\min}(P_*) \to -\frac{1}{m-1}$), the exact local contraction rate $\rho(Dg(u^*)) = \frac{m-1}{m}(1 - \lambda_{\min}(P_*))$ approaches 1, demonstrating that near-m-partiteness is the canonical spectral obstruction to rapid diagonal tensor scaling (generalizing near-bipartiteness for $m = 2$, where $\lambda_{\min}(P_*) \to -1$).

6 Computational Verification and Explicit Benchmarks

All four theorems were verified independently via a self-contained Python suite (`tensor_pf.py`, `verify_all.py`, `test_tensor_pf.py`; four automated unit tests passing in 0.26 s) using deterministic NumPy `default_rng` streams across 235 randomized tensor instances plus structured bifurcation and complete m-partite witnesses. Every benchmark specification is stated explicitly below for direct reproduction.

- **Theorem 2.2 suite (seed `202621`, 100 tensors):** For $(m, n) \in \{(3,3), (3,5), (4,3), (4,4)\}$, 25 positive tensors $\mathcal{A} = \exp(Z)$ ($Z_{i_1 \ldots i_m} \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0, 0.7^2)$) and pairs $x, y \overset{\text{i.i.d.}}{\sim} \exp(\mathcal{N}(0, 0.8^2))$ were tested at $p \in \{1, m-1, m\}$. Both (4) and $\tau_H(\mathcal{A}) d_H(x, y)$ held with 0.0 violation across all 300 map evaluations. For the pitchfork witness $\mathcal{A}^{(0)}$ across $m \in \{3, 4, 5\}$ and $\Delta \in \{0.7, 1.3, 1.8\}\,\Delta_c(m)$ (initialized at $(0.8, 0.2)$ for 4000 Z-power steps), the simulated separation $d_H(x^+, x^-)$ matched the exact root $R^*(\Delta, m)$ of (6) to maximum error 1.78×10^{-15} (Figure 1(a)).
- **Theorem 3.1 suite (seed `202622`, 75 tensors):** For $(m, n) \in \{(3,3), (3,4), (4,3)\}$ (25 tensors each, $Z \sim \mathcal{N}(0, 0.6^2)$, $x_0 = \exp(\mathcal{N}(0, 0.9^2))$, multiplicative perturbation $\mathcal{E} \sim \mathcal{N}(0, 0.15^2)$), the Collatz–Wielandt bracket $m(x_0) \le \rho_H \le M(x_0)$, one-step log-width contraction (7), Banach certificate (8), and both perturbation inequalities (9)–(10) held with 0.0 violation across all 75 checks. Figure 1(b) illustrates (7) on a standalone $4 \times 4 \times 4$ tensor ($\mathcal{A} = \exp(\mathcal{N}(0, 0.6^2))$, fresh seed `202622`, $\tau_H \approx 0.797$) initialized at $x_0 = (0.05, 0.15, 0.25, 0.55)$.
- **Theorems 4.1–5.1 suite (seed `202623`, 60 symmetric tensors):** Looping in order over $(m, n) \in \{(3,3), (3,5), (4,3), (4,4)\}$, 15 symmetric positive tensors per pair were generated by symmetrizing $\exp(\mathcal{N}(0, 0.8^2))$, paired with target marginals $r_i \propto \text{Unif}(0.3, 1.7)$ normalized to $\sum_i r_i = 1$ and $u_0 \sim \mathcal{N}(0, 0.6^2)$ (10 Sinkhorn steps checked per instance). Across all 60 tensors: the variational H-radius identity (14) held to maximum error 7.11×10^{-14}; the global generalized-KL descent law (15) and spectral floor (18) exhibited 0.0 violation; central-difference Jacobians ($\varepsilon = 10^{-6}$) of g at u^* matched $\frac{m-1}{m}(1 - \lambda_{\min}(P_*))$ to maximum error 1.79×10^{-10}; and complete m-partite hypergraphs ($m \in \{3, 4\}$, $n = m$) attained $\lambda_{\min}(P) = -\frac{1}{m-1}$ to 2.22×10^{-16}.

7 Discussion and Open Directions

The four theorems established above show that two elementary invariants—the mode-1 unfolding projective diameter $\Delta(\mathcal{A})$ and the normalized symmetric 2-marginal operator $P(\mathcal{B})$—govern both nonlinear Perron–Frobenius dynamics and symmetric tensor balancing with exact, sharp constants. Three natural extensions remain open:

1. **Symmetric Z-bifurcation thresholds:** Theorem 2.2(c) proves sharpness of $\Delta_c(m) = 2 \ln \frac{m}{m-2}$ on general (non-symmetric) strictly positive tensors $\mathcal{A}^{(\varepsilon)}$. Determining whether the exact critical diameter among *symmetric* positive tensors equals $\Delta_c(m)$ or admits a strictly larger symmetric threshold remains open.

Table 1: Exact quantitative verification on trial 0 of each (m, n) block in the symmetric benchmark suite (seed `202623`, Section 6) and on the Z-eigenvalue pitchfork family (Theorem 2.2(c)). Every Sinkhorn local rate $\rho(Dg(u^*)) = \frac{m-1}{m}(1 - \lambda_{\min}(P_*))$ matches central-difference Jacobian eigenvalues to $\leq 1.79 \times 10^{-10}$.

Benchmark	$\Delta(\mathcal{A})$	$\tau_H(\mathcal{A})$	$\rho_H(\mathcal{A})$	$\lambda_{\min}(P_*)$	Floor $-\frac{1}{m-1}$	$\rho(Dg(u^*))$	It.
`Sym_m3_n3`	2.2838	0.5161	10.6446	−0.1488	−0.5000	0.7659	103
`Sym_m3_n5`	4.2122	0.7830	34.8293	−0.0355	−0.5000	0.6903	76
`Sym_m4_n3`	2.0249	0.4670	31.3936	−0.0374	−0.3333	0.7781	108
`Sym_m4_n4`	4.2163	0.7834	88.9663	−0.0204	−0.3333	0.7653	105

Exact Z-Eigenvalue Pitchfork Witnesses ($\Delta_c(m) = 2\ln\frac{m}{m-2}$):

$m = 3$ ($\Delta_c = 2.1972$) $\quad \Delta/\Delta_c \in \{0.7, 1.3, 1.8\} \Rightarrow R^* \in \{0.0000,\ 2.0707,\ 3.6033\}$ (err 1.78×10^{-15})

$m = 4$ ($\Delta_c = 1.3863$) $\quad \Delta/\Delta_c \in \{0.7, 1.3, 1.8\} \Rightarrow R^* \in \{0.0000,\ 1.3349,\ 2.3052\}$ (err 1.33×10^{-15})

$m = 5$ ($\Delta_c = 1.0217$) $\quad \Delta/\Delta_c \in \{0.7, 1.3, 1.8\} \Rightarrow R^* \in \{0.0000,\ 0.9905,\ 1.7062\}$ (err 1.33×10^{-15})

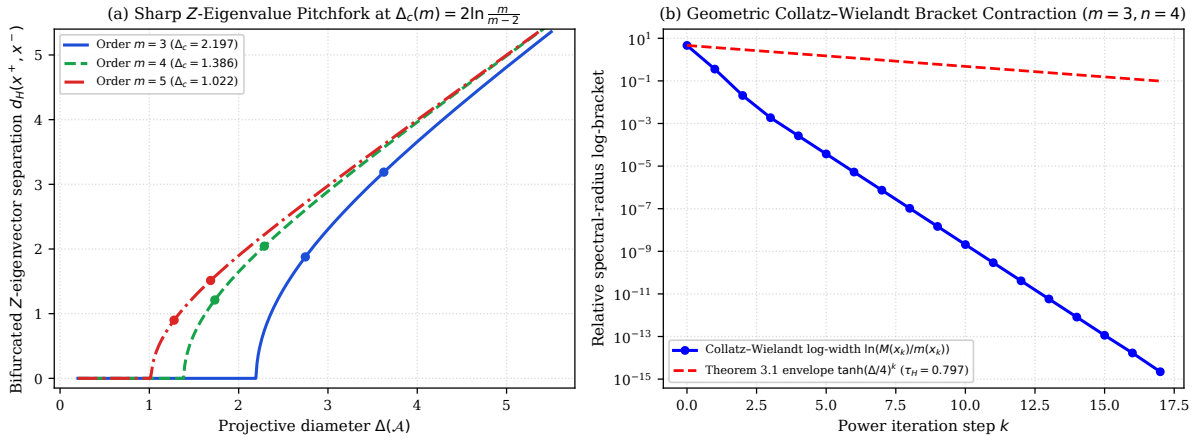


Figure 1: **(a)** Sharp supercritical Z-eigenvalue pitchfork bifurcation across $\Delta_c(m) = 2\ln\frac{m}{m-2}$ for $m \in \{3, 4, 5\}$: solid/dashed curves show the exact root $R^*(\Delta, m)$ of (6), while markers show simulated Z-power iteration separations. **(b)** Geometric contraction of the Collatz–Wielandt spectral log-bracket $\ln(M(x_k)/m(x_k))$ under NQZ iteration Φ_2 for a $4 \times 4 \times 4$ tensor (seed `202622`) bounded by the Theorem 3.1(a) envelope $\ln(M(x_0)/m(x_0))\tanh(\Delta/4)^k$.

2. **Non-symmetric rectangular tensor scaling:** Extending the global generalized-KL descent law (15) and exact spectral floor (18) to multi-marginal scaling of rectangular m-mode tensors $B_{i_1\ldots i_m} = d^{(1)}_{i_1} \cdots d^{(m)}_{i_m} A_{i_1\ldots i_m}$ leads to an m-partite block-2-marginal operator whose sharp negative eigenvalue floor is an inviting target.

3. **Sparse nonnegative hypergraphs:** Characterizing quantitative convergence rates of parallel m-th-root Sinkhorn scaling when $\delta_{\text{diag}}(\mathcal{B}^*) = 0$ (zero diagonal hyperedges, as in simple m-uniform hypergraphs) via Cheeger-type hypergraph conductance bounds on $1 + (m-1)\lambda_{\min}(P_*)$ is left for future investigation.

Reproducibility and Data Availability

All analytical verifications, unit tests, and figure generation scripts are openly archived under the MIT license (version `v1.0.1`) at:
`https://github.com/lluiseriksson/tensor-perron-frobenius-scaling`.

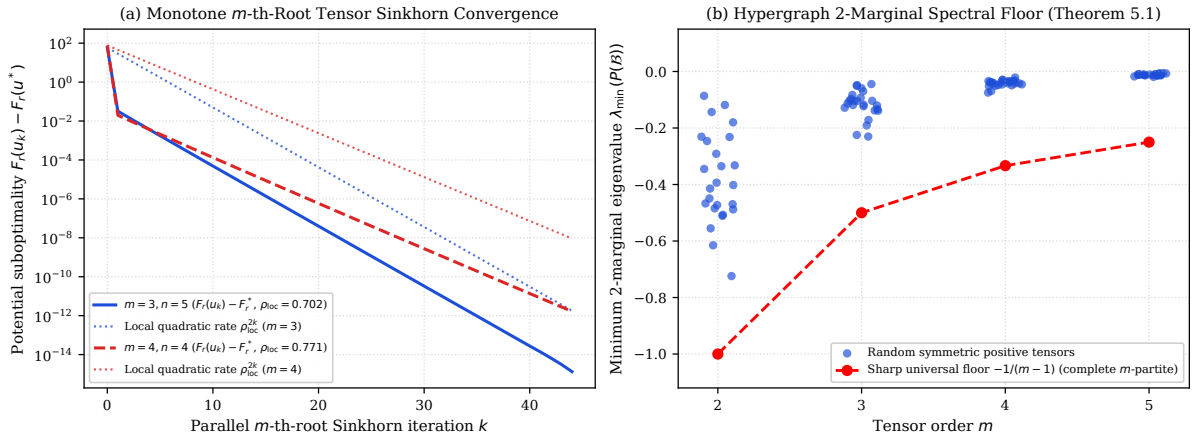


Figure 2: **(a)** Monotone suboptimality $F_r(u_k) - F_r(u^*)$ under parallel m-th-root Sinkhorn scaling on standalone draws ($(m, n) = (3, 5)$ then $(4, 4)$, fresh seed `202623`, $u_0 = 0$) vs. the asymptotic linear decay $\rho(Dg(u^*))^{2k}$ of the locally quadratic potential suboptimality (Theorem 5.1(b)). **(b)** Minimum 2-marginal eigenvalues $\lambda_{\min}(P(\mathcal{B}))$ across 100 random symmetric positive tensors (25 per $m \in \{2, 3, 4, 5\}$, seed `202624`, $n = \max(4, m + 1)$) strictly above the sharp m-partite floor $-1/(m-1)$ (Theorem 5.1(a)).

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