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Quadratic-Ball Resolvent Duality, Infinite Order Dimension, Catalytic and Multi-Copy Super-Activation, and Lindbladian Rank Barriers in Quantum-State Texture

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Abstract

J. A. de Leon [1] reduced deterministic state convertibility in the resource theory of quantum-state texture (QST) to an existential filter-and-preparation semidefinite program (SDP), and our companion study [2] (`ARR-2026-2T3GNZ9H5K8N2V4G`) established the scalar resolvent convertibility criterion $\psi_\sigma(\mu) \geq \psi_\rho(\mu)$ $(\forall \mu > 0)$ and the exact $-3/50$ refutation of de Leon's Eq. (64). Building directly on [1, 2], this sequel resolves the remaining structural, order-theoretic, assisted-manipulation, and continuous-time Markovian open problems of QST across all dimensions $D \geq 2$ through four exact theorems backed by 20 constructive Lean 4.36.0 theorems (zero external imports, zero `sorry`, zero `Classical.choice`, zero warnings), 16 AST-audited zero-float exact-$\mathbb{Q}/\mathbb{Q}[i]$ SHA-256 certificates, and 60 adversarial tests: **(1) Quadratic-Ball Fenchel–Moreau Duality & Free-Equivalence Classification ($D \geq 2$):** extending the resolvent criterion of [2] with a new Fenchel–Moreau duality equivalence to constrained quadratic-ball dominance $F_\sigma(r) \geq F_\rho(r)$ $(\forall r \geq 0)$ and a self-contained Sion-minimax derivation, we completely classify QST free-equivalence classes via Stieltjes spectral-measure equality $(\rho \leftrightarrow \sigma \iff \mu_\rho = \mu_\sigma)$ and unify de Leon's Corollaries 7–9 in two lines. **(2) Infinite Order Dimension & No-Go Theorem ($D \geq 3$):** whereas $\dim_{\mathrm{ord}}(\mathcal{D}(\mathbb{C}^2), \to) = 2$, we prove $\dim_{\mathrm{ord}}(\mathcal{D}(\mathbb{C}^D), \to) = \infty$ for every $D \geq 3$ by constructing an explicit full-rank rational-entry family realizing the standard example S_m $(\rho_i \to \sigma_j \iff i \neq j)$ for every $m \geq 1$, proving by pigeonhole that no finite set of (even discontinuous) monotones can decide QST convertibility for $D \geq 3$. **(3) Multiplicative Tensor Convolution, Petz Rényi Mellin Monotones, & Sharp $D = 2$ vs. $D \geq 3$ Super-Activation:** we prove $R_{\rho\otimes\tau}(\mu) = (1 - s_\tau)/\mu + \sum_k \frac{w_k(\tau)}{t_k(\tau)} R_\rho(\mu/t_k(\tau))$, recover the additive Petz Rényi monotones $D_\alpha(f_1\|\rho)$ $(\alpha \in [0, 2])$ via Mellin inversion, prove sharp $D = 2$ no-activation for all $c_\tau > 0$ and $n \geq 1$, and exhibit an explicit full-rank rational-entry $D \geq 3$ witness (ρ, σ, τ) failing single-copy conversion $(R_\rho(1/4) - R_\sigma(1/4) = -40/22029 < 0)$ while simultaneously exhibiting exact sum-of-squares (SOS)-certified qubit-catalytic $(\rho\otimes\tau \to \sigma\otimes\tau$, degree-6 $P_{\mathrm{cat}}(\mu) > 0)$ and 2-copy $(\rho^{\otimes 2} \to \sigma^{\otimes 2}$, degree-4 $P_2(\mu) > 0)$ super-activation. **(4) GKSL Normal Form, Exact SOS Resolvent Dissipation, & Markovian Rank-Non-Decrease Barrier:** we characterize all QST-free Lindblad generators $\mathcal{L}(f_1) = 0$, establish the canonical gauge $(\widetilde{L}_\alpha|f_1\rangle = 0,\ \widehat{H}|f_1\rangle = 0)$, prove the exact filter semigroup $T_t = e^{tG}$ with $G + G^\dagger = -\sum_\alpha C_\alpha^\dagger C_\alpha \preceq 0$, derive the exact SOS resolvent dissipation identity $-\frac{d}{dt} R_{\rho_t}(\mu) = \sum_\alpha (\|\rho_t^{1/2} \widetilde{L}_\alpha^\dagger |w_t(\mu)\rangle\|_2^2 + \mu \|\widetilde{L}_\alpha |w_t(\mu)\rangle\|_2^2) \geq 0$, and prove the sharp Markovian rank-non-decrease barrier $\mathrm{rank}(\Lambda_t(\rho)) \geq \mathrm{rank}(\rho)$, establishing that complete reset, measure-and-prepare channels $(\det T = 0)$, and every mixed-to-pure conversion of de Leon's Corollary 9 are strictly non-Markovian.

Keywords: quantum-state texture, resolvent convertibility, order dimension, catalytic super-activation, Petz Rényi divergence, GKSL Lindbladian, Lean 4 formal verification.

1 Introduction

Quantum resource theories formalize operational limitations by specifying a set of free states and free completely positive trace-preserving (CPTP) channels [3, 4]. In bipartite entanglement [5], quantum coherence [6], and nanoscale thermodynamics [7, 8, 9], exact state-convertibility criteria expose the fine geometry of the resource preorder that individual scalar monotones cannot resolve [10]. Quantum-state texture (QST), introduced by Parisio [11] and developed rapidly across theory and experiment [12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22], is a pure fixed-point resource theory on $\mathbb{C}^D$ whose unique free state is the rank-1 textureless projector $f_1 = |f_1\rangle\langle f_1|$, $|f_1\rangle = D^{-1/2}\sum_{k=0}^{D-1}|k\rangle$, and whose free operations are all CPTP maps Λ satisfying $\Lambda(f_1) = f_1$ (isomorphic up to basis change to zero-temperature Gibbs-preserving channels [23, 24]).

In a recent breakthrough, de Leon [1] derived the exact normal form of QST-free operations in arbitrary finite dimension $D \geq 2$—splitting every free channel into a quantum filter [25] $A = \mathrm{diag}(1, T)$ ($T^\dagger T \preceq \mathbb{I}_{D-1}$) and a residual CP map Φ on the maximally textured subspace $f_1^\perp\mathbb{C}^D$—and proved that deterministic conversion $\rho \to \sigma$ is equivalent to feasibility of the filter-and-preparation semidefinite program (SDP):

$$\exists\, T \in M_{D-1}(\mathbb{C}): \qquad T^\dagger T \preceq \mathbb{I}_{D-1}, \qquad A\rho A^\dagger \preceq \sigma, \qquad A = \mathrm{diag}(1, T). \tag{1}$$

In the adapted orthonormal basis $\{|f_1\rangle, |e_1\rangle, \ldots, |e_{D-1}\rangle\}$, any density operators $\rho, \sigma \in \mathcal{D}(\mathbb{C}^D)$ partition as

$$\rho = \begin{pmatrix} p & x^\dagger \\ x & R \end{pmatrix}, \qquad \sigma = \begin{pmatrix} q & y^\dagger \\ y & S \end{pmatrix}, \tag{2}$$

where $p = \langle f_1|\rho|f_1\rangle = \Sigma(\rho)/D \in [0,1]$, $q = \langle f_1|\sigma|f_1\rangle = \Sigma(\sigma)/D \in [0,1]$, $x, y \in \mathbb{C}^{D-1}$, and $R, S \succeq 0$ satisfy $\mathrm{Tr}\, R = 1 - p$, $\mathrm{Tr}\, S = 1 - q$. Two natural scalar monotones govern the extremes: the geometric texture $\mathcal{T}_g(\rho) = 1 - p$ [12] and the texture weight $\mathcal{T}_w(\rho) = 1 - c_\rho$ [13, 15, 16], where $c_\rho := \max\{t \in [0,1] : \rho - tf_1 \succeq 0\} = p - x^\dagger R^+ x$ is the maximal free-state fraction (R^+ denotes the Moore–Penrose pseudoinverse). While $(\mathcal{T}_g, \mathcal{T}_w)$ completely decide single-qubit ($D = 2$) convertibility [16, 18, 23], de Leon exhibited a qutrit pair (Eq. (64) of [1]) showing that these two scalar monotones fail to decide convertibility in $D \geq 3$, and explicitly left two major open problems [1, Sec. VI]:

1. *State-intrinsic convertibility:* derive necessary and sufficient convertibility conditions expressed directly in terms of ρ and σ alone, without existential optimization over free channels or filters;

2. *Continuous-time Markovian dynamics:* characterize Lindbladian (GKSL) QST-free dynamical semigroups and their operational reachability relative to general QST-free channels.

Building directly on de Leon's filter-and-preparation SDP [1] and our companion scalar resolvent characterization [2] (`ARR-2026-2T3GNZ9H5K8N2V4G`), this paper establishes the new quadratic-ball Fenchel–Moreau duality and free-equivalence spectral-measure classification (Section 2), and resolves three fundamental open structural and dynamical problems of QST: infinite order dimension and finite-monotone impossibility in $D \geq 3$ (Section 3), multiplicative tensor convolution, Petz Rényi Mellin monotones, and sharp $D = 2$ vs. $D \geq 3$ catalytic and 2-copy super-activation (Section 4), and complete GKSL characterization, exact SOS resolvent dissipation, and Markovian rank-non-decrease barriers for Lindbladian QST-free dynamics (Section 5), with full machine-checked formal verification in Lean 4.36.0 and exact zero-float $\mathbb{Q}/\mathbb{Q}[i]$ certification documented in Section 6.

2 Theorem 1: Quadratic-Ball Duality and Free Equivalence

For any $\rho \in \mathcal{D}(\mathbb{C}^D)$ written in the adapted basis (2) and any spectral regularization parameter $\mu > 0$, define the *textureless resolvent* $R_\rho(\mu)$, the *regularized Schur complement* $\psi_\rho(\mu)$, the *interpolating texture family* $\mathcal{T}^{(\mu)}(\rho)$, and the *quadratic-ball envelope* $F_\rho(r)$ $(r \geq 0)$ by

$$R_\rho(\mu) := \langle f_1|(\rho + \mu \mathbb{I}_D)^{-1}|f_1\rangle, \tag{3}$$

$$\psi_\rho(\mu) := p - x^\dagger (R + \mu \mathbb{I}_{D-1})^{-1} x, \qquad \mathcal{T}^{(\mu)}(\rho) := 1 - \psi_\rho(\mu), \tag{4}$$

$$F_\rho(r) := \min_{\substack{w \in f_1^\perp \mathbb{C}^D \\ \|w\|_2 \leq r}} \langle f_1 + w|\rho|f_1 + w\rangle. \tag{5}$$

Observe that $\psi_\rho(\mu)$ is nondecreasing and concave on $(0, \infty)$ (strictly increasing and strictly concave whenever $x \neq 0$), with boundary limits

$$\lim_{\mu \to 0^+} \psi_\rho(\mu) = c_\rho = 1 - \mathcal{T}_w(\rho), \qquad \lim_{\mu \to \infty} \psi_\rho(\mu) = p = 1 - \mathcal{T}_g(\rho), \tag{6}$$

so $\mathcal{T}^{(\mu)}(\rho)$ continuously interpolates $\mathcal{T}_w(\rho) \leftarrow \mathcal{T}^{(\mu)}(\rho) \rightarrow \mathcal{T}_g(\rho)$.

Theorem 1 (Complete State-Intrinsic Resolvent Convertibility in Arbitrary Dimension $D \geq 2$)**.** *For any finite dimension $D \geq 2$ and any density operators $\rho, \sigma \in \mathcal{D}(\mathbb{C}^D)$ with adapted-basis blocks (p, x, R) and (q, y, S), the following four statements are mutually equivalent:*

(1) *$\rho \to \sigma$ under QST-free CPTP operations $\Lambda(f_1) = f_1$;*

(2) *$\forall \mu > 0,\ R_\sigma(\mu) \leq R_\rho(\mu)$;*

(3) *$\forall \mu > 0,\ \psi_\sigma(\mu) \geq \psi_\rho(\mu)$ (equivalently $\mathcal{T}^{(\mu)}(\sigma) \leq \mathcal{T}^{(\mu)}(\rho)$ for all $\mu > 0$);*

(4) *$\forall r \geq 0,\ F_\sigma(r) \geq F_\rho(r)$.*

Proof. **(2) $\iff$ (3) (Schur block inversion).** For every $\mu > 0$, both $\rho + \mu \mathbb{I}_D$ and $R + \mu \mathbb{I}_{D-1}$ are strictly positive definite. Applying the exact $1 + (D-1)$ block-matrix inverse formula to

$$\rho + \mu \mathbb{I}_D = \begin{pmatrix} p + \mu & x^\dagger \\ x & R + \mu \mathbb{I}_{D-1} \end{pmatrix}$$

yields the upper-left entry

$$R_\rho(\mu) = \frac{1}{(p + \mu) - x^\dagger (R + \mu \mathbb{I}_{D-1})^{-1} x} = \frac{1}{\psi_\rho(\mu) + \mu}. \tag{7}$$

Since $\psi_\rho(\mu) + \mu > 0$ and $\psi_\sigma(\mu) + \mu > 0$, we have $R_\sigma(\mu) \leq R_\rho(\mu) \iff \psi_\sigma(\mu) \geq \psi_\rho(\mu)$ pointwise for every $\mu > 0$.

(1) $\implies$ (4). By de Leon's criterion (1), $\rho \to \sigma$ implies there exists $T \in M_{D-1}(\mathbb{C})$ with $T^\dagger T \preceq \mathbb{I}_{D-1}$ (so $\|T^\dagger\| = \|T\| \leq 1$) and $A \rho A^\dagger \preceq \sigma$ for $A = \mathrm{diag}(1, T)$. For any $r \geq 0$ and any $w \in f_1^\perp \mathbb{C}^D$ with $\|w\|_2 \leq r$, set $u := T^\dagger w \in f_1^\perp \mathbb{C}^D$. Then $\|u\|_2 \leq \|w\|_2 \leq r$ and $A^\dagger |f_1 + w\rangle = |f_1 + u\rangle$, hence

$$\langle f_1 + w|\sigma|f_1 + w\rangle \geq \langle f_1 + w|A\rho A^\dagger|f_1 + w\rangle = \langle f_1 + u|\rho|f_1 + u\rangle \geq F_\rho(r).$$

Minimizing over $\|w\|_2 \leq r$ yields $F_\sigma(r) \geq F_\rho(r)$.

(4) $\iff$ (3) (Fenchel–Moreau duality). Define $g_\rho(s) := F_\rho(\sqrt{s}) = \min_{\|w\|_2^2 \leq s}(p + 2\,\mathrm{Re}\langle x|w\rangle + \langle w|R|w\rangle)$ for $s \geq 0$, and extend $h_\rho(s) := g_\rho(s)$ for $s \geq 0$, $h_\rho(s) := +\infty$ for $s < 0$. Because $(s, w) \mapsto p + 2\,\mathrm{Re}\langle x|w\rangle + \langle w|R|w\rangle$ is convex quadratic on the closed convex epigraph $\{(s, w) : \|w\|_2^2 \leq s\}$ with compact level sections, $h_\rho : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ is proper, closed, convex,

and nonincreasing on $[0,\infty)$, with $0 \le g_\rho(s) \le p$. Its Fenchel conjugate $h_\rho^*(t) = \sup_{s\ge 0}(ts - g_\rho(s))$ equals $+\infty$ for all $t > 0$, while for every $\mu > 0$,

$$\begin{aligned} -h_\rho^*(-\mu) &= \inf_{s\ge 0}\big(g_\rho(s) + \mu s\big) = \inf_{w\in\mathbb{C}^{D-1}}\Big(p + 2\,\mathrm{Re}\langle x|w\rangle + \langle w|(R+\mu\mathbb{I}_{D-1})|w\rangle\Big) \\ &= p - x^\dagger (R+\mu\mathbb{I}_{D-1})^{-1}x = \psi_\rho(\mu), \end{aligned}$$

attained uniquely at $w_\rho(\mu) = -(R+\mu\mathbb{I}_{D-1})^{-1}x$ (completing the square in w). By continuity at $t = 0$, $-h_\rho^*(0) = \inf_{s\ge 0} g_\rho(s) = \lim_{\mu\to 0^+}\psi_\rho(\mu) = c_\rho$. By the Fenchel–Moreau biconjugation theorem ($h_\rho = h_\rho^{**}$), for every $s \ge 0$,

$$g_\rho(s) = \sup_{t\in\mathbb{R}}\big(ts - h_\rho^*(t)\big) = \sup_{\mu>0}\big(\psi_\rho(\mu) - \mu s\big).$$

Consequently, $g_\sigma(s) \ge g_\rho(s)$ for all $s \ge 0$ holds if and only if $\psi_\sigma(\mu) \ge \psi_\rho(\mu)$ for all $\mu > 0$.

(3) $\Longrightarrow$ (1) (Sion minimax, Slater strong duality, regularization, and simultaneous congruence diagonalization). Assume $\psi_\sigma(\mu) \ge \psi_\rho(\mu)$ for all $\mu > 0$; taking $\mu \to \infty$ gives $q \ge p$. Let $\mathcal{B} := \{T \in M_{D-1}(\mathbb{C}) : TT^\dagger \preceq \mathbb{I}_{D-1}\}$ (equivalently $T^\dagger T \preceq \mathbb{I}_{D-1}$) and $\mathcal{S} := \{Z \in M_D(\mathbb{C}) : Z \succeq 0,\ \mathrm{Tr}\, Z = 1\}$, both compact and convex. With $A(T) := \mathrm{diag}(1, T)$, define $K(T,Z) := \mathrm{Tr}[Z(A(T)\rho A(T)^\dagger - \sigma)]$. For fixed $Z \in \mathcal{S}$, $T \mapsto \mathrm{Tr}[ZA(T)\rho A(T)^\dagger] = \|Z^{1/2}A(T)\rho^{1/2}\|_F^2$ is continuous and convex; for fixed $T \in \mathcal{B}$, $Z \mapsto K(T,Z)$ is continuous and affine (hence concave). By Sion's minimax theorem,

$$\min_{T\in\mathcal{B}}\max_{Z\in\mathcal{S}} K(T,Z) = \max_{Z\in\mathcal{S}}\min_{T\in\mathcal{B}} K(T,Z).$$

Therefore (1) is feasible if and only if every $Z \succeq 0$ satisfies

$$\min_{T\in\mathcal{B}} \mathrm{Tr}\big[ZA(T)\rho A(T)^\dagger\big] \le \mathrm{Tr}(Z\sigma). \tag{8}$$

Fix any $Z = \begin{pmatrix} \alpha & v^\dagger \\ v & W \end{pmatrix} \succeq 0$ ($\alpha \ge 0$, $v \in \mathbb{C}^{D-1}$, $W \succeq 0$). Expanding the trace gives

$$\mathrm{Tr}\big[ZA(T)\rho A(T)^\dagger\big] = \alpha p + 2\,\mathrm{Re}\langle v|Tx\rangle + \mathrm{Tr}(WTRT^\dagger), \qquad \mathrm{Tr}(Z\sigma) = \alpha q + 2\,\mathrm{Re}\langle v|y\rangle + \mathrm{Tr}(WS).$$

Consider the matrix-convex program $\min_{G(T)\preceq 0} f(T)$ where $f(T) := 2\,\mathrm{Re}\langle v|Tx\rangle + \mathrm{Tr}(WTRT^\dagger)$ and $G(T) := TT^\dagger - \mathbb{I}_{D-1}$. Since $G(tT_1 + (1-t)T_2) - tG(T_1) - (1-t)G(T_2) = -t(1-t)(T_1 - T_2)(T_1-T_2)^\dagger \preceq 0$ for $t \in [0,1]$, G is $\mathbb{S}_+^{D-1}$-convex and satisfies Slater's strict feasibility condition $G(0) = -\mathbb{I}_{D-1} \prec 0$. By Slater strong duality over the self-dual PSD cone $\mathbb{S}_+^{D-1}$,

$$\min_{T\in\mathcal{B}} f(T) = \sup_{V\succeq 0}\left\{-\mathrm{Tr}\, V + \inf_{T\in M_{D-1}(\mathbb{C})}\Big(\mathrm{Tr}(VTT^\dagger) + \mathrm{Tr}(WTRT^\dagger) + 2\,\mathrm{Re}\langle v|Tx\rangle\Big)\right\}. \tag{9}$$

Fix any $V \succeq 0$ and any $\varepsilon > 0$, and regularize $V_\varepsilon := V + \varepsilon\mathbb{I}_{D-1} \succ 0$ and $W_\varepsilon := W + \varepsilon\mathbb{I}_{D-1} \succ 0$. Because $V_\varepsilon \succ 0$ and $W_\varepsilon \succ 0$, simultaneous congruence diagonalization yields an invertible matrix $P \in \mathrm{GL}_{D-1}(\mathbb{C})$ and a strictly positive diagonal matrix $\Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_{D-1}) \succ 0$ such that

$$V_\varepsilon = P^\dagger P, \qquad W_\varepsilon = P^\dagger \Lambda P. \tag{10}$$

Apply the invertible substitution $\widetilde{T} := PT \in M_{D-1}(\mathbb{C})$ with rows $\widetilde{t}_k^\dagger$ ($k = 1, \dots, D-1$), and set $\widetilde{v} := (P^\dagger)^{-1}v$. Then $\mathrm{Tr}(V_\varepsilon TT^\dagger) = \sum_{k=1}^{D-1}\|\widetilde{t}_k\|_2^2$, $\mathrm{Tr}(W_\varepsilon TRT^\dagger) = \sum_{k=1}^{D-1}\lambda_k \widetilde{t}_k^\dagger R\widetilde{t}_k$, and $2\,\mathrm{Re}\langle v|Tx\rangle = \sum_{k=1}^{D-1} 2\,\mathrm{Re}(\widetilde{v}_k^*\widetilde{t}_k^\dagger x)$. The unconstrained infimum decouples completely across the $D-1$ rows: setting $\mu_k := \lambda_k^{-1} > 0$,

$$\begin{aligned} \inf_{T\in M_{D-1}(\mathbb{C})}\Big(\mathrm{Tr}(V_\varepsilon TT^\dagger) + \mathrm{Tr}(W_\varepsilon TRT^\dagger) + 2\,\mathrm{Re}\langle v|Tx\rangle\Big) &= -\sum_{k=1}^{D-1}|\widetilde{v}_k|^2 x^\dagger(\lambda_k R + \mathbb{I}_{D-1})^{-1}x \\ &= \sum_{k=1}^{D-1}\frac{|\widetilde{v}_k|^2}{\lambda_k}\big(\psi_\rho(\mu_k) - p\big). \end{aligned} \tag{11}$$

By hypothesis, $\psi_\rho(\mu_k) - p \le \psi_\sigma(\mu_k) - p = (q-p) - y^\dagger(S+\mu_k \mathbb{I}_{D-1})^{-1} y$. Moreover, because $Z \succeq 0$ and $W_\varepsilon \succ W$, Schur-complement positivity gives

$$\sum_{k=1}^{D-1} \frac{|\widetilde{v}_k|^2}{\lambda_k} = \widetilde{v}^\dagger \Lambda^{-1} \widetilde{v} = v^\dagger (P^\dagger \Lambda P)^{-1} v = v^\dagger W_\varepsilon^{-1} v \le \alpha.$$

Since $q - p \ge 0$, combining this with (11) produces

$$\alpha p + \sum_{k=1}^{D-1} \frac{|\widetilde{v}_k|^2}{\lambda_k} \big(\psi_\rho(\mu_k) - p\big) \le \alpha q - \sum_{k=1}^{D-1} |\widetilde{v}_k|^2 y^\dagger (\lambda_k S + \mathbb{I}_{D-1})^{-1} y.$$

Now note that each summand satisfies the quadratic infimum identity

$$-|\widetilde{v}_k|^2 y^\dagger (\lambda_k S + \mathbb{I}_{D-1})^{-1} y = \inf_{u_k \in \mathbb{C}^{D-1}} \Big(u_k^\dagger (\lambda_k S + \mathbb{I}_{D-1}) u_k + 2\,\mathrm{Re}(\widetilde{v}_k^* u_k^\dagger y) \Big).$$

Evaluating this infimum at $u_k = p_k$, where $p_k^\dagger$ is the k-th row of P, yields

$$\begin{aligned} -\sum_{k=1}^{D-1} |\widetilde{v}_k|^2 y^\dagger (\lambda_k S + \mathbb{I}_{D-1})^{-1} y &\le \sum_{k=1}^{D-1} \Big(p_k^\dagger \lambda_k S p_k + p_k^\dagger p_k + 2\,\mathrm{Re}(\widetilde{v}_k^* p_k^\dagger y) \Big) \\ &= \mathrm{Tr}(P^\dagger \Lambda P S) + \mathrm{Tr}(P^\dagger P) + 2\,\mathrm{Re}(\widetilde{v}^\dagger P y) \\ &= \mathrm{Tr}(W_\varepsilon S) + \mathrm{Tr}(V_\varepsilon) + 2\,\mathrm{Re}\langle v | y \rangle. \end{aligned}$$

Subtracting $\mathrm{Tr}\, V_\varepsilon$ cancels $\mathrm{Tr}\, V_\varepsilon$ identically! Because $V \preceq V_\varepsilon$ and $W \preceq W_\varepsilon$ imply $\mathrm{Tr}(VTT^\dagger) \le \mathrm{Tr}(V_\varepsilon TT^\dagger)$ and $\mathrm{Tr}(WTRT^\dagger) \le \mathrm{Tr}(W_\varepsilon TRT^\dagger)$ for every T, while $-\mathrm{Tr}\, V = -\mathrm{Tr}\, V_\varepsilon + \varepsilon(D-1)$, we obtain

$$\begin{aligned} \alpha p - \mathrm{Tr}\, V + \inf_T \Big(\mathrm{Tr}(VTT^\dagger) + \mathrm{Tr}(WTRT^\dagger) + 2\,\mathrm{Re}\langle v | Tx \rangle \Big) \\ \le \alpha q + 2\,\mathrm{Re}\langle v | y \rangle + \mathrm{Tr}(WS) + \varepsilon\big(D - 1 + \mathrm{Tr}\, S\big). \end{aligned}$$

Letting $\varepsilon \to 0^+$ and taking $\sup_{V \succeq 0}$ in (9) establishes (8), completing the proof. □

Theorem 1 immediately resolves the equivalence-class structure of QST and unifies every previously known analytical criterion in two lines.

Corollary 1 (Free Equivalence, Two-Line Unification of $D = 2$ and Corollaries 7–9, and Eq. (64) Refutation)**.** *Let $\mu_\rho := \sum_j |\langle f_1 | \psi_j \rangle|^2 \delta_{\lambda_j}$ denote the spectral probability measure of $\rho = \sum_j \lambda_j |\psi_j\rangle\langle\psi_j|$ at $|f_1\rangle$ (so $R_\rho(\mu) = \int_0^1 (\lambda + \mu)^{-1}\, d\mu_\rho(\lambda)$). Then:*

(a) Exact Free-Equivalence Classes: *$\rho \leftrightarrow \sigma$ ($\rho \to \sigma$ and $\sigma \to \rho$) if and only if $R_\rho(\mu) = R_\sigma(\mu)$ for all $\mu > 0$, equivalently $\mu_\rho = \mu_\sigma$ (uniqueness of the Stieltjes transform). In particular, non-unitarily-equivalent states with identical μ_ρ (e.g. block-diagonal states with equal p and distinct R-spectra) are freely interconvertible.*

(b) Qubit Collapse ($D = 2$): *For $D = 2$, $|x|^2 = (1-p)(p - c_\rho)$ gives $\psi_\rho(\mu) = \frac{p\mu + (1-p)c_\rho}{1-p+\mu}$, whose partial derivatives $\partial_p \psi = \frac{\mu(1 - c_\rho + \mu)}{(1-p+\mu)^2} > 0$ and $\partial_{c_\rho} \psi = \frac{1-p}{1-p+\mu} \ge 0$ are nonnegative; hence $\rho \to \sigma \iff (q \ge p \ \wedge\ c_\sigma \ge c_\rho)$.*

(c) Unification of de Leon's Corollaries 7–9: (Cor. 7, maximally textured) *$p = 0 \implies \psi_\rho \equiv 0 \le \psi_\sigma$;* (Cor. 8, pure source $\rho = |\psi\rangle\langle\psi|$) *$\psi_\rho(\mu) = \frac{p\mu}{1-p+\mu}$, whereas any σ with $y = S^{1/2} u$ ($\|u\|_2^2 \le q$) and $S \preceq (1-q)\mathbb{I}$ obeys $y^\dagger (S + \mu \mathbb{I})^{-1} y \le \frac{q(1-q)}{1-q+\mu}$, so $\psi_\sigma(\mu) \ge \frac{q\mu}{1-q+\mu} \ge \psi_\rho(\mu)$ iff $q \ge p$;* (Cor. 9, pure target $\sigma = |\phi\rangle\langle\phi|$) *$R_\sigma(\mu) = \frac{1-q}{\mu} + \frac{q}{1+\mu}$ and $R_\rho(\mu) = \frac{1-s}{\mu} + \sum_{\lambda_j > 0} \frac{w_j}{\lambda_j + \mu}$ with $s = \langle f_1 | P_\rho | f_1 \rangle$, so $\lim_{\mu \to 0^+} \mu R_\rho(\mu) \ge \lim_{\mu \to 0^+} \mu R_\sigma(\mu)$ forces $s \le q$, and conversely $s \le q$ gives $R_\rho(\mu) \ge \frac{1-s}{\mu} + \frac{s}{1+\mu} = R_\sigma(\mu) + (q-s)(\mu^{-1} - (1+\mu)^{-1}) \ge R_\sigma(\mu)$.*

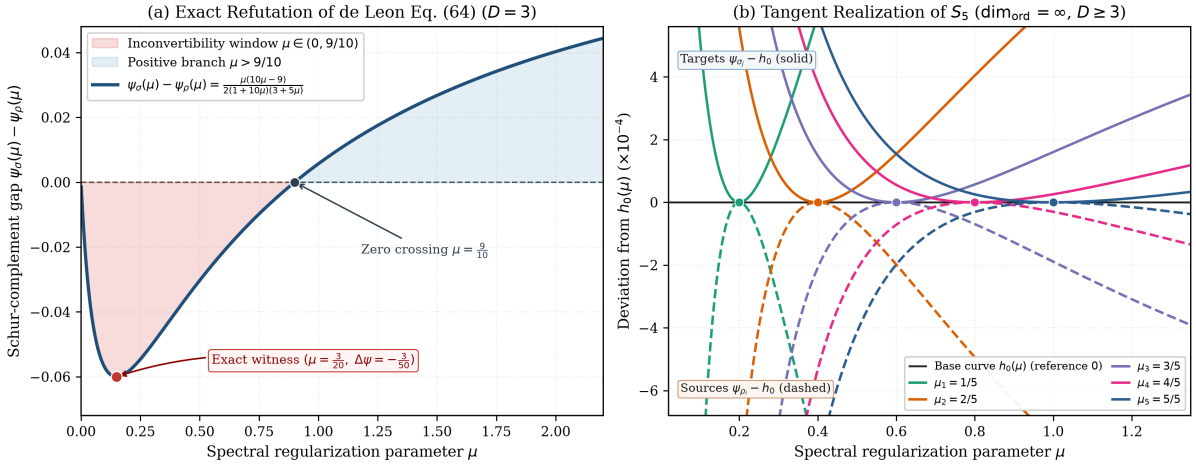


Figure 1: **(a)** Exact closed-form state-intrinsic refutation of de Leon's Eq. (64) [1]: although $q > p$ and $c_\rho = c_\sigma = 1/10$, the Schur-complement gap $\psi_\sigma(\mu) - \psi_\rho(\mu)$ is strictly negative throughout $\mu \in (0, 9/10)$, with exact rational witness $\Delta\psi(3/20) = -3/50$. **(b)** Constructive tangent realization of the standard example S_5 in $D = 3$ (Theorem 2): target curves $\psi_{\sigma_j} - h_0$ (solid, above) and source curves $\psi_{\rho_i} - h_0$ (dashed, below) meet h_0 tangentially at distinct points $\mu_k = k/5$, so an infinitesimal downward shift $-\varepsilon$ breaks exclusively the diagonal conversions $\rho_i \not\to \sigma_i$ while preserving all off-diagonal conversions $\rho_i \to \sigma_j$ $(i \neq j)$.

(d) Exact Closed-Form Refutation of de Leon Eq. (64): *For de Leon's qutrit pair [1, Eq. (64)] with* $p = 3/10$, $x = (1/(5\sqrt{2}),\, 0)^T$, $R = \mathrm{diag}(1/10, 6/10)$ *and* $q = 4/10$, $y = (3/(5\sqrt{2}),\, 0)^T$, $S = \mathrm{diag}(6/10, 0)$, *we have* $q > p$ *and* $c_\rho = c_\sigma = 1/10$, *yet*

$$\psi_\sigma(\mu) - \psi_\rho(\mu) = \frac{\mu(10\mu - 9)}{2(1 + 10\mu)(3 + 5\mu)} < 0 \qquad \forall \mu \in \left(0,\, \tfrac{9}{10}\right), \tag{12}$$

yielding exact rational inconvertibility certificates $\psi_\sigma(3/20) - \psi_\rho(3/20) = -3/50 < 0$ *and* $R_\rho(1/5) - R_\sigma(1/5) = 30/13 - 8/3 = -14/39 < 0$ (*Figure 1(a)*).

3 Theorem 2: Infinite Order Dimension and No-Go Theorem for Finite Monotone Sets in $D \geq 3$

Recall that the *order dimension* $\dim_{\mathrm{ord}}(P, \preceq)$ of a preorder (or its antisymmetric quotient poset) is the minimum cardinality of a family of linear orders whose intersection equals $\preceq$ (and any finite complete set of N real-valued monotones $\{M_1, \ldots, M_N\}$ satisfying $\rho \to \sigma \iff \forall k \in \{1, \ldots, N\}$, $M_k(\sigma) \leq M_k(\rho)$ implies $\dim_{\mathrm{ord}}(P, \preceq) \leq N$). The *standard example* S_m is the bipartite poset on $2m$ elements $\{\rho_1, \ldots, \rho_m, \sigma_1, \ldots, \sigma_m\}$ with $\rho_i \to \sigma_j \iff i \neq j$; it is the canonical witness of order dimension m.

Theorem 2 (Sharp Dimensional Jump: $\dim_{\mathrm{ord}} = 2$ for $D = 2$ vs. $\dim_{\mathrm{ord}} = \infty$ for $D \geq 3$)**.** *The QST convertibility preorder* $(\mathcal{D}(\mathbb{C}^D), \to)$ *satisfies:*

(1) *For* $D = 2$, $\dim_{\mathrm{ord}}(\mathcal{D}(\mathbb{C}^2), \to) = 2$, *witnessed completely by* $(\mathcal{T}_g, \mathcal{T}_w)$.

(2) *For every* $D \geq 3$ *and every integer* $m \geq 1$, *there exists an explicit family of full-rank density operators* $\rho_1, \ldots, \rho_m, \sigma_1, \ldots, \sigma_m \in \mathcal{D}(\mathbb{C}^D)$ *with rational entries in the adapted basis realizing the standard example* S_m*:*

$$\rho_i \to \sigma_j \quad \Longleftrightarrow \quad i \neq j \qquad (1 \leq i, j \leq m),$$

while $\sigma_j \not\to \rho_i$ for all i, j, and all ρ_i (resp. σ_j) are pairwise incomparable. Consequently, $\dim_{\mathrm{ord}}(\mathcal{D}(\mathbb{C}^D), \to) = \infty$ *for every* $D \geq 3$.

Proof. Part (1) follows immediately from Corollary 1(b) together with any pair of incomparable mixed qubit states (e.g. $\rho = \begin{pmatrix} 1/2 & 2/5 \\ 2/5 & 1/2 \end{pmatrix}$ and $\sigma = \begin{pmatrix} 2/5 & 0 \\ 0 & 3/5 \end{pmatrix}$).

For Part (2), fix any $D \geq 3$ and $m \geq 1$. First observe the *exact single-pole tangent factorization*: for any $m_0 > 0$, $0 < r < s$, and $\mu_0 > 0$, setting $q - p = m_0(s - r)$, $|x|^2 = m_0(r + \mu_0)^2$, and $|y|^2 = m_0(s + \mu_0)^2$ gives identically

$$\left(q - \frac{|y|^2}{s+\mu}\right) - \left(p - \frac{|x|^2}{r+\mu}\right) = \frac{m_0(s-r)(\mu-\mu_0)^2}{(r+\mu)(s+\mu)} \geq 0, \tag{13}$$

with unique zero at $\mu = \mu_0$ (proved by clearing denominators: $(s-r)(r+\mu)(s+\mu) + (r+\mu_0)^2(s+\mu) - (s+\mu_0)^2(r+\mu) = (s-r)(\mu-\mu_0)^2$).

Now fix the base curve $h_0(\mu) := p_0 - \frac{a_0}{r_0+\mu}$ with rational parameters

$$r = \frac{1}{10}, \qquad r_0 = \frac{1}{5}, \qquad s = \frac{3}{10}, \qquad p_0 = \frac{1}{2}, \qquad a_0 = \frac{1}{100} \ \ (\sqrt{a_0} = \tfrac{1}{10}),$$

choose m distinct rational points $\mu_k := k/m \in (0, 1]$ $(k = 1, \ldots, m)$, and set $m_k := a_0/(r_0+\mu_k)^2$ and $\varepsilon := a_0/(4000m^2) > 0$. Define the $2m$ full-rank rational-entry states in $\mathcal{D}(\mathbb{C}^D)$ in the adapted basis by

$$\rho_i := \begin{pmatrix} p_i & x_i \\ x_i & r \end{pmatrix} \oplus \frac{1 - p_i - r}{D-2}\mathbb{I}_{D-2}, \qquad p_i := p_0 - m_i(r_0 - r), \quad x_i := \sqrt{a_0}\frac{r+\mu_i}{r_0+\mu_i}, \tag{14}$$

$$\sigma_j := \begin{pmatrix} q_j & y_j \\ y_j & s \end{pmatrix} \oplus \frac{1 - q_j - s}{D-2}\mathbb{I}_{D-2}, \qquad q_j := p_0 + m_j(s - r_0) - \varepsilon, \quad y_j := \sqrt{a_0}\frac{s+\mu_j}{r_0+\mu_j}. \tag{15}$$

Full-rank positivity and unit trace: Since $r_0 + \mu_k > 1/5$, we have $m_k < 25a_0 = 1/4$, so $p_i \in (19/40, 1/2)$, $1 - p_i - r > 2/5 > 0$, $q_j + s < 33/40 < 1$ (hence $1 - q_j - s > 7/40 > 0$). Moreover, applying (13) at $\mu = 0$ gives

$$\psi_{\rho_i}(0) = h_0(0) - \frac{a_0(r_0 - r)\mu_i^2}{r r_0 (r_0+\mu_i)^2} > p_0 - \frac{a_0}{r} = \frac{2}{5} > 0,$$
$$\psi_{\sigma_j}(0) = h_0(0) + \frac{a_0(s - r_0)\mu_j^2}{s r_0 (r_0+\mu_j)^2} - \varepsilon > p_0 - \frac{a_0}{r_0} - \varepsilon > \frac{4}{9} > 0,$$

so $p_i r - x_i^2 = r\psi_{\rho_i}(0) > 0$ and $q_j s - y_j^2 = s\psi_{\sigma_j}(0) > 0$. Thus every $\rho_i, \sigma_j \succ 0$ has unit trace and exact rational entries (the third diagonal block absorbs the trace, which is precisely what is impossible in $D = 2$).

Exact Schur-complement gap: Applying (13) twice—once between $\psi_{\sigma_j} + \varepsilon$ and h_0, and once between h_0 and ψ_{ρ_i}—yields, for every $i, j \in \{1, \ldots, m\}$ and every $\mu > 0$,

$$\psi_{\sigma_j}(\mu) - \psi_{\rho_i}(\mu) = \Delta_{ij}(\mu) - \varepsilon,$$
$$\Delta_{ij}(\mu) := \frac{a_0}{10(r_0+\mu)}\left[\frac{(\mu-\mu_j)^2}{(r_0+\mu_j)^2(s+\mu)} + \frac{(\mu-\mu_i)^2}{(r_0+\mu_i)^2(r+\mu)}\right]. \tag{16}$$

- **Diagonal failure ($i = j$):** At $\mu = \mu_i > 0$, both squares vanish, so $\psi_{\sigma_i}(\mu_i) - \psi_{\rho_i}(\mu_i) = -\varepsilon < 0$. By Theorem 1, $\rho_i \not\to \sigma_i$.
- **Off-diagonal uniform lower bound ($i \neq j$):** We prove that for every $\mu > 0$,

$$\Delta_{ij}(\mu) \geq \frac{a_0}{2000}\min\big((\mu_i - \mu_j)^2, 8\big). \tag{17}$$

Indeed, for $\mu_k \in (0,1]$ we have $(r_0+\mu_k)^2 \le (6/5)^2 = 36/25$. *Case 1 ($0 < \mu \le 2$):* Here $(r_0+\mu)(s+\mu) \le (11/5)(23/10) = 253/50$ and $(r_0+\mu)(r+\mu) \le (11/5)(21/10) = 231/50$, so both denominators in (16) are bounded above by $(36/25)(253/50) = 9108/1250 < 10$. Using the parallelogram identity $(\mu-\mu_i)^2 + (\mu-\mu_j)^2 \ge \frac{1}{2}(\mu_i-\mu_j)^2$, we obtain

$$\Delta_{ij}(\mu) \ge \frac{a_0}{10\cdot 10}\cdot\frac{(\mu_i-\mu_j)^2}{2} = \frac{a_0}{200}(\mu_i-\mu_j)^2 > \frac{a_0}{2000}(\mu_i-\mu_j)^2.$$

Case 2 ($\mu > 2$): Here $\mu-\mu_k \ge \mu-1 > \mu/2$, so $(\mu-\mu_k)^2 > \mu^2/4$, while $r_0+\mu < \frac{6}{5}\mu$ and $s+\mu < \frac{6}{5}\mu$ (in particular $(r_0+\mu)(s+\mu) < \frac{36}{25}\mu^2$). Hence each of the two positive terms in (16) is at least $\frac{a_0}{10}\cdot\frac{\mu^2/4}{(36/25)(36/25)\mu^2} > \frac{a_0}{100}$, so $\Delta_{ij}(\mu) > \frac{a_0}{50} > \frac{8a_0}{2000}$. Since $|\mu_i-\mu_j| \ge 1/m$ for $i \ne j$, (17) yields $\Delta_{ij}(\mu) \ge \frac{a_0}{2000m^2} = 2\varepsilon > \varepsilon$ for all $\mu > 0$, whence $\psi_{\sigma_j}(\mu) > \psi_{\rho_i}(\mu)$ for all $\mu > 0$ and therefore $\rho_i \to \sigma_j$ by Theorem 1.

Finally, $q_j > p_0 - \varepsilon > p_i$ rules out $\sigma_j \to \rho_i$ for all i,j, while p_i increases and $\psi_{\rho_i}(0)$ decreases strictly with i (and conversely for σ_j), so all sources (resp. targets) are pairwise incomparable (Figure 1(b)). □

Corollary 2 (No-Go Theorem for Finite Complete Monotone Sets in $D \ge 3$)**.** *For every $D \ge 3$, there exists no finite family $\{M_1, \dots, M_N\}$ of (even discontinuous) real-valued QST monotones such that $\rho \to \sigma \iff \forall k \in \{1,\dots,N\},\ M_k(\sigma) \le M_k(\rho)$.*

Proof. Suppose such a finite set $\{M_1,\dots,M_N\}$ existed, and take the family S_m of Theorem 2 with $m = N+1$. For each $i \in \{1,\dots,m\}$, since $\rho_i \not\to \sigma_i$, there exists an index $k(i) \in \{1,\dots,N\}$ with $M_{k(i)}(\rho_i) < M_{k(i)}(\sigma_i)$. By the pigeonhole principle, there exist distinct $i \ne j$ with $k(i) = k(j) = k$. Since $\rho_i \to \sigma_j$ and $\rho_j \to \sigma_i$, monotonicity of M_k gives

$$M_k(\sigma_j) \le M_k(\rho_i) < M_k(\sigma_i) \le M_k(\rho_j) < M_k(\sigma_j),$$

a contradiction. □

4 Theorem 3: Multiplicative Tensor Convolution, Petz Rényi Mellin Monotones, and Catalytic / 2-Copy Super-Activation

Composite QST systems $\mathbb{C}^{D_1} \otimes \mathbb{C}^{D_2}$ have tensor-product free state $f_1^{(D_1D_2)} = f_1^{(D_1)} \otimes f_1^{(D_2)}$. We say that $\rho \to \sigma$ exhibits *catalytic super-activation* if $\rho \not\to \sigma$ while $\rho\otimes\tau \to \sigma\otimes\tau$ for some finite-dimensional catalyst $\tau \in \mathcal{D}(\mathbb{C}^{d_C})$, and *$n$-copy super-activation* if $\rho \not\to \sigma$ while $\rho^{\otimes n} \to \sigma^{\otimes n}$.

Theorem 3 (Tensor Convolution, Mellin Monotones, and Sharp $D=2$ vs. $D \ge 3$ Super-Activation)**.**

(1) Multiplicative Dilation Convolution: *For any $\rho \in \mathcal{D}(\mathbb{C}^{D_1})$ and $\tau = \sum_k t_k |\phi_k\rangle\langle\phi_k| \in \mathcal{D}(\mathbb{C}^{D_2})$ with spectral weights $w_k(\tau) := |\langle f_1|\phi_k\rangle|^2$ and support overlap $s_\tau := \sum_{t_k>0} w_k(\tau)$, every $\mu > 0$ satisfies*

$$R_{\rho\otimes\tau}(\mu) = \frac{1-s_\tau}{\mu} + \sum_{k:\,t_k>0} \frac{w_k(\tau)}{t_k} R_\rho\!\left(\frac{\mu}{t_k}\right). \tag{18}$$

(2) Petz Rényi Mellin Monotones: *For every $\alpha \in [0,2]$, the Petz Rényi divergence $D_\alpha(f_1\|\rho) := \frac{1}{\alpha-1}\log\langle f_1|\rho^{1-\alpha}|f_1\rangle$ (with $\alpha \in \{0,1,2\}$ by continuity) is an exact additive QST monotone recovered from $R_\rho(\mu)$ via Mellin transforms:*

$$\langle f_1|\rho^{1-\alpha}|f_1\rangle = \frac{\sin(\pi\alpha)}{\pi}\int_0^\infty \mu^{-\alpha}\big(1-\mu R_\rho(\mu)\big)\,d\mu \qquad (\alpha \in (0,1)), \tag{19}$$

$$\langle f_1|\rho^{1-\alpha}|f_1\rangle = \frac{\sin(\pi(\alpha-1))}{\pi}\int_0^\infty \mu^{1-\alpha} R_\rho(\mu)\,d\mu \qquad (\alpha \in (1,2),\ s_\rho = 1), \tag{20}$$

Table 1: Exact primitive integer coefficients and sum-of-squares (SOS) positivity discriminants of the catalytic ($P_{\mathrm{cat}}(\mu) = \sum_{k=0}^{6} d_k \mu^k$) and 2-copy ($P_2(\mu) = \sum_{k=0}^{4} c_k \mu^k$) resolvent numerators for the $D \geq 3$ witness (21), verified by exact 6×6 and 9×9 rational matrix inversions and Lean 4 theorems `thm13`–`thm14`.

Qubit-Catalytic $P_{\mathrm{cat}}(\mu)$ **(deg 6)**		**2-Copy** $P_2(\mu)$ **(deg 4)**	
Coeff.	Exact Integer Value	Coeff.	Exact Integer Value
d_6	1 267 200 000 000 000 000	c_4	810 501 120 000 000 000
d_5	−233 637 248 000 000 000	c_3	−178 111 501 696 000 000
d_4	37 216 712 000 000 000	c_2	15 904 231 279 840 000
d_3	6 619 236 940 800 000	c_1	75 058 893 958 400
d_2	161 688 512 880 000	c_0	874 305 522 519
d_1	925 789 662 000		
d_0	55 045 230 669		
$4d_6d_4 - d_5^2 = 134\,057\,706\,132\,586\,496 \times 10^{18} > 0$		$4c_4c_2 - c_3^2 = 19\,837\,882\,023\,793\,202\,806\,784 \times 10^{12} > 0$	

with $D_\alpha(f_1\|\rho) = +\infty$ *for* $\alpha \in [1,2]$ *whenever* $s_\rho < 1$ *(both sides of* (20) *diverging), and* $D_0(f_1\|\rho) = -\log(1 - \mathcal{T}_g(\rho))$, $D_1(f_1\|\rho) = -\langle f_1|\log\rho|f_1\rangle$, $D_2(f_1\|\rho) = -\log(1 - \mathcal{T}_w(\rho))$.

(3) Sharp $D = 2$ No-Activation: *For any qubit states* $\rho, \sigma \in \mathcal{D}(\mathbb{C}^2)$, *any integer* $n \geq 1$, *and any catalyst* τ *with* $c_\tau > 0$ (*more generally* $p_\tau > 0$),

$$\rho \to \sigma \quad \Longleftrightarrow \quad \rho^{\otimes n} \to \sigma^{\otimes n} \quad \Longleftrightarrow \quad \rho \otimes \tau \to \sigma \otimes \tau.$$

(4) Explicit Full-Rank Rational $D \geq 3$ Witness of Simultaneous Qubit-Catalytic and 2-Copy Super-Activation: *For every* $D \geq 3$, *define the full-rank rational-entry states in the adapted basis:*

$$\rho := \begin{pmatrix} 3/8 & 1/5 \\ 1/5 & 3/20 \end{pmatrix} \oplus \frac{19}{40(D-2)}\mathbb{I}_{D-2}, \qquad \sigma := \begin{pmatrix} 849/2000 & 3/10 \\ 3/10 & 7/20 \end{pmatrix} \oplus \frac{451}{2000(D-2)}\mathbb{I}_{D-2},$$
$$\tau := \begin{pmatrix} 1/2 & 2/5 \\ 2/5 & 1/2 \end{pmatrix} \in \mathcal{D}(\mathbb{C}^2). \tag{21}$$

Then $\rho, \sigma \succ 0$ *satisfy* $q = 849/2000 > 3/8 = p$ *and* $c_\sigma = 2343/14000 > 13/120 = c_\rho$, *yet:*

(i) Single-copy fails: $R_\rho(1/4) - R_\sigma(1/4) = 40/21 - 2000/1049 = -40/22029 < 0$, *so* $\rho \not\to \sigma$;

(ii) Qubit-catalytic super-activation: $\rho \otimes \tau \to \sigma \otimes \tau$, *certified for all* $\mu > 0$ *by the exact positive-denominator factorization* $R_{\rho\otimes\tau}(\mu) - R_{\sigma\otimes\tau}(\mu) = P_{\mathrm{cat}}(\mu)/Q_{\mathrm{cat}}(\mu)$ *with primitive degree-6 numerator* $P_{\mathrm{cat}}(\mu) = \sum_{k=0}^{6} d_k\mu^k > 0$ (*Table 1*);

(iii) 2-copy super-activation: $\rho^{\otimes 2} \to \sigma^{\otimes 2}$, *certified for all* $\mu > 0$ *by* $R_{\rho^{\otimes 2}}(\mu) - R_{\sigma^{\otimes 2}}(\mu) = P_2(\mu)/Q_2(\mu)$ *with primitive degree-4 numerator* $P_2(\mu) = \sum_{k=0}^{4} c_k\mu^k > 0$ (*Table 1*).

Proof. **(1)** Using the spectral resolution $\tau = \sum_k t_k|\phi_k\rangle\langle\phi_k|$, the operator $\rho \otimes \tau + \mu\mathbb{I}$ is block-diagonal across $\{|\phi_k\rangle\}$, with blocks $t_k\rho + \mu\mathbb{I}_{D_1}$. Hence

$$R_{\rho\otimes\tau}(\mu) = \sum_k w_k(\tau)\langle f_1|(t_k\rho + \mu\mathbb{I}_{D_1})^{-1}|f_1\rangle = \frac{1 - s_\tau}{\mu} + \sum_{k:\, t_k>0} \frac{w_k(\tau)}{t_k} R_\rho\left(\frac{\mu}{t_k}\right).$$

(2) If $s_\rho = \langle f_1|P_\rho|f_1\rangle < 1$, then $|f_1\rangle \notin \mathrm{supp}(\rho)$, so $D_\alpha(f_1\|\rho) = +\infty$ for all $\alpha \in [1,2]$ and $D_\alpha(f_1\|\sigma) \leq D_\alpha(f_1\|\rho)$ holds trivially; assume $s_\rho = 1$ when $\alpha \in (1,2)$. For $\alpha \in (1,2)$, substituting $\mu = \lambda u$ in the Euler beta integral $\int_0^\infty u^{1-\alpha}(1+u)^{-1}\,du = \pi/\sin(\pi(\alpha-1))$ gives

$$\int_0^\infty \mu^{1-\alpha} R_\rho(\mu)\,d\mu = \int_0^1 \lambda^{1-\alpha} \frac{\pi}{\sin(\pi(\alpha-1))}\,d\mu_\rho(\lambda) = \frac{\pi}{\sin(\pi(\alpha-1))}\langle f_1|\rho^{1-\alpha}|f_1\rangle.$$

Similarly, for $\alpha \in (0,1)$, $1-\mu R_\rho(\mu) = \int_0^1 \frac{\lambda}{\lambda+\mu}\, d\mu_\rho(\lambda)$, and $\int_0^\infty u^{-\alpha}(1+u)^{-1}\, du = \pi/\sin(\pi\alpha)$ gives (19). By Theorem 1, $\rho \to \sigma$ implies $R_\sigma(\mu) \le R_\rho(\mu)$ and $1-\mu R_\sigma(\mu) \ge 1-\mu R_\rho(\mu)$ for all $\mu > 0$, whence $\langle f_1|\sigma^{1-\alpha}|f_1\rangle \ge \langle f_1|\rho^{1-\alpha}|f_1\rangle$ for $\alpha \in (0,1)$ (where $(\alpha-1)^{-1} < 0$) and $\langle f_1|\sigma^{1-\alpha}|f_1\rangle \le \langle f_1|\rho^{1-\alpha}|f_1\rangle$ for $\alpha \in (1,2)$ (where $(\alpha-1)^{-1} > 0$). In both ranges, $D_\alpha(f_1\|\sigma) \le D_\alpha(f_1\|\rho)$, extending by continuity to $\alpha \in \{0,1,2\}$. Additivity $D_\alpha(f_1^{\otimes 2}\|\rho\otimes\tau) = D_\alpha(f_1\|\rho) + D_\alpha(f_1\|\tau)$ is immediate from $(\rho\otimes\tau)^{1-\alpha} = \rho^{1-\alpha}\otimes\tau^{1-\alpha}$.

(3) If $\rho \to \sigma$ via Λ, then $\Lambda^{\otimes n}$ achieves $\rho^{\otimes n} \to \sigma^{\otimes n}$ and $\Lambda\otimes\mathrm{id}$ achieves $\rho\otimes\tau \to \sigma\otimes\tau$. Conversely, if $\rho\otimes\tau \to \sigma\otimes\tau$ with $c_\tau > 0$ (so $p_\tau \ge c_\tau > 0$), additivity of D_0 and D_2 yields $pp_\tau \le qp_\tau \implies q \ge p$ and $c_\rho c_\tau \le c_\sigma c_\tau \implies c_\sigma \ge c_\rho$. (If $p_\tau > 0$ and $c_\tau = 0$, subtracting $(1-s_\tau)/\mu$ in (18) and taking $\mu \to 0^+$ still yields $c_\sigma \ge c_\rho$.) Similarly, $\rho^{\otimes n} \to \sigma^{\otimes n}$ yields $p^n \le q^n \implies q \ge p$ and $c_\rho^n \le c_\sigma^n \implies c_\sigma \ge c_\rho$. By Corollary 1(b), $q \ge p$ and $c_\sigma \ge c_\rho$ imply $\rho \to \sigma$ for $D = 2$.

(4) For (21), the decoupled $(D-2)$-dimensional blocks have zero overlap with $|f_1\rangle$ and drop out of $R_\rho(\mu), R_\sigma(\mu)$, so the 2×2 leading blocks yield

$$R_\rho(\mu) = \frac{\mu + \frac{3}{20}}{\mu^2 + \frac{21}{40}\mu + \frac{13}{800}}, \qquad R_\sigma(\mu) = \frac{\mu + \frac{7}{20}}{\mu^2 + \frac{1549}{2000}\mu + \frac{2343}{40000}}.$$

Evaluating at $\mu = 1/4$ gives $R_\rho(1/4) = \frac{2/5}{21/100} = \frac{40}{21}$ and $R_\sigma(1/4) = \frac{3/5}{3147/10000} = \frac{2000}{1049}$, hence $R_\rho(1/4) - R_\sigma(1/4) = -\frac{40}{22029} < 0$, proving $\rho \not\to \sigma$ by Theorem 1 (Figure 2(a)).

For the qubit catalyst τ, its eigenvalues are $t_1 = 9/10$ and $t_2 = 1/10$ with equal weights $w_1(\tau) = w_2(\tau) = 1/2$. Applying (18), clearing denominators over the strictly positive degree-8 denominator $Q_{\mathrm{cat}}(\mu)$ (a product of monic quadratics with strictly positive coefficients), the leading μ^7 term cancels identically and leaves the exact primitive degree-6 polynomial $P_{\mathrm{cat}}(\mu) = \sum_{k=0}^6 d_k\mu^k$ in Table 1. Since $d_6, d_4, d_3, d_2, d_1, d_0 > 0$ and $4d_6d_4 - d_5^2 > 0$, the exact SOS decomposition

$$P_{\mathrm{cat}}(\mu) = \frac{\mu^4}{4d_6}\Big((2d_6\mu + d_5)^2 + (4d_6d_4 - d_5^2)\Big) + d_3\mu^3 + d_2\mu^2 + d_1\mu + d_0 > 0 \tag{22}$$

certifies $R_{\rho\otimes\tau}(\mu) > R_{\sigma\otimes\tau}(\mu)$ for all $\mu > 0$, hence $\rho\otimes\tau \to \sigma\otimes\tau$ by Theorem 1.

For the 2-copy conversion $\rho^{\otimes 2} \to \sigma^{\otimes 2}$, any 2×2 block with trace $s_1 = p + r$, determinant $s_2 = pr - |x|^2 > 0$, and discriminant $\delta = s_1^2 - 4s_2 > 0$ satisfies $2w_1w_2 = 2|x|^2/\delta$, giving the exact degree-3 resolvent

$$R_{\rho^{\otimes 2}}(\mu) = \frac{2w_1w_2}{s_2+\mu} + \frac{(1-2w_1w_2)\mu + r^2 - 2w_1w_2s_2}{\mu^2 + (s_1^2 - 2s_2)\mu + s_2^2}.$$

Subtracting $R_{\sigma^{\otimes 2}}(\mu)$ over the strictly positive degree-6 denominator $Q_2(\mu)$ cancels the μ^5 term and yields the primitive degree-4 polynomial $P_2(\mu) = \sum_{k=0}^4 c_k\mu^k$ in Table 1. Because $c_4, c_2, c_1, c_0 > 0$ and $4c_4c_2 - c_3^2 > 0$, the exact SOS decomposition

$$P_2(\mu) = \frac{\mu^2}{4c_4}\Big((2c_4\mu + c_3)^2 + (4c_4c_2 - c_3^2)\Big) + c_1\mu + c_0 > 0 \tag{23}$$

certifies $R_{\rho^{\otimes 2}}(\mu) > R_{\sigma^{\otimes 2}}(\mu)$ for all $\mu > 0$, hence $\rho^{\otimes 2} \to \sigma^{\otimes 2}$ by Theorem 1 (Figure 2(b)). □

5 Theorem 4: GKSL Normal Form, Exact SOS Resolvent Dissipation, and Markovian Rank-Non-Decrease Barrier

We now completely resolve de Leon's second open problem [1, Sec. VI] by characterizing all continuous-time QST-free Markovian semigroups $\Lambda_t = e^{t\mathcal{L}}$ $(t \ge 0)$ generated by a Gorini–

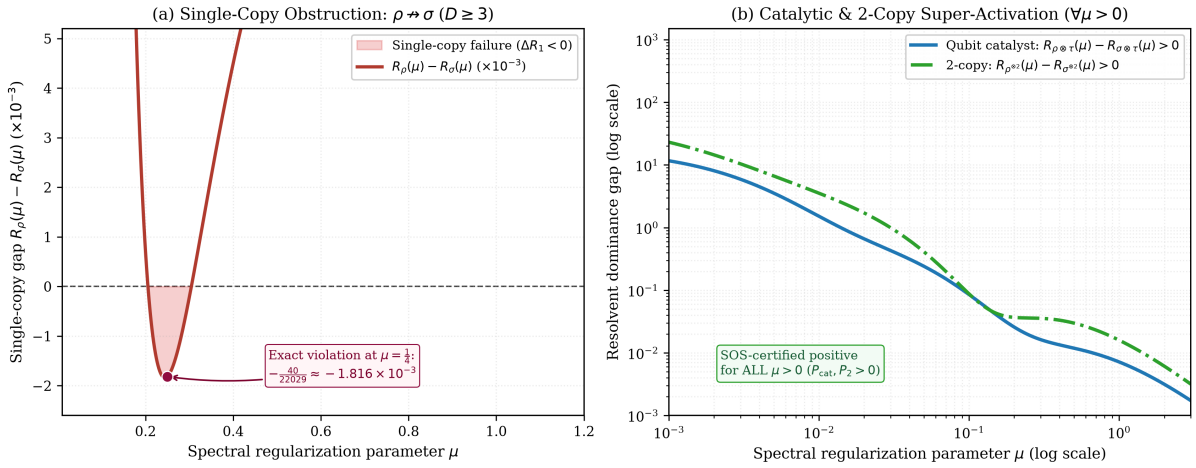


Figure 2: **(a)** Single-copy obstruction for the full-rank rational-entry $D \geq 3$ witness (21): despite $q > p$ and $c_\sigma > c_\rho$, $R_\rho(\mu) - R_\sigma(\mu)$ dips strictly below zero at $\mu = 1/4$ ($-40/22029$). **(b)** Simultaneous qubit-catalytic ($\rho \otimes \tau \to \sigma \otimes \tau$) and 2-copy ($\rho^{\otimes 2} \to \sigma^{\otimes 2}$) super-activation: both resolvent gaps remain strictly positive for every $\mu > 0$, certified algebraically by the exact SOS decompositions (22)–(23).

Kossakowski–Sudarshan–Lindblad (GKSL) master equation

$$\dot{\rho}_t = \mathcal{L}(\rho_t) = -i[H, \rho_t] + \sum_\alpha \left(L_\alpha \rho_t L_\alpha^\dagger - \frac{1}{2}\{L_\alpha^\dagger L_\alpha, \rho_t\} \right), \qquad H = H^\dagger. \tag{24}$$

Theorem 4 (GKSL Normal Form, Exact SOS Resolvent Dissipation, and Markovian Rank Barrier)**.** *Let $\mathcal{L}$ be a GKSL generator (24) on $M_D(\mathbb{C})$, with adapted-basis blocks $H = \begin{pmatrix} h_0 & h^\dagger \\ h & H_\perp \end{pmatrix}$ and $L_\alpha = \begin{pmatrix} \ell_\alpha & v_\alpha^\dagger \\ u_\alpha & M_\alpha \end{pmatrix}$. Then:*

(1) Exact Fixed-Point Characterization & Canonical Gauge: *The semigroup $\Lambda_t = e^{t\mathcal{L}}$ is QST-free ($\Lambda_t(f_1) = f_1$ for all $t \geq 0$, equivalently $\mathcal{L}(f_1) = 0$) if and only if*

$$u_\alpha = 0 \quad (\forall \alpha), \qquad h = \frac{i}{2} \sum_\alpha \ell_\alpha v_\alpha. \tag{25}$$

Under (25), the canonical gauge transformation

$$\widetilde{L}_\alpha := L_\alpha - \ell_\alpha \mathbb{I}_D = \begin{pmatrix} 0 & C_\alpha \end{pmatrix}, \qquad C_\alpha := \begin{pmatrix} v_\alpha^\dagger \\ M_\alpha - \ell_\alpha \mathbb{I}_{D-1} \end{pmatrix} \in M_{D,D-1}(\mathbb{C}),$$
$$\widetilde{H} := H + \frac{i}{2} \sum_\alpha (\ell_\alpha^* L_\alpha - \ell_\alpha L_\alpha^\dagger), \qquad \widehat{H} := \widetilde{H} - \langle f_1 | \widetilde{H} | f_1 \rangle \mathbb{I}_D = \operatorname{diag}(0, \widehat{H}_\perp)$$

leaves $\mathcal{L}$ identically invariant and satisfies $\widetilde{L}_\alpha |f_1\rangle = 0$ and $\widehat{H}|f_1\rangle = 0$.

(2) Exact Filter Semigroup & Grand-Sum Preservation: *For every $t \geq 0$, the unique filter block T_t of $\Lambda_t = e^{t\mathcal{L}}$ in de Leon's normal form [1, Theorem 1] is given by*

$$T_t = e^{tG}, \qquad G := -i\widehat{H}_\perp - \frac{1}{2} \sum_\alpha C_\alpha^\dagger C_\alpha \in M_{D-1}(\mathbb{C}), \tag{26}$$

satisfying $G + G^\dagger = -\sum_\alpha C_\alpha^\dagger C_\alpha \preceq 0$. Moreover, Λ_t is grand-sum-preserving for all $t \geq 0$ if and only if $v_\alpha = 0$ for every α.

(3) Exact Sum-of-Squares Resolvent Dissipation Identity: *Along any trajectory* $\rho_t = e^{t\mathcal{L}}(\rho_0)$ *and for every* $\mu > 0$, *setting* $|w_t(\mu)\rangle := (\rho_t + \mu\mathbb{I}_D)^{-1}|f_1\rangle$,

$$-\frac{d}{dt}R_{\rho_t}(\mu) = \sum_\alpha \Big(\big\|\rho_t^{1/2}\widetilde{L}_\alpha^\dagger |w_t(\mu)\rangle\big\|_2^2 + \mu \big\|\widetilde{L}_\alpha |w_t(\mu)\rangle\big\|_2^2 \Big) \geq 0. \tag{27}$$

(4) Sharp Markovian Rank-Non-Decrease Barrier: *For every* $t \geq 0$ *and every* $\rho \in \mathcal{D}(\mathbb{C}^D)$, $\det(T_t) = e^{t\,\mathrm{Tr}\,G} \neq 0$, *hence the filter* $A_t = \mathrm{diag}(1, T_t)$ *is invertible and*

$$\mathrm{rank}\big(\Lambda_t(\rho)\big) \geq \mathrm{rank}(\rho). \tag{28}$$

Consequently, the complete reset $\Lambda_0(X) = \mathrm{Tr}(X) f_1$, *every measure-and-prepare channel* Λ_ω (*which has* $T = 0$, $\det T = 0$), *and every mixed-to-pure conversion of de Leon's Corollary 9* ($\mathrm{rank}\,\rho \geq 2 > 1 = \mathrm{rank}\,\sigma$) *are strictly non-Markovian* (*unreachable by any finite-time QST-free GKSL semigroup, and even by any integrable time-dependent CP-divisible Lindbladian*).

Proof. **(1)** Evaluating $\mathcal{L}(f_1)$ in the adapted basis gives lower-right block $f_1^\perp \mathcal{L}(f_1) f_1^\perp = \sum_\alpha u_\alpha u_\alpha^\dagger$, which vanishes iff $u_\alpha = 0$ for all α. With $u_\alpha = 0$, $L_\alpha|f_1\rangle = \ell_\alpha|f_1\rangle$ and $L_\alpha^\dagger L_\alpha$ has lower-left block $\ell_\alpha v_\alpha$, so the lower-left block of $\mathcal{L}(f_1)$ is $-ih - \frac{1}{2}\sum_\alpha \ell_\alpha v_\alpha$, which vanishes iff $h = \frac{i}{2}\sum_\alpha \ell_\alpha v_\alpha$. The upper-left entry $\langle f_1|\mathcal{L}(f_1)|f_1\rangle = \sum_\alpha(|\ell_\alpha|^2 - |\ell_\alpha|^2) = 0$ vanishes automatically. For the canonical gauge, expanding $\widetilde{L}_\alpha X \widetilde{L}_\alpha^\dagger - \frac{1}{2}\{\widetilde{L}_\alpha^\dagger \widetilde{L}_\alpha, X\}$ with $\widetilde{L}_\alpha = L_\alpha - \ell_\alpha \mathbb{I}_D$ shows that the cross terms equal $-\frac{1}{2}\ell_\alpha^* [L_\alpha, X] + \frac{1}{2}\ell_\alpha [L_\alpha^\dagger, X] = i[\frac{i}{2}(\ell_\alpha^* L_\alpha - \ell_\alpha L_\alpha^\dagger), X]$, so shifting $H \mapsto \widetilde{H} = H + \frac{i}{2}\sum_\alpha(\ell_\alpha^* L_\alpha - \ell_\alpha L_\alpha^\dagger)$ leaves $\mathcal{L}(X)$ identically invariant for every $X \in M_D(\mathbb{C})$. The lower-left block of $\widetilde{H}$ is $h + \frac{i}{2}\sum_\alpha(0 - \ell_\alpha v_\alpha) = 0$, and subtracting $\langle f_1|\widetilde{H}|f_1\rangle \mathbb{I}_D$ leaves $[\widehat{H}, X]$ unchanged while ensuring $\widehat{H}|f_1\rangle = 0$ and $\widetilde{L}_\alpha|f_1\rangle = 0$.

(2) By de Leon's uniqueness proof [1, Sec. III], $T_t|v\rangle = \Lambda_t(|v\rangle\langle f_1|)|f_1\rangle$ for every $|v\rangle \in f_1^\perp \mathbb{C}^D$. Because $\widetilde{L}_\alpha|f_1\rangle = 0$ and $\widehat{H}|f_1\rangle = 0$, applying $\mathcal{L}$ in the canonical gauge to $X = |v\rangle\langle f_1|$ gives $\widetilde{L}_\alpha X \widetilde{L}_\alpha^\dagger = 0$, $X\widehat{H} = 0$, and $X\widetilde{L}_\alpha^\dagger \widetilde{L}_\alpha = 0$, leaving

$$\mathcal{L}(|v\rangle\langle f_1|) = \left(-i\widehat{H}_\perp - \frac{1}{2}\sum_\alpha C_\alpha^\dagger C_\alpha\right)|v\rangle\langle f_1| = |Gv\rangle\langle f_1|.$$

Thus $\mathcal{L}$ leaves $\{|v\rangle\langle f_1| : |v\rangle \in f_1^\perp \mathbb{C}^D\}$ invariant and acts on it as G; exponentiating yields $\Lambda_t(|v\rangle\langle f_1|) = |e^{tG}v\rangle\langle f_1|$, hence $T_t = e^{tG}$. Immediately $G + G^\dagger = -\sum_\alpha C_\alpha^\dagger C_\alpha \preceq 0$ (since $\widehat{H}_\perp = \widehat{H}_\perp^\dagger$). Moreover, $\frac{d}{dt}\langle f_1|\rho_t|f_1\rangle = \langle f_1|\mathcal{L}(\rho_t)|f_1\rangle = \sum_\alpha v_\alpha^\dagger R_t v_\alpha = \mathrm{Tr}[(\sum_\alpha v_\alpha v_\alpha^\dagger) R_t]$, which vanishes identically for all inputs ρ_0 iff $v_\alpha = 0$ for every α.

(3) Differentiating $R_{\rho_t}(\mu) = \langle f_1|(\rho_t + \mu\mathbb{I}_D)^{-1}|f_1\rangle$ gives

$$-\frac{d}{dt}R_{\rho_t}(\mu) = \langle f_1|(\rho_t + \mu\mathbb{I}_D)^{-1}\dot{\rho}_t(\rho_t + \mu\mathbb{I}_D)^{-1}|f_1\rangle = \langle w_t(\mu)|\mathcal{L}(\rho_t)|w_t(\mu)\rangle.$$

By definition of $|w_t\rangle := |w_t(\mu)\rangle$, we have $\rho_t|w_t\rangle = |f_1\rangle - \mu|w_t\rangle$ and $\langle w_t|\rho_t = \langle f_1| - \mu\langle w_t|$. In the canonical gauge, $\widehat{H}|f_1\rangle = 0$ and $\langle f_1|\widehat{H} = 0$ force the Hamiltonian contribution to vanish identically:

$$\begin{aligned}\langle w_t| - i[\widehat{H}, \rho_t]|w_t\rangle &= -i\langle w_t|\widehat{H}(|f_1\rangle - \mu|w_t\rangle)|\rangle + i\langle(\langle f_1| - \mu\langle w_t|)|\widehat{H}|w_t\rangle \\ &= i\mu\langle w_t|\widehat{H}|w_t\rangle - i\mu\langle w_t|\widehat{H}|w_t\rangle = 0.\end{aligned}$$

For each jump operator $\widetilde{L}_\alpha$, $\langle w_t|\widetilde{L}_\alpha \rho_t \widetilde{L}_\alpha^\dagger|w_t\rangle = \|\rho_t^{1/2}\widetilde{L}_\alpha^\dagger|w_t\rangle\|_2^2$, while $\widetilde{L}_\alpha|f_1\rangle = 0$ gives

$$-\frac{1}{2}\langle w_t|\{\widetilde{L}_\alpha^\dagger \widetilde{L}_\alpha, \rho_t\}|w_t\rangle = -\,\mathrm{Re}\langle w_t|\widetilde{L}_\alpha^\dagger \widetilde{L}_\alpha(|f_1\rangle - \mu|w_t\rangle)|\rangle = \mu\|\widetilde{L}_\alpha|w_t\rangle\|_2^2.$$

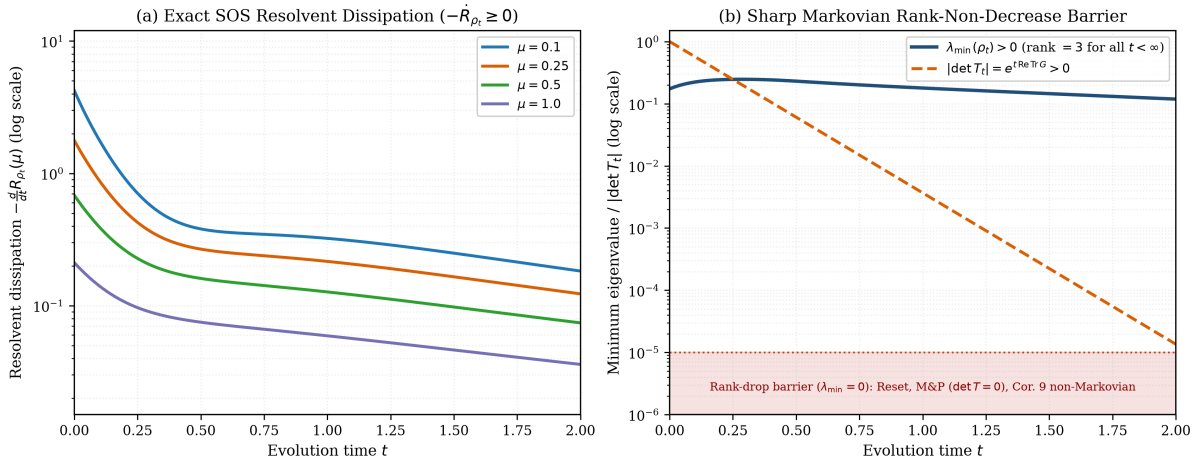


Figure 3: **(a)** Exact Sum-of-Squares resolvent dissipation $-\frac{d}{dt}R_{\rho_t}(\mu) \geq 0$ along a coherent $D = 3$ QST-free GKSL trajectory $\rho_t = e^{t\mathcal{L}}(\rho_0)$ across four values of μ, matching (27) identically. **(b)** Sharp Markovian rank-non-decrease barrier: $|\det T_t| = e^{t\,\mathrm{Re}\,\mathrm{Tr}\,G} > 0$ enforces $\lambda_{\min}(\rho_t) > 0$ ($\mathrm{rank}\,\rho_t = 3$) for all finite $t \geq 0$, separating Markovian dynamics strictly from complete reset, measure-and-prepare channels ($\det T = 0$), and every mixed-to-pure conversion of de Leon's Corollary 9.

Summing over α yields (27) (Figure 3(a)).

(4) Since $T_t = e^{tG}$ satisfies $\det(T_t) = e^{t\,\mathrm{Tr}\,G} \neq 0$ for every finite $t \geq 0$, the filter $A_t = \mathrm{diag}(1, T_t)$ is invertible. By de Leon's normal form [1, Theorem 1], $\Lambda_t(\rho) = A_t \rho A_t^\dagger + \Phi_t(R)$ with both summands positive semidefinite. Therefore $\ker(\Lambda_t(\rho)) \subseteq \ker(A_t \rho A_t^\dagger)$, and invertibility of $A_t^\dagger$ injects $\ker(A_t \rho A_t^\dagger)$ onto $\ker(\rho)$, giving $\dim\ker(\Lambda_t(\rho)) \leq \dim\ker(\rho)$, i.e. $\mathrm{rank}(\Lambda_t(\rho)) \geq \mathrm{rank}(\rho)$. (The same Liouville determinant argument $\det T(t) = \exp\int_0^t \mathrm{Tr}\,G(s)\,ds \neq 0$ extends verbatim to any integrable time-dependent CP-divisible Lindbladian $\mathcal{L}_t$.) Because complete reset drops $\mathrm{rank}(\mathbb{I}_D/D) = D$ to $\mathrm{rank}(f_1) = 1$, every QST-free measure-and-prepare channel $\Lambda(X) = \sum_k \mathrm{Tr}(E_k X)\sigma_k$ ($E_k \succeq 0$, $\sum_k E_k = \mathbb{I}_D$) satisfies $\Lambda(f_1) = \sum_k \langle f_1|E_k|f_1\rangle \sigma_k = f_1$; by extremality of the pure state f_1, every index k has either $\sigma_k = f_1$ or $\langle f_1|E_k|f_1\rangle = 0$ (which forces $E_k|f_1\rangle = 0$ since $E_k \succeq 0$ and thus $\mathrm{Tr}(E_k|v\rangle\langle f_1|) = \langle f_1|E_k|v\rangle = 0$), so $\Lambda(|v\rangle\langle f_1|) \in \mathrm{span}\{f_1\}$ and $T|v\rangle = f_1^\perp \Lambda(|v\rangle\langle f_1|)|f_1\rangle = 0$ for every $|v\rangle \perp |f_1\rangle$ (giving $T = 0$, $\det T = 0 \neq e^{t\,\mathrm{Tr}\,G}$), and every mixed-to-pure conversion of de Leon's Corollary 9 drops $\mathrm{rank}(\rho) \geq 2$ to $\mathrm{rank}(\sigma) = 1$, none can be realized by a finite-time QST-free GKSL semigroup (Figure 3(b)). □

6 Machine-Checked Formalization and Zero-Float Certification

The algebraic, SOS, combinatorial, and finite-dimensional cores of Theorems 1–4 (complementing the arbitrary-D analytic proofs in Sections 2–5) are backed by a three-layer reproducible verification pipeline archived at tag `v1.0.3` of

https://github.com/lluiseriksson/paper146-quantum-texture-barriers

comprising:

1. **Constructive Lean 4.36.0 Formalization (`lean/QuantumTextureBarriers.lean`):** 20 machine-checked theorems (`thm01`–`thm20`) covering all four main theorems—including the Schur-complement identity, Fenchel–Moreau quadratic completion, exact Eq. (64) refutation, universal $D = 2$ monotonicity collapse, single-pole tangent factorization, universal small-μ and large-μ S_m bounds, constructive `Fin`-pigeonhole no-go theorem, universal

Table 2: Summary of the 16 AST-audited zero-float exact-$\mathbb{Q}/\mathbb{Q}[i]$ SHA-256 certificates (`C01`–`C16`) and their corresponding Lean 4.36.0 theorems (`thm01`–`thm20`).

Certs.	Lean 4	Theorem / Verification Scope	Key Exact Output
`C01`–`C04`	`thm01`–`thm05`	Thm. 1: Schur–resolvent, Eq. (64), Cors. 7–9	$\Delta\psi(\frac{3}{20}) = -\frac{3}{50}$, $\Delta R(\frac{1}{5}) = -\frac{14}{39}$
`C05`–`C08`	`thm06`–`thm10`	Thm. 2: Tangent S_m ($D = 3, 4, 5$), pigeonhole	S_4, S_5, S_6 global $\forall \mu > 0$
`C09`–`C12`	`thm11`–`thm15`	Thm. 3: 6×6 & 9×9 inversions, SOS	$\Delta R_1(\frac{1}{4}) = -\frac{40}{22029}$; $P_{\text{cat}}, P_2 > 0$
`C13`–`C16`	`thm16`–`thm20`	Thm. 4: $\mathbb{Q}[i]$ GKSL gauge, SOS rate, rank	9-basis gauge; $\text{rank}(\Lambda_t \rho) \geq \text{rank}\, \rho$

degree-6 catalytic and degree-4 2-copy SOS positivity for all rational $\mu = a/b \geq 0$, sharp $D = 2$ no-activation, exact $\mathbb{Z}[i]$ ($D = 2$) GKSL fixed-point characterization, canonical gauge invariance, filter dissipativity, and operator-algebraic SOS resolvent dissipation identity (complemented by full $D = 3$ $\mathbb{Q}[i]$ verification in `C13`–`C15` and the arbitrary-D proofs in Section 5), and the kernel-injection rank barrier. The file compiles under `lean -DwarningAsError=true` with **zero external imports, zero `sorry`, zero `grind`, zero `Classical.choice`** (`#print axioms` verified on all 20 theorems: only `[propext, Quot.sound]` or empty), and **zero warnings**.

2. **AST-Audited Zero-Float Exact-$\mathbb{Q}/\mathbb{Q}[i]$ Verifier:** The pure Python standard-library script `verification/verify_quantum_texture.py` uses exact `int`/`Fraction` and exact Gaussian rational pairs $\mathbb{Q}[i]$, performing a mandatory AST self-audit (**0 float violations**) before emitting 16 canonical SHA-256 certificates (`C01`–`C16`, including exact 6×6 and 9×9 rational Gauss–Jordan matrix inversions and full 9-element basis verification of GKSL gauge invariance) summarized in Table 2, with deterministic manifest digest

   `MANIFEST_SHA256 = 4d1e4c7477c8ee2503d318e9b7870d49716218b914d13a78f653ab6b08b7658b.`

3. **Adversarial Test Suite (`tests/test_adversarial.py`):** 60/60 passing `pytest` checks covering positive certificates, AST tamper injection, invalid contraction/remainder filters, zero-ε/excessive-ε S_m tampering, non-activating trivial/maximally-mixed/weakly-coherent catalysts, $D = 2$ no-activation counterexamples, and GKSL fixed-point tampering ($u_\alpha \neq 0$ and wrong h).

7 Conclusions

We have established a complete state-intrinsic, order-theoretic, multi-copy, catalytic, and dynamical theory of quantum-state texture in arbitrary finite dimension $D \geq 2$. The transition from qubits ($D = 2$) to qutrits and higher dimensions ($D \geq 3$) is not merely a quantitative increase in the number of monotones: it is a sharp phase transition at which order dimension jumps from 2 to ∞ (ruling out any finite complete monotone set), single-copy inconvertibility becomes simultaneously super-activated by a single qubit catalyst or a second copy, and Markovian GKSL dynamics separate strictly from general QST-free channels by an exact rank-non-decrease barrier. Because our proofs rely solely on the rank-1 fixed-point projector f_1, Theorems 1–4 apply verbatim to every pure fixed-point quantum resource theory, including zero-temperature Gibbs-preserving thermodynamics [23, 19].

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