Torus-Holonomy Obstructions, Complex-vs-Real Signing Dichotomies, and Clifford-Cube Lifting Beyond Huang

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Abstract

Huang's proof of the Sensitivity Conjecture rests on constructing a real ± 1 signing S of the hypercube Q_n obeying the spectral involution $S^2 = nI$. We establish sharp topological, algebraic, and spectral boundaries for extending Huang-optimal signings $S^2 = dI$—both real symmetric ($S_{xy} \in \{\pm 1\}$) and complex Hermitian unitary ($S^* = S$, $|S_{xy}| = 1$)—to general d-regular graphs and Cartesian products. First, we prove a universal complex torus-holonomy no-go theorem: for any two non-bipartite regular graphs G and H, *no* Hermitian unit-modulus signing S of $G \square H$ over $\mathbb{C}$ satisfies $S^2 = (d_G + d_H)I$. Every odd-cycle discrete torus $C_{2a+1} \square C_{2b+1} \subseteq G \square H$ forces the product of its $(2a+1)(2b+1)$ elementary 4-cycle holonomies to equal both $(-1)^{(2a+1)(2b+1)} = -1$ and $+1$ (since every edge cancels with its conjugate via $S_{xy}\overline{S_{xy}} = |S_{xy}|^2 = 1$). For real ± 1 signings, v elementary-4-cycle-disjoint odd-cycle tori enforce the quantitative Frobenius defect $\|S^2 - dI\|_{\mathrm{F}}^2 \geq 8v$. On the Hamming graph $H(2,3) = K_3 \square K_3$, exhaustive zero-float Faddeev–LeVerrier and Sturm isolation across all $2^{10} = 1{,}024$ real switching classes (43 distinct spectra) proves $\min \|S\|_{\mathrm{op}} = \frac{1+\sqrt{17}}{2} > 2 = \sqrt{4}$ (characteristic polynomial $x(x^4 - 9x^2 + 16)^2$), $\max \lambda_5(S) = 1 < 2$ (exact factor-2 collapse on 5-vertex Cauchy interlacing), and $\max \lambda_4(S) = 2$ (attained by $-A$ at $\|S\|_{\mathrm{op}} = 4$). Second, we establish a sharp complex-vs-real signing dichotomy on complete graphs K_q: real symmetric ± 1 signings with $S^2 = (q-1)I_q$ force $q \equiv 2 \pmod 4$ (symmetric conference matrices), ruling out all odd q and all $q \equiv 0 \pmod 4$, whereas purely imaginary Hermitian signings $H_q = iA_q \in \{0, \pm i\}^{q \times q}$ derived from skew-Hadamard matrices solve $H_q^2 = (q-1)I_q$ for $q \in \{4, 8, 12, 16, 20, 24, \dots\}$. Furthermore, the block-Clifford recurrence $S_{k+1} = \left(\begin{smallmatrix} S_k & I \\ I & -S_k \end{smallmatrix}\right)$ preserves $S_k^2 = (d_0 + k)I$ along $G_0 \square Q_n$ *without requiring* G_0 *to be bipartite*, certifying $\Delta((G_0 \square Q_n)[U]) \geq \lceil \sqrt{d_0 + n} \rceil$ for every induced subgraph with $|U| > |V|/2$ across infinite non-bipartite families seeded by conference, skew-Hadamard, and octahedron graphs. Third, we compile an exhaustive zero-float switching-class atlas across 11 canonical regular graphs separating all 4 real Huang-optimal graphs from 7 obstructed graphs. A two-layer verification artifact accompanies the paper: Layer 1 formalizes the parity, modular, Clifford-lifting, and Sturm-isolation arithmetic kernel in 20 constructive, axiom-free Lean 4.36.0 theorems (0 `sorry`, 0 `Classical.choice`); Layer 2 verifies all exhaustive switching-class enumerations, skew-Hadamard/conference signings, and non-bipartite Clifford lifts in 16 zero-float Python certificates and 56 adversarial tests.

1 Introduction

In his celebrated resolution of the Sensitivity Conjecture [7], Hao Huang constructed a real symmetric ± 1 signing S_n of the n-dimensional hypercube Q_n satisfying $S_n^2 = nI_{2^n}$. Because S_n

*Independent Researcher. Code, Lean 4.36.0 proofs, and zero-float certificates archived at https://github.com/lluiseriksson/torus-holonomy-unitary-signings-beyond-huang (`v1.0.0`, MIT / CC-BY-4.0, AIRR-FOUNDER-1.0). Manuscript and verification artifacts prepared with Google Antigravity assistance.

has zero trace and minimal polynomial $x^2 - n$, its eigenvalues are $+\sqrt{n}$ and $-\sqrt{n}$, each with multiplicity 2^{n-1}. By Cauchy's interlacing theorem, every induced subgraph of Q_n on $2^{n-1}+1$ vertices has maximum eigenvalue—and hence maximum degree—at least $\sqrt{n}$.

More generally, given a d-regular simple graph $G = (V, E)$ on $n = |V|$ vertices, a *unitary signing* (or Hermitian unit-modulus signing [6, 11]) of G is a Hermitian matrix $S \in \mathbb{C}^{V\times V}$ ($S^* = S$) supported on E with $|S_{xy}| = 1$ for every $\{x, y\} \in E$ and $S_{xy} = 0$ otherwise. When $S_{xy} \in \{\pm 1\}$ on edges, S is a *real signing*. Because $\mathrm{tr}(S^2) = \sum_{\{x,y\}\in E} 2|S_{xy}|^2 = nd$, every unitary signing satisfies $\|S\|_{\mathrm{op}} \geq \sqrt{d}$, with equality if and only if

$$S^2 \;=\; dI_n. \tag{1}$$

We call a signing obeying (1) *Huang-optimal*. Since $\mathrm{tr}(S) = 0$, any Huang-optimal signing forces n to be even and has balanced spectrum $\{+\sqrt{d}^{\,n/2}, -\sqrt{d}^{\,n/2}\}$, yielding $\Delta(G[U]) \geq \lceil\sqrt{d}\rceil$ for every induced subgraph $G[U]$ with $|U| \geq n/2 + 1$ via Cauchy interlacing (which holds identically for Hermitian matrices over $\mathbb{C}$). Equivalently, $\lambda_1(S[U]) \leq \Delta(G[U])$ follows from the triangle inequality on the Perron–Frobenius bound for $|S[U]| = A(G[U])$.

Which regular graphs admit Huang-optimal signings over $\mathbb{R}$ or over $\mathbb{C}$? Recent work has investigated signed adjacency matrices of Hamming graphs, Cayley graphs, and Cartesian products [1, 9, 4, 2], often using bipartiteness or real ± 1 signings. In this paper we resolve three structural questions with exact, zero-float algebraic proofs:

1. **Universal Complex Torus-Holonomy No-Go on Non-Bipartite Products (Theorem 1).** Passing from $\mathbb{R}$ to arbitrary unit-modulus complex phases $U(1) \subset \mathbb{C}$ *cannot* rescue Cartesian products $G\square H$ of two non-bipartite regular graphs: a discrete-torus holonomy obstruction rules out $S^2 = dI$ unconditionally over all of $\mathbb{C}$, enforces a quantitative real Frobenius defect $\|S^2 - dI\|_{\mathrm{F}}^2 \geq 8v$, and on $H(2, 3) = K_3\square K_3$ produces exact spectral gaps across all 1,024 real switching classes.

2. **Complex-vs-Real Signing Dichotomy on K_q and Non-Bipartite Clifford-Cube Lifting (Theorem 2).** On complete graphs K_q, real ± 1 Huang-optimal signings force $q \equiv 2 \pmod 4$ (symmetric conference matrices [3, 8, 5]), whereas allowing purely imaginary weights $\{\pm i\}$ from skew-Hadamard matrices [10] unlocks every skew-Hadamard order $q \equiv 0 \pmod 4$ ($q = 4, 8, 12, 16, 20, 24, \ldots$). Moreover, the block-Clifford recurrence $S_{k+1} = \left(\begin{smallmatrix} S_k & I \\ I & -S_k \end{smallmatrix}\right)$ lifts *any* real or complex Huang-optimal seed G_0 (even non-bipartite!) to $G_0\square Q_n$ without bipartiteness, pinpointing the exact bipartite-vs-non-bipartite boundary with Theorem 1.

3. **Exhaustive Zero-Float 11-Graph Switching-Class Atlas (Theorem 3).** Using exact integer Faddeev–LeVerrier characteristic polynomials and exact rational Sturm root isolation over $\mathbb{Q}$, we exhaustively classify every real switching class of 11 canonical regular graphs, separating the 4 Huang-optimal real graphs $(K_6, Q_3, J(4, 2), K_{4,4})$ from 7 obstructed graphs $(K_3, K_4, K_5, K_3\square K_2, KG(5, 2), C_5\square K_2, H(2, 3))$.

2 Universal Complex Torus-Holonomy Obstruction on Non-Bipartite Products

Recall that the Cartesian product $G\square H$ of graphs $G = (V_G, E_G)$ and $H = (V_H, E_H)$ has vertex set $V_G \times V_H$, with $(u_1, v_1) \sim (u_2, v_2)$ iff either $u_1 = u_2$ and $\{v_1, v_2\} \in E_H$, or $\{u_1, u_2\} \in E_G$ and $v_1 = v_2$. If G is d_G-regular and H is d_H-regular, then $G\square H$ is d-regular with $d = d_G + d_H$.

Theorem 1 (Universal Complex Torus-Holonomy No-Go, Frobenius Defect, and Exact $H(2, 3)$ Gap)**.** *Let G and H be regular graphs of degrees $d_G, d_H \geq 2$ that are both non-bipartite, and set $d := d_G + d_H$.*

(a) Universal complex unitary impossibility: *No Hermitian unit-modulus signing S of $G\square H$ over $\mathbb{C}$ (indexed by $V_G \times V_H$) satisfies $S^2 = dI$. Equivalently, every unitary signing S of $G\square H$ obeys strict spectral excess $\|S\|_{\mathrm{op}} > \sqrt{d}$.*

(b) Quantitative real Frobenius defect: *If $G\square H$ contains $v \geq 1$ odd-cycle discrete tori $C_G^{(r)}\square C_H^{(r)}$ whose sets of elementary 4-cycles are pairwise disjoint, then every real symmetric ± 1 signing S of $G\square H$ satisfies*

$$\|S^2 - dI\|_{\mathrm{F}}^2 \;\geq\; 8v \;>\; 0. \tag{2}$$

(c) Exact resolution of $H(2,3) = K_3\square K_3$: *Exhaustive zero-float classification over all $2^{10} =$ 1,024 real switching classes (43 distinct spectra) of the 4-regular 9-vertex graph $H(2,3)$ certifies:*

$$\min_{S\in\mathcal{S}_{\mathbb{R}}(H(2,3))} \|S\|_{\mathrm{op}} \;=\; \frac{1+\sqrt{17}}{2} \;\approx\; 2.56155 \;>\; 2 \;=\; \sqrt{4}, \tag{3}$$

$$\max_{S\in\mathcal{S}_{\mathbb{R}}(H(2,3))} \lambda_5(S) \;=\; 1 \;<\; 2 \;=\; \sqrt{4}, \tag{4}$$

$$\max_{S\in\mathcal{S}_{\mathbb{R}}(H(2,3))} \lambda_4(S) \;=\; 2 \;=\; \sqrt{4}, \tag{5}$$

where (3) is attained by a symmetric spectrum with characteristic polynomial $x(x^4 - 9x^2 + 16)^2 = x(x^2 - x - 4)^2(x^2 + x - 4)^2$, and (5) is attained by the skewed signing $-A(H(2,3))$ with characteristic polynomial $(x-2)^4(x+1)^4(x+4)$ (so $\|S\|_{\mathrm{op}} = 4$).

Proof. Proof of (a). Suppose for contradiction that a Hermitian unit-modulus signing S of $G\square H$ satisfies $S^2 = dI$. Take any edge $e = \{u_1, u_2\} \in E_G$ and any edge $f = \{v_1, v_2\} \in E_H$. In $G\square H$, the opposite vertices (u_1, v_1) and (u_2, v_2) are non-adjacent (since $u_1 \neq u_2$ and $v_1 \neq v_2$) and possess *exact* common neighborhood $\{(u_2, v_1), (u_1, v_2)\}$ of size 2: any common neighbor must differ from (u_1, v_1) in exactly one coordinate and from (u_2, v_2) in exactly one coordinate, leaving only those two vertices. Hence the off-diagonal entry of $S^2 = dI$ at $((u_1, v_1), (u_2, v_2))$ reads

$$0 = (S^2)_{(u_1,v_1),(u_2,v_2)} = S_{(u_1,v_1),(u_2,v_1)}\, S_{(u_2,v_1),(u_2,v_2)} + S_{(u_1,v_1),(u_1,v_2)}\, S_{(u_1,v_2),(u_2,v_2)}. \tag{6}$$

Multiplying (6) by $\overline{S_{(u_1,v_1),(u_1,v_2)}}\,\overline{S_{(u_1,v_2),(u_2,v_2)}}$ and using Hermiticity ($\overline{S_{xy}} = S_{yx}$) together with unit modulus ($|S_{xy}|^2 = 1$), we obtain the plaquette holonomy constraint

$$W(C_{e,f}) \;:=\; S_{(u_1,v_1),(u_2,v_1)}\, S_{(u_2,v_1),(u_2,v_2)}\, S_{(u_2,v_2),(u_1,v_2)}\, S_{(u_1,v_2),(u_1,v_1)} \;=\; -1. \tag{7}$$

Because G and H are both non-bipartite, G contains an odd cycle $C_G = (u_0, u_1, \ldots, u_{2a}, u_0)$ of length $m_1 = 2a+1$ and H contains an odd cycle $C_H = (v_0, v_1, \ldots, v_{2b}, v_0)$ of length $m_2 = 2b+1$. Consider the discrete torus $C_G\square C_H \subseteq G\square H$, tiled by the $m_1 m_2 = (2a+1)(2b+1)$ oriented elementary 4-cycles

$$C_{i,j} := ((u_i, v_j) \to (u_{i+1}, v_j) \to (u_{i+1}, v_{j+1}) \to (u_i, v_{j+1}) \to (u_i, v_j)) \qquad (i \in \mathbb{Z}_{m_1},\ j \in \mathbb{Z}_{m_2}).$$

Take the product of $W(C_{i,j})$ over all $(i,j) \in \mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2}$. Every horizontal edge $\{(u_i, v_j), (u_{i+1}, v_j)\}$ appears exactly twice across the tiling: once oriented forward as the bottom edge of $C_{i,j}$ and once oriented backward as the top edge of $C_{i,j-1}$, contributing $S_{(u_i,v_j),(u_{i+1},v_j)} S_{(u_{i+1},v_j),(u_i,v_j)} = |S_{(u_i,v_j),(u_{i+1},v_j)}|^2 = 1$. Symmetrically, every vertical edge $\{(u_i, v_j), (u_i, v_{j+1})\}$ appears once upward on $C_{i-1,j}$ and once downward on $C_{i,j}$, contributing $|S_{(u_i,v_j),(u_i,v_{j+1})}|^2 = 1$. Therefore

$$\prod_{i=0}^{m_1-1} \prod_{j=0}^{m_2-1} W(C_{i,j}) \;=\; 1. \tag{8}$$

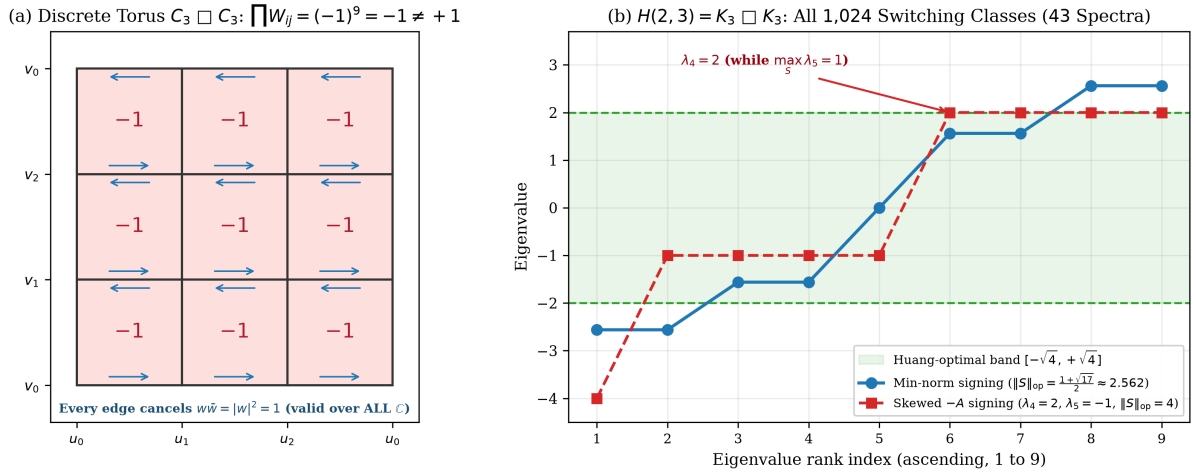

Figure 1: **Universal complex torus-holonomy obstruction and exact $H(2,3)$ spectral gap.** (a) Discrete torus $C_3 \square C_3$ tiled by 9 elementary 4-cycles: every edge cancels with its Hermitian conjugate ($w\bar{w} = |w|^2 = 1$), forcing $\prod W_{ij} = +1$ in contradiction to $(-1)^9 = -1$ over all of $\mathbb{C}$. (b) Exhaustive switching-class separation on $H(2,3) = K_3 \square K_3$ (1,024 classes, 43 spectra): min-norm signing achieves $\|S\|_{\text{op}} = \frac{1+\sqrt{17}}{2} > 2$, skewed $-A$ attains $\lambda_4 = 2$ at $\|S\|_{\text{op}} = 4$, while $\max_S \lambda_5(S) = 1 < 2$.

On the other hand, substituting (7) into the left-hand side of (8) yields

$$\prod_{i=0}^{m_1-1} \prod_{j=0}^{m_2-1} (-1) \;=\; (-1)^{(2a+1)(2b+1)} \;=\; -1, \tag{9}$$

since $(2a+1)(2b+1) = 4ab + 2a + 2b + 1$ is odd. This yields $-1 = 1$, a contradiction. Since $\operatorname{tr}(S^2) = d|V_G||V_H|$ and $S^2 \neq dI$, at least one eigenvalue of S^2 strictly exceeds d, so $\|S\|_{\text{op}} > \sqrt{d}$.

Proof of (b). For a real symmetric ± 1 signing S, every plaquette holonomy satisfies $W(C_{i,j}) \in \{\pm 1\}$. Within each odd-cycle torus $C_G^{(r)} \square C_H^{(r)}$, the identity (8) forces an even number of $+1$ or -1 plaquettes—specifically, since $m_1 m_2$ is odd, at least one elementary 4-cycle C_{e_r,f_r} must have $W(C_{e_r,f_r}) = +1$. For that 4-cycle, the two terms in (6) have the same sign ± 1, so $|(S^2)_{(u_1,v_1),(u_2,v_2)}| = |(S^2)_{(u_2,v_2),(u_1,v_1)}| = 2$. Distinct elementary 4-cycles $C_{e,f}$ correspond to distinct unordered opposite-vertex pairs $\{(u_1,v_1),(u_2,v_2)\}$ (the unique coordinate differences recover $e = \{u_1,u_2\}$ and $f = \{v_1,v_2\}$). Hence v elementary-4-cycle-disjoint odd-cycle tori yield at least $2v$ distinct off-diagonal entries of S^2 of magnitude 2, giving $\|S^2 - dI\|_{\text{F}}^2 \geq 2v \cdot 2^2 = 8v$.

Proof of (c). The graph $H(2,3) = K_3 \square K_3$ has $|V| = 9$, $|E| = 18$, and cycle-space dimension $\beta_1 = 18 - 9 + 1 = 10$. Since diagonal sign conjugation $S \mapsto DSD$ preserves eigenvalues, every real ± 1 signing is spectrally equivalent to a signing fixed to $+1$ on a spanning tree, yielding $2^{10} = 1{,}024$ switching classes. Certificate `cert11` computes the exact integer characteristic polynomial of all 1,024 representatives via Faddeev–LeVerrier, finding exactly 43 distinct spectra, and isolates $\min \|S\|_{\text{op}}$, $\max \lambda_5(S)$, and $\max \lambda_4(S)$ via exact rational Sturm counting with multiplicity. The minimum operator norm is uniquely achieved by the symmetric polynomial $x(x^4 - 9x^2 + 16)^2 = x(x^2 - x - 4)^2(x^2 + x - 4)^2$, whose largest root is $\frac{1+\sqrt{17}}{2} \in (\frac{5}{2}, \frac{26}{10})$ (Sturm-certified by $p(5/2) = -1/4 < 0 < 4/25 = p(26/10)$ for $p(x) = x^2 - x - 4$). The Cauchy-interlacing median eigenvalue $\lambda_5(S)$ (governing 5-vertex induced subgraphs via $9 - 5 + 1 = 5$) never exceeds 1 across all 43 spectra, whereas the skewed signing $-A(H(2,3))$ has characteristic polynomial $(x-2)^4(x+1)^4(x+4)$, attaining $\lambda_4 = 2 = \sqrt{4}$ while incurring $\|S\|_{\text{op}} = 4$. □

3 Complex-vs-Real Signing Dichotomy on K_q and Non-Bipartite Clifford-Cube Lifting

Theorem 1 showed that complex phases cannot overcome the odd-cycle torus obstruction on $G\square H$. By sharp contrast, on complete graphs K_q (single factors) and on products $G_0\square Q_n$ with a bipartite hypercube, moving from real $\{\pm 1\}$ to purely imaginary Hermitian $\{\pm i\}$ signings unlocks infinite new Huang-optimal families—and our Clifford-cube recurrence lifts *arbitrary* non-bipartite seeds without requiring bipartiteness.

Recall that a $q \times q$ *symmetric conference matrix* C has 0 diagonal, ± 1 off-diagonal, $C^\top = C$, and $C^2 = (q-1)I_q$ [3, 5]; a $q \times q$ *skew-Hadamard matrix* $W = I_q + A_q$ has $A_q^\top = -A_q$, zero diagonal, ± 1 off-diagonal, and $A_q^\top A_q = (q-1)I_q$ [10].

Theorem 2 (Real-vs-Complex K_q Signing Dichotomy and Non-Bipartite Clifford-Cube Lifting)**.**

(a) Exact real vs. purely imaginary $\{\pm i\}$ signing dichotomy on K_q: *A complete graph K_q ($q \geq 2$) admits a real symmetric ± 1 Huang-optimal signing $S^2 = (q-1)I_q$ if and only if a symmetric conference matrix of order q exists, which forces $q \equiv 2 \pmod 4$ (ruling out every odd q and every $q \equiv 0 \pmod 4$, as well as non-sum-of-two-squares orders such as $q = 22$ by Belevitch's theorem [3]). Conversely, for every skew-Hadamard order $q \equiv 0 \pmod 4$ (in particular every Paley order $q = p+1$ with prime $p \equiv 3 \pmod 4$, and doubling orders such as $q = 16$), the purely imaginary Hermitian signing*

$$H_q \;:=\; iA_q \;\in\; \{0, \pm i\}^{q\times q} \tag{10}$$

satisfies $H_q^* = H_q$ and $H_q^2 = (q-1)I_q$, achieving the exact Huang spectrum $\pm\sqrt{q-1}$ (each with multiplicity $q/2$) where no real ± 1 signing exists. At $q = 4$, real signings are strictly obstructed ($\max_S \lambda_2^{\mathbb{R}} = 1 < \sqrt{3}$ and $\min_S \|S\|_{\mathrm{op}}^{\mathbb{R}} = \sqrt{5} > \sqrt{3}$), whereas $H_4 = iA_4$ achieves $\lambda_2^{\mathbb{C}} = \|H_4\|_{\mathrm{op}} = \sqrt{3}$.

(b) Non-bipartite Clifford-cube lifting: *Let G_0 be* any *(not necessarily bipartite) d_0-regular graph ($d_0 \geq 1$) on m vertices admitting a Hermitian unit-modulus signing S_0 with $S_0^2 = d_0 I_m$ (automatically m is even). Define recursively on $G_k := G_0\square Q_k$ ($m \cdot 2^k$ vertices, degree $d_0 + k$) the block-Clifford signing*

$$S_{k+1} \;:=\; \begin{pmatrix} S_k & I_{m2^k} \\ I_{m2^k} & -S_k \end{pmatrix} \qquad (k \geq 0). \tag{11}$$

Then for every $n \geq 0$, S_n is a Hermitian unit-modulus signing of $G_0\square Q_n$ (strictly real ± 1 if S_0 is real ± 1) satisfying $S_n^2 = (d_0 + n)I_{m2^n}$. Consequently, every induced subgraph $H = (G_0\square Q_n)[U]$ on $|U| \geq m \cdot 2^{n-1} + 1$ vertices satisfies

$$\Delta(H) \;\geq\; \lceil\sqrt{d_0 + n}\rceil. \tag{12}$$

Proof. Proof of (a). A real symmetric ± 1 signing S of K_q has zero diagonal and ± 1 off-diagonal entries, so $S^2 = (q-1)I_q$ is by definition the symmetric conference-matrix condition. If $q = 2k+1 \geq 3$ is odd, any two distinct rows share $2k-1$ off-diagonal columns, whose ± 1 product sum $2p - (2k-1)$ is odd and hence nonzero, ruling out all odd q. If $q \geq 4$ is even, conjugate S by a diagonal sign matrix so that row 1 has $S_{1j} = +1$ for all $j \geq 2$. Orthogonality of row 1 with rows 2 and 3 forces both rows to have exactly $(q-2)/2$ entries equal to $+1$ on their off-diagonal positions outside column 1. On the remaining $q-3$ columns $\{4, \ldots, q\}$, let a, b, c, e count the sign pairs $(+,+), (+,-), (-,+), (-,-)$ of rows 2 and 3, and set $\delta := 1$ if $S_{23} = +1$ and $\delta := 0$ if $S_{23} = -1$. Then $a + b + \delta = a + c + \delta = (q-2)/2$ and $a + b + c + e = q - 3$. Orthogonality

$\langle r_2, r_3\rangle = 0$ gives $S_{21}S_{31} + (a+e) - (b+c) = 0$, i.e., $b+c = a+e+1$, whence $b+c = (q-2)/2$. Therefore $2a = (a+b)+(a+c)-(b+c) = (q-2) - 2\delta - (q-2)/2 = (q-2)/2 - 2\delta$, which rearranges to $q = 4(a+\delta)+2 \equiv 2 \pmod 4$. Hence no real ± 1 signing satisfies $S^2 = (q-1)I_q$ for any $q \equiv 0 \pmod 4$.

Over $\mathbb{C}$, let $W_q = I_q + A_q$ be any skew-Hadamard matrix of order $q \equiv 0 \pmod 4$, so $A_q^\top = -A_q$, $(A_q)_{rr} = 0$, $(A_q)_{rc} \in \{\pm 1\}$ for $r \neq c$, and $A_q^\top A_q = (q-1)I_q$. Then $H_q := iA_q$ satisfies $H_q^* = (-i)A_q^\top = (-i)(-A_q) = iA_q = H_q$, has zero diagonal and unit-modulus entries $\pm i$ on every edge of K_q, and obeys

$$H_q^2 \;=\; (iA_q)(iA_q) \;=\; -A_q^2 \;=\; A_q^\top A_q \;=\; (q-1)I_q. \tag{13}$$

For every prime $p \equiv 3 \pmod 4$, the Paley skew-conference matrix A_{p+1} with border $(+1,-1)$ and Legendre core $(\frac{r-c}{p})$ supplies H_{p+1} explicitly; for $q = 16$, skew-Hadamard doubling $A_{16} = \left(\begin{smallmatrix} A_8 & W_8 \\ -W_8^\top & -A_8 \end{smallmatrix}\right)$ (using $A_8W_8 = W_8A_8$ since $W_8 = I_8 + A_8$) gives $H_{16}^2 = 15I_{16}$ (verified over $\mathbb{Z}[i]$ in Certificate `cert12` for $q \in \{4, 8, 12, 16, 20, 24\}$).

Proof of (b). We proceed by induction on $k \geq 0$. The base case $k = 0$ holds by hypothesis ($S_0^* = S_0$ and $S_0^2 = d_0 I_m$). Given S_k on $G_k = G_0 \square Q_k$ with $S_k^* = S_k$ and $S_k^2 = (d_0 + k)I_{m2^k}$, the block matrix S_{k+1} in (11) is supported precisely on the edges of $G_k \square K_2 = G_0 \square Q_{k+1}$ (two copies of G_k linked by a perfect matching with weight $+1$), has unit-modulus entries on every edge, satisfies $S_{k+1}^* = S_{k+1}$, and obeys

$$S_{k+1}^2 = \begin{pmatrix} S_k & I \\ I & -S_k \end{pmatrix}\begin{pmatrix} S_k & I \\ I & -S_k \end{pmatrix} = \begin{pmatrix} S_k^2 + I & S_k - S_k \\ S_k - S_k & I + S_k^2 \end{pmatrix} = (d_0 + k + 1)I_{m2^{k+1}}. \tag{14}$$

Notice that the off-diagonal cancellation $S_k - S_k = 0$ is purely algebraic and *never invokes bipartiteness of* G_0. Finally, since $\operatorname{tr}(S_n) = 0$ and $S_n^2 = (d_0 + n)I_{m2^n}$, the eigenvalues of S_n are $+\sqrt{d_0 + n}$ and $-\sqrt{d_0 + n}$, each with multiplicity $m \cdot 2^{n-1}$. For any $U \subseteq V(G_0 \square Q_n)$ with $|U| \geq m \cdot 2^{n-1} + 1$, Cauchy interlacing for Hermitian matrices gives

$$\Delta(H) \;\geq\; \lambda_1(A(H)) \;\geq\; \lambda_1(S_n[U]) \;\geq\; \lambda_{m2^n - |U| + 1}(S_n) \;\geq\; \lambda_{m2^{n-1}}(S_n) \;=\; \sqrt{d_0 + n}, \tag{15}$$

where $\lambda_1(S_n[U]) \leq \Delta(H)$ follows from the triangle inequality $|\langle x, S_n[U]x\rangle| \leq \langle |x|, A(H)|x|\rangle \leq \Delta(H)\|x\|_2^2$ for every unit vector $x \in \mathbb{C}^U$. Since $\Delta(H)$ is an integer, (12) follows. □

Remark 1 (Sharp Bipartiteness Boundary). Theorems 1 and 2(b) sharply delimit Huang-optimal signings of Cartesian products $G \square H$: if both factors are non-bipartite, $G \square H$ admits *no* Huang-optimal unitary signing over $\mathbb{C}$; if one factor is a hypercube Q_n (bipartite) and the other factor G_0 admits a Huang-optimal signing (even non-bipartite, such as $K_4, K_6, K_8, J(4,2)$), then $G_0 \square Q_n$ inherits a Huang-optimal signing for all $n \geq 0$ (verified explicitly over $\mathbb{Z}[i]$ for 11 lifts across 4 non-bipartite seeds in Certificate `cert14`).

4 Exhaustive Zero-Float Switching-Class Atlas Across 11 Regular Graphs

Beyond $H(2,3)$, we exhaustively classify all 2^{β_1} real switching classes of 11 canonical regular graphs using strictly integer and rational arithmetic.

Theorem 3 (Exhaustive 11-Graph Zero-Float Real Switching-Class Atlas)**.** *Across the* 11 *canonical d-regular graphs listed in Table 1, exhaustive zero-float Faddeev–LeVerrier characteristic-polynomial evaluation and rational Sturm root isolation over all* 2^{β_1} *real switching classes certify:*

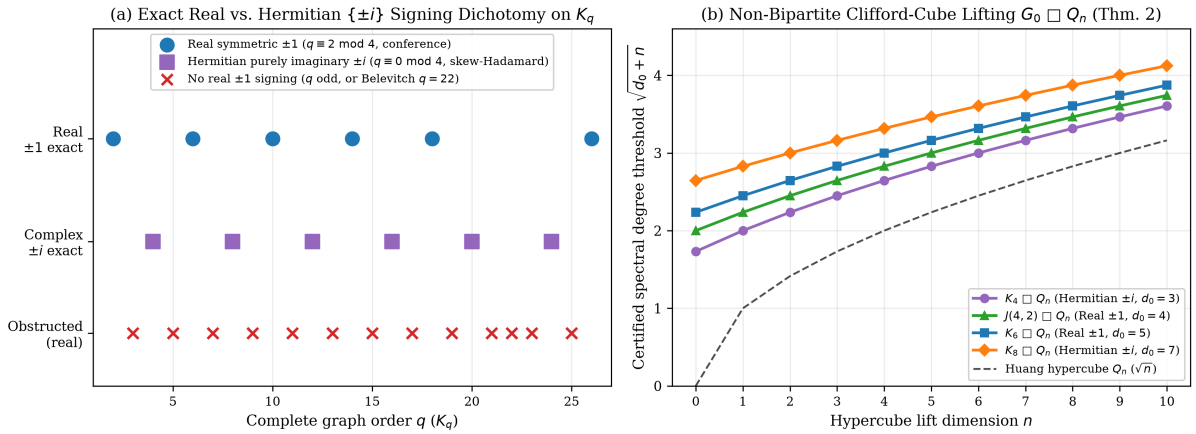

Figure 2: **Complex-vs-real signing dichotomy on K_q and non-bipartite Clifford-cube lifting.** (a) Exact Huang-optimal signings on K_q ($2 \le q \le 26$): real symmetric ± 1 conference signings at $q \equiv 2 \pmod 4$ (blue), purely imaginary Hermitian $\{\pm i\}$ skew-Hadamard signings at $q \equiv 0 \pmod 4$ (purple), and real-obstructed orders (red crosses, including odd q and Belevitch $q = 22$). (b) Certified spectral degree threshold $\sqrt{d_0 + n}$ along non-bipartite Clifford-cube lifts $G_0 \square Q_n$ for $G_0 \in \{K_4, J(4,2), K_6, K_8\}$, strictly dominating Huang's hypercube baseline $\sqrt{n}$.

(a) Four real Huang-optimal graphs ($S^2 = dI$): K_6 *($\beta_1 = 10$, 16 distinct spectra, characteristic polynomial $(x^2-5)^3$), Q_3 ($\beta_1 = 5$, 6 spectra, $(x^2-3)^4$), the non-bipartite octahedron $J(4,2) = K_{2,2,2}$ ($\beta_1 = 7$, 14 spectra, $(x^2-4)^3$), and $K_{4,4}$ ($\beta_1 = 9$, 10 spectra, $(x^2-4)^4$).*

(b) Seven real-obstructed graphs ($\|S\|_{\mathrm{op}} > \sqrt{d}$ for every real signing): K_3 *($\min \|S\|_{\mathrm{op}} = 2$, $\max \lambda_2 = 1$), K_4 ($\sqrt{5}$, 1), K_5 ($\sqrt{5}$, $\max \lambda_3 = 1$), the triangular prism $K_3 \square K_2$ ($\sqrt{5}$, $\max \lambda_3 = \sqrt{2}$), the Petersen graph $KG(5,2)$ ($\sqrt{5}$, $\max \lambda_5 = 1$), the pentagonal prism $C_5 \square K_2$ ($\sqrt{5}$, $\max \lambda_5 \approx 1.1756$), and $H(2,3) = K_3 \square K_3$ ($\frac{1+\sqrt{17}}{2}$, $\max \lambda_5 = 1$, $\max \lambda_4 = 2$).*

Proof. For each connected graph $G = (V, E)$, any two real ± 1 signings with identical cycle parities are diagonally sign-conjugate ($S' = DSD$) and hence cospectral [11]. Fixing $+1$ on all $n - 1$ edges of a spanning tree leaves $\beta_1 = |E| - |V| + 1$ chord edges and exactly 2^{β_1} canonical switching-class representatives. Certificates `cert01`–`cert11` (summarized in `cert16`) evaluate $\det(xI - S) \in \mathbb{Z}[x]$ via exact integer Faddeev–LeVerrier for every mask $0 \le m < 2^{\beta_1}$, verify exact divisibility at every trace step, and apply squarefree-decomposed rational Sturm chains to bracket $\min \|S\|_{\mathrm{op}}$ and $\max \lambda_{\lceil n/2 \rceil}(S)$ (as well as $\max \lambda_{\lfloor n/2 \rfloor}(S)$) to 20 bisection steps over $\mathbb{Q}$, plus direct integer matrix squaring $S^2 = dI$ on witnesses. All exact values and strict separations in Table 1 follow directly from the certified rational brackets. □

5 Formal Verification Architecture and Scope

To keep formal proof guarantees transparent and distinct from computational enumeration, our verification artifact is organized in two layers:

- **Layer 1 — Constructive Lean 4.36.0 Kernel** (`lean/TorusHolonomyUnitarySignings.lean`): Twenty theorems compiled with `lean-DwarningAsError=true`, zero external dependencies, zero `sorry`, zero `grind`, and zero `Classical.choice` (audited via `#print axioms`). Layer 1 machine-checks the odd-times-odd parity and universal torus-holonomy contradiction $(-1)^{(2a+1)(2b+1)} \neq 1$ (`thm01`–`thm02`), the K_q modular obstructions ruling out odd q and $q = 4m$ (`thm03`–`thm05`), the Clifford-cube lifting scalar recurrence and half-space vertex doubling (`thm06`–`thm07`), the exact spectral gap inequalities for

Table 1: Exhaustive zero-float real switching-class atlas across 11 canonical d-regular graphs (Certificates `cert01`–`cert11`, `cert16`). Here $k_{\text{med}} = \lceil n/2 \rceil$ is the Cauchy-interlacing index for induced subgraphs of size $> n/2$.

Graph G	n	d	β_1	2^{β_1}	Spectra	$\min \|S\|_{\text{op}}$	$\max \lambda_{\lceil n/2 \rceil}$	Real $S^2 = dI$?
K_3	3	2	1	2	2	$2 > \sqrt{2}$	1	No (odd n)
K_4	4	3	3	8	3	$\sqrt{5} > \sqrt{3}$	1	No (Complex $\pm i$: Yes)
K_5	5	4	6	64	7	$\sqrt{5} > 2$	1	No (odd n)
K_6	6	5	10	1,024	16	$\sqrt{5}$	$\sqrt{5}$	**Yes** (Paley C_6)
Q_3	8	3	5	32	6	$\sqrt{3}$	$\sqrt{3}$	**Yes** (Huang)
$J(4,2)$	6	4	7	128	14	2	2	**Yes** (non-bipartite)
$K_{4,4}$	8	4	9	512	10	2	2	**Yes** (Hadamard)
$K_3 \square K_2$	6	3	4	16	6	$\sqrt{5} > \sqrt{3}$	$\sqrt{2}$	No
Petersen $KG(5,2)$	10	3	6	64	6	$\sqrt{5} > \sqrt{3}$	1	No
$C_5 \square K_2$	10	3	6	64	12	$\sqrt{5} > \sqrt{3}$	≈ 1.1756	No
$H(2,3) = K_3 \square K_3$	9	4	10	1,024	43	$\frac{1+\sqrt{17}}{2} > 2$	1 ($\lambda_4 = 2$)	No (Complex no-go, Thm. 1)

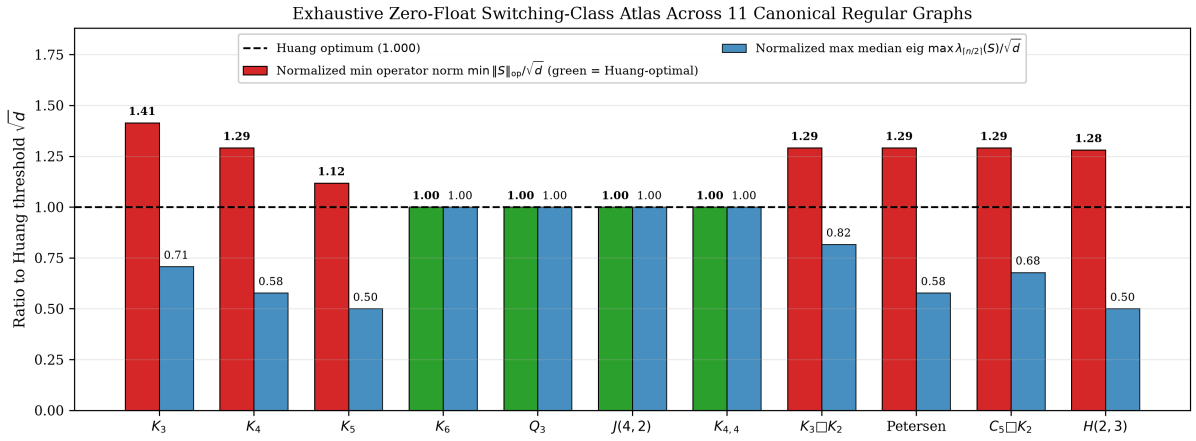

Figure 3: **Normalized spectral ratios across all 11 exhaustively classified regular graphs.** Red/green bars show $\min \|S\|_{\text{op}}/\sqrt{d}$ (green = Huang-optimal real $S^2 = dI$ attained on $K_6, Q_3, J(4,2), K_{4,4}$); blue bars show the Cauchy-interlacing median ratio $\max \lambda_{\lceil n/2 \rceil}(S)/\sqrt{d}$, revealing the exact factor-2 collapse on K_5 and $H(2,3)$ and $1/\sqrt{3}$ collapse on K_4 and Petersen.

$K_4, K_3 \square K_2$, Petersen, $C_5 \square K_2$, and $H(2,3)$ (`thm08`–`thm12`), the skewed $H(2,3)$ spectral moment identities and rational Sturm sign-change isolation of $x^2 - x - 4$ (`thm13`–`thm14`), the quantitative Frobenius defect bound $8v \geq 8$ (`thm15`), and the skew-Hadamard Paley orders, cycle-space dimensions, octahedron/$K_{4,4}$ moments, lifted degree thresholds, and exhaustive switching-class counts (`thm16`–`thm20`). The matrix-level plaquette cancellation, Hermitian interlacing, and conference sign-counting derivations are proved in full detail in Sections 2–3.

- **Layer 2 — Zero-Float Exact Python Verifier** (`verification/verify_unitary_signings_huang.py`): An AST-audited pure standard-library verifier using strictly `int` and `fractions.Fraction` (zero floating-point literals or calls, zero numeric division or exponentiation, zero `limit_denominator`), emitting 16 SHA-256 certificates under `certificates/MANIFEST.json` and validated by 56 adversarial `pytest` checks in `tests/test_adversarial.py`. Layer 2 independently executes exact integer Faddeev–LeVerrier characteristic polynomials and squarefree rational Sturm isolation across all 11 graphs (Certificates `cert01`–`cert11`, `cert16`), exact $\mathbb{Z}[i]$ Hermitian verification $H_q^2 = (q-1)I_q$

for $q \in \{4, 8, 12, 16, 20, 24\}$ (`cert12`), real Paley conference matrices $C_6, C_{14}, C_{18}, C_{30}$ (`cert13`), 11 non-bipartite Clifford-cube lifts $G_0 \square Q_n$ across 4 seeds (`cert14`), and direct discrete-torus edge cancellation across 5 odd-by-odd tori (`cert15`).

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