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Repulsion-Laplacian Variance Decomposition, Sharp Signed/Unsigned Sub-Gaussian Concentration, Canonical-Correlation Conditioning, and Trace-Finsler Stability of Determinantal Point Processes

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Abstract

Let $Y \sim \mathrm{DPP}(K)$ be a finite symmetric Determinantal Point Process on $[n] := \{1, \dots, n\}$ ($n \geq 2$) with real symmetric marginal kernel $0 \preceq K \preceq I_n$ (encompassing both L-ensembles $K = L(I_n + L)^{-1}$ for $L \succeq 0$ and fixed-cardinality projection DPPs $K^2 = K$). We establish four exact, non-asymptotic structural theorems uniting spectral graph theory, exponential-tilting concentration, Schur-complement conditioning, and symmetric-cone Finsler geometry.

First, for any linear statistic $X_f := \sum_{i \in Y} f_i$ ($f \in \mathbb{R}^n$), we prove the exact *Repulsion-Laplacian Variance Decomposition*

$$\mathrm{Var}_{\mathrm{DPP}}(X_f) \;=\; f^\top \big(K \circ (I_n - K)\big) f \;=\; f^\top D_{K-K^2} f + f^\top \mathcal{L}_{K \circ K} f \;=\; \big\| (I_n - K)^{1/2} D_f K^{1/2} \big\|_F^2,$$

where $\mathcal{L}_{K \circ K}$ is the combinatorial Laplacian of the repulsion graph $W_{ij} = K_{ij}^2$ ($i \neq j$). For projection DPPs ($K^2 = K$), this reduces identically to pure Dirichlet cut hyperuniformity $\mathrm{Var}(|Y \cap S|) = \|P_S K P_{S^c}\|_F^2$. Using spectral symmetrization over $\mathrm{spec}(K) \subseteq [\alpha, \beta] \subseteq [0, 1]$, we derive a sharp three-phase variance ceiling $\nu(\alpha, \beta)\|f\|_2^2$ establishing an exact *factor-of-2 signed vs. unsigned dichotomy*: $\mathrm{Var}_{\mathrm{DPP}}(X_f) \leq \frac{1}{2}\|f\|_2^2$ for arbitrary signed $f \in \mathbb{R}^n$ (attained on $K = \frac{1}{2}\mathbf{1}\mathbf{1}^\top$, $f = (1, -1)^\top$, where negative dependence *doubles* independent Bernoulli variance), versus $\mathrm{Var}_{\mathrm{DPP}}(X_f) \leq \frac{1}{4}\|f\|_2^2$ for all sign-definite $f \in \mathbb{R}^n_{\geq 0}$.

Second, we prove exact closure of arbitrary DPPs (including projection DPPs without L-ensemble representations) under exponential tilting, with explicit symmetric marginal kernel $K^{(\theta)} = e^{\frac{\theta}{2} D_f} K (I_n - K + e^{\theta D_f} K)^{-1} e^{\frac{\theta}{2} D_f}$ ($0 \preceq K^{(\theta)} \preceq I_n$), yielding $\Lambda_f''(\theta) = \mathrm{Var}_{K^{(\theta)}}(X_f)$ and immediate sharp sub-Gaussian tails $\mathbb{P}(|X_f - \mathbb{E}X_f| \geq t) \leq 2\exp(-t^2/\|f\|_2^2)$ for signed f and $2\exp(-2t^2/\|f\|_2^2)$ for unsigned $f \geq 0$.

Third, for conditioning on $A \subseteq Y$ ($\det(K_{A,A}) > 0$) and any disjoint $B \subseteq [n]$, we prove the exact repulsion trace drop $\mathbb{E}[|Y \cap B|] - \mathbb{E}[|Y \cap B| \mid A \subseteq Y] = \|K_{B,A} K_{A,A}^{-1/2}\|_F^2$ and the exact canonical-correlation inclusion law $\mathbb{P}(A \cup B \subseteq Y) = \mathbb{P}(A \subseteq Y)\mathbb{P}(B \subseteq Y) \prod_{j=1}^r (1 - \rho_j^2)$ in terms of the singular values ρ_j of $K_{A,A}^{-1/2} K_{A,B} K_{B,B}^{-1/2}$.

Fourth, equipping the positive-definite cone $\mathbb{P}_n^{++}$ with the affine-invariant trace-Finsler metric $d_1(L_1, L_2) := \|\log(L_1^{-1/2} L_2 L_1^{-1/2})\|_{S_1}$, we prove via Poincaré eigenvalue interlacing that the L-ensemble map $L \mapsto \mathrm{DPP}_L(L)$ is universally 1-Lipschitz into Hilbert's projective metric, $d_H(\mathrm{DPP}_L(L_1), \mathrm{DPP}_L(L_2)) \leq d_1(L_1, L_2)$—and an *exact isometry* on all coordinate-diagonal pairs and all Loewner-ordered non-commuting pairs ($L_2 \succeq L_1$ or $L_1 \succeq L_2$)—yielding sharp nonlinear Total Variation and Jeffreys-divergence perturbation bounds $\tanh(d_1/4)$ and $d_1 \tanh(d_1/4)$.

Keywords: determinantal point processes, negative dependence, hyperuniformity, sub-Gaussian concentration, Schur complement, canonical correlations, trace-Finsler metric, Hilbert projective metric.

1 Introduction

Finite Determinantal Point Processes (DPPs) [7, 9, 6, 5] model repulsive random subsets $Y \subseteq [n]$ whose inclusion probabilities are principal minors of a symmetric marginal kernel $K \in \mathbb{R}^{n\times n}$ satisfying $0 \preceq K \preceq I_n$:

$$\mathbb{P}(S \subseteq Y) \;=\; \det(K_{S,S}) \qquad \forall S \subseteq [n] \quad (\det(K_{\emptyset,\emptyset}) := 1). \tag{1}$$

When $K \prec I_n$, Y admits an L-ensemble representation $L := K(I_n - K)^{-1} \succeq 0$ (conversely $K = L(I_n + L)^{-1}$) with explicit subset probabilities $\mathbb{P}_L(Y = S) = \det(L_{S,S})/\det(I_n + L)$; when $K^2 = K$ (orthogonal projection of rank r), Y is a projection DPP supported on subsets of exact size $|Y| = r$ with no finite L-ensemble.

DPPs are canonical prototypes of strong Rayleigh negative dependence [2], random matrix eigenvalue statistics, randomized numerical linear algebra, and diverse subset selection [5, 8]. While concentration for Lipschitz or monotone functions of DPPs has been studied via stochastic dominance and strong Rayleigh geometry [8, 4], four fundamental non-asymptotic structural laws have remained separate or unsharpened:

1. **Graph-Laplacian repulsion structure and signed vs. unsigned variance dichotomy.** How does $\mathrm{Var}_{\mathrm{DPP}}(X_f)$ decompose into diagonal randomness and pairwise repulsion energy, and why can negative dependence *double* variance for signed statistics while strictly reducing variance for unsigned statistics?
2. **Unified exponential-tilting closure for projection and L-ensemble DPPs.** Can arbitrary linear statistics X_f (signed or unsigned, projection or L-ensemble) be analyzed through a single closed-form tilted DPP kernel $K^{(\theta)}$ yielding sharp sub-Gaussian tails without strong-Rayleigh induction?
3. **Exact canonical-correlation factorization under Schur conditioning.** How does conditioning on $A \subseteq Y$ quantitatively contract both expected counts and joint inclusion probabilities on a disjoint set B?
4. **Intrinsic geometric stability of L-ensembles under non-commuting kernel perturbations.** What affine-invariant metric on $\mathbb{P}_n^{++}$ governs exact Hilbert-projective, Total Variation, and symmetrized KL stability of $L \mapsto \mathrm{DPP}_L(L)$ without requiring commutativity?

Summary of Main Results

We resolve all four questions through four tightly linked, self-contained theorems:

- **Theorem 2.1 (Repulsion-Laplacian Decomposition and Factor-of-2 Variance Law):** Proves $\mathrm{Var}_{\mathrm{DPP}}(X_f) = f^\top D_{K-K^2} f + f^\top \mathcal{L}_{K\circ K} f = \|(I_n - K)^{1/2} D_f K^{1/2}\|_F^2$, projection cut hyperuniformity, a three-phase spectral ceiling $\nu(\alpha,\beta)\|f\|_2^2$, and the sharp factor-of-2 dichotomy ($\frac{1}{2}\|f\|_2^2$ signed vs. $\frac{1}{4}\|f\|_2^2$ unsigned).
- **Theorem 3.1 (Exponential-Tilting Closure and Sharp Sub-Gaussian Tails):** Proves exact DPP closure under $e^{\theta X_f}$ for all $0 \preceq K \preceq I_n$ via $K^{(\theta)} = e^{\frac{\theta}{2}D_f} K (I_n - K + e^{\theta D_f} K)^{-1} e^{\frac{\theta}{2}D_f}$, immediately yielding exact CGF second derivatives $\Lambda_f''(\theta) = \mathrm{Var}_{K^{(\theta)}}(X_f)$ and sharp sub-Gaussian tails.
- **Theorem 4.1 (Schur Conditioning Trace Drop and Canonical-Correlation Law):** Establishes the exact repulsion trace drop $\|K_{B,A} K_{A,A}^{-1/2}\|_F^2$ and exact canonical-correlation factorization $\mathbb{P}(A \cup B \subseteq Y) = \mathbb{P}(A \subseteq Y)\mathbb{P}(B \subseteq Y) \prod_{j=1}^r (1 - \rho_j^2)$.
- **Theorem 5.1 (Trace-Finsler Isometry and L-Ensemble Perturbation Stability):** Proves universal 1-Lipschitz stability $d_H(\mathrm{DPP}_L(L_1), \mathrm{DPP}_L(L_2)) \le d_1(L_1, L_2)$, exact isometry

for all diagonal and all Loewner-ordered non-commuting pairs, and sharp hyperbolic Total Variation ($\tanh(d_1/4)$) and Jeffreys-divergence ($d_1 \tanh(d_1/4)$) bounds.

All proofs and exact 2^n-enumeration benchmarks (4/4 pytest modules) are openly archived at: https://github.com/lluiseriksson/dpp-repulsion-geometry.

2 Repulsion-Laplacian Decomposition and Factor-of-2 Variance Law

For any symmetric $K \in \mathbb{R}^{n\times n}$ with $0 \preceq K \preceq I_n$, let $K \circ M$ denote the Hadamard (entrywise) product, $D_f := \operatorname{diag}(f)$, $D_{K-K^2} := \operatorname{diag}(K_{11} - (K^2)_{11}, \dots, K_{nn} - (K^2)_{nn})$, and let $\mathcal{L}_{K\circ K}$ be the combinatorial graph Laplacian of the nonnegative undirected repulsion graph with edge weights $W_{ij} := K_{ij}^2$ $(i \neq j)$, i.e., $(\mathcal{L}_{K\circ K})_{ii} = \sum_{j\neq i} K_{ij}^2$ and $(\mathcal{L}_{K\circ K})_{ij} = -K_{ij}^2$ $(i \neq j)$. Let $Y^{\mathrm{ind}} \sim \bigotimes_{i=1}^n \mathrm{Bern}(K_{ii})$ denote an independent Bernoulli vector matching the marginals of $Y \sim \mathrm{DPP}(K)$.

Theorem 2.1 (Repulsion-Laplacian Variance Decomposition and Three-Phase Spectral Ceiling)**.** *Let $0 \preceq K \preceq I_n$ $(n \geq 2)$ with $\operatorname{spec}(K) \subseteq [\alpha, \beta] \subseteq [0, 1]$, and let $Y \sim \mathrm{DPP}(K)$ and $X_f := \sum_{i\in Y} f_i$ $(f \in \mathbb{R}^n)$.*

(a) ***Exact Repulsion-Laplacian and Operator-Conjugate Identities.*** *For every $f \in \mathbb{R}^n$,*

$$\mathrm{Var}_{\mathrm{DPP}}(X_f) \;=\; f^\top (K\circ(I_n - K))f \;=\; f^\top D_{K-K^2} f + f^\top \mathcal{L}_{K\circ K} f \;=\; \|(I_n - K)^{1/2} D_f K^{1/2}\|_F^2, \tag{2}$$

where $D_{K-K^2} \succeq 0$ and $\mathcal{L}_{K\circ K} \succeq 0$. In particular, for any projection DPP ($K^2 = K$), $D_{K-K^2} = 0$, $\mathrm{Var}_{\mathrm{DPP}}(X_f) = \frac{1}{2}\sum_{i\neq j} K_{ij}^2 (f_i - f_j)^2$, and for every spatial window $S \subseteq [n]$ with coordinate projection $P_S := \operatorname{diag}(\mathbf{1}_S)$ and complement $P_{S^c} := I_n - P_S$,

$$\mathrm{Var}_{\mathrm{DPP}}(|Y \cap S|) \;=\; \sum_{i\in S,\, j\notin S} K_{ij}^2 \;=\; \|P_S K P_{S^c}\|_F^2. \tag{3}$$

(b) ***Three-Phase Spectral Ceiling and Sharp Signed/Unsigned Dichotomy.*** *Define*

$$\nu(\alpha,\beta) \;:=\; \begin{cases} \beta(1-\beta), & \beta \leq \frac{1}{2}, \\ \alpha(1-\alpha), & \alpha \geq \frac{1}{2}, \\ \frac{1}{2}(\alpha+\beta-2\alpha\beta), & \alpha \leq \frac{1}{2} \leq \beta. \end{cases} \tag{4}$$

Then for every signed $f \in \mathbb{R}^n$,

$$\mathrm{Var}_{\mathrm{DPP}}(X_f) \;\leq\; \nu(\alpha,\beta)\,\|f\|_2^2 \;\leq\; \frac{1}{2}\,\|f\|_2^2, \tag{5}$$

with $\frac{1}{2}\|f\|_2^2$ attained identically on the rank-1 projection DPP $K = \frac{1}{2}\mathbf{1}_2\mathbf{1}_2^\top$ with $f = (1, -1)^\top$ (zero-padded by 0 for $n > 2$, where $\mathrm{Var}_{\mathrm{DPP}}(X_f) = 1 = 2\,\mathrm{Var}_{\mathrm{ind}}(X_f)$). Conversely, for every sign-definite $f \in \mathbb{R}^n_{\geq 0}$ (or $\mathbb{R}^n_{\leq 0}$),

$$\mathrm{Var}_{\mathrm{DPP}}(X_f) \;\leq\; \mathrm{Var}_{\mathrm{ind}}(X_f) \;=\; \sum_{i=1}^n f_i^2 K_{ii}(1 - K_{ii}) \;\leq\; \frac{1}{4}\,\|f\|_2^2, \tag{6}$$

and whenever $\sum_i K_{ii}(1 - K_{ii}) > 0$, the exact cardinality variance-reduction ratio is

$$\frac{\mathrm{Var}_{\mathrm{DPP}}(|Y|)}{\mathrm{Var}_{\mathrm{ind}}(|Y|)} \;=\; \frac{\mathrm{Tr}(K - K^2)}{\sum_{i=1}^n K_{ii}(1 - K_{ii})} \;=\; 1 - \frac{\|K - \operatorname{diag}(K)\|_F^2}{\sum_{i=1}^n K_{ii}(1 - K_{ii})} \;\in\; [0, 1]. \tag{7}$$

Proof. **(a)** From (1), $\mathbb{P}(i \in Y) = K_{ii}$ and $\mathbb{P}(\{i,j\} \subseteq Y) = K_{ii}K_{jj} - K_{ij}^2$ $(i \neq j)$, so $\mathrm{Cov}(\mathbf{1}_{i\in Y}, \mathbf{1}_{j\in Y}) = K_{ii}(1-K_{ii})\delta_{ij} - K_{ij}^2(1-\delta_{ij}) = (K \circ (I_n - K))_{ij}$. Hence

$$\mathrm{Var}_{\mathrm{DPP}}(X_f) = \sum_{i=1}^n f_i^2 K_{ii}(1-K_{ii}) - \sum_{i\neq j} f_i f_j K_{ij}^2.$$

Using $(K^2)_{ii} = K_{ii}^2 + \sum_{j\neq i} K_{ij}^2$, rewrite $K_{ii}(1-K_{ii}) = (K_{ii} - (K^2)_{ii}) + \sum_{j\neq i} K_{ij}^2$. Substituting gives

$$\mathrm{Var}_{\mathrm{DPP}}(X_f) = \sum_{i=1}^n (K_{ii} - (K^2)_{ii}) f_i^2 + \frac{1}{2}\sum_{i\neq j} K_{ij}^2 (f_i - f_j)^2 = f^\top D_{K-K^2} f + f^\top \mathcal{L}_{K\circ K} f.$$

Since $0 \preceq K \preceq I_n$ implies $K - K^2 = K(I_n - K) \succeq 0$, every diagonal entry $(K-K^2)_{ii} \geq 0$, so $D_{K-K^2} \succeq 0$. Moreover,

$$\|(I_n - K)^{1/2} D_f K^{1/2}\|_F^2 = \mathrm{Tr}(D_f K D_f (I_n - K)) = \sum_{i=1}^n f_i^2 K_{ii} - \sum_{i,j=1}^n f_i f_j K_{ij}^2 = \mathrm{Var}_{\mathrm{DPP}}(X_f),$$

proving (2). When $K^2 = K$, $D_{K-K^2} = 0$, and setting $f = \mathbf{1}_S$ yields (3).

(b) Let $K = \sum_{a=1}^n \lambda_a u_a u_a^\top$ be an orthonormal eigendecomposition with $\lambda_a \in [\alpha, \beta] \subseteq [0,1]$, and set $M_{ab} := u_a^\top D_f u_b = M_{ba}$. Expanding the Frobenius norm in (2) and symmetrizing $(a,b) \leftrightarrow (b,a)$ gives

$$\mathrm{Var}_{\mathrm{DPP}}(X_f) \;=\; \sum_{a,b=1}^n (1-\lambda_a)\lambda_b\, M_{ab}^2 \;=\; \sum_{a,b=1}^n \frac{\lambda_a + \lambda_b - 2\lambda_a\lambda_b}{2}\, M_{ab}^2.$$

The bilinear function $h(u,v) := \frac{1}{2}(u + v - 2uv)$ is affine in each variable separately on $[\alpha,\beta]^2$, so its maximum is attained at a vertex (α,α), (β,β), or (α,β). Comparing values via $h(\alpha,\beta) - h(\alpha,\alpha) = (\beta-\alpha)(\frac{1}{2} - \alpha)$ and $h(\alpha,\beta) - h(\beta,\beta) = (\beta-\alpha)(\beta - \frac{1}{2})$ yields $\max_{[\alpha,\beta]^2} h = \nu(\alpha,\beta) \leq \nu(0,1) = \frac{1}{2}$. Since $\sum_{a,b} M_{ab}^2 = \|D_f\|_F^2 = \|f\|_2^2$, (5) follows immediately. For $K = \frac{1}{2}\mathbf{1}_2\mathbf{1}_2^\top$ and $f = (1,-1)^\top$, Y is $\{1\}$ or $\{2\}$ with probability $\frac{1}{2}$ each, so $X_f \in \{+1,-1\}$ symmetrically and $\mathrm{Var}_{\mathrm{DPP}}(X_f) = 1 = \frac{1}{2}\|f\|_2^2 = 2\,\mathrm{Var}_{\mathrm{ind}}(X_f)$. For $f \in \mathbb{R}^n_{\geq 0}$ (or $\mathbb{R}^n_{\leq 0}$), $f_i f_j K_{ij}^2 \geq 0$ for all $i \neq j$, so dropping $-\sum_{i\neq j} f_i f_j K_{ij}^2 \leq 0$ gives (6) (since $t(1-t) \leq \frac{1}{4}$ on $[0,1]$). Setting $f = \mathbf{1}_n$ in (2) gives $\mathrm{Var}_{\mathrm{DPP}}(|Y|) = \mathrm{Tr}(K - K^2) = \sum_i K_{ii}(1-K_{ii}) - \sum_{i\neq j} K_{ij}^2$, proving (7). □

3 Exponential-Tilting DPP Closure and Sharp Sub-Gaussian Concentration

For any $0 \preceq K \preceq I_n$ and $f \in \mathbb{R}^n$, recall the cumulant generating function (CGF) of $X_f = \sum_{i\in Y} f_i$ [4, 5]:

$$\Lambda_f(\theta) \;:=\; \log \mathbb{E}\big[e^{\theta X_f}\big] \;=\; \log\det\Big(I_n + \big(e^{\theta D_f} - I_n\big)K\Big), \qquad \theta \in \mathbb{R}. \tag{8}$$

Theorem 3.1 (Unified Exponential-Tilting DPP Closure and Sharp Sub-Gaussian Tails)**.** *Let $0 \preceq K \preceq I_n$ ($n \geq 2$, allowing both L-ensembles and projection DPPs) and $f \in \mathbb{R}^n$.*

(a) ***Exact Tilted DPP Closure.*** *For every $\theta \in \mathbb{R}$, the matrix $I_n - K + e^{\theta D_f} K$ is nonsingular, and the exponentially tilted law $\mathbb{P}_\theta(Y = S) := e^{\theta\sum_{i\in S} f_i - \Lambda_f(\theta)}\mathbb{P}(Y = S)$ is again a symmetric DPP with marginal kernel*

$$K^{(\theta)} \;:=\; e^{\frac{\theta}{2}D_f}\, K\big(I_n - K + e^{\theta D_f}K\big)^{-1} e^{\frac{\theta}{2}D_f}, \qquad 0 \;\preceq\; K^{(\theta)} \;\preceq\; I_n. \tag{9}$$

When $K = UU^\top$ is a rank-r projection ($r \geq 1$, $U^\top U = I_r$), (9) simplifies to the orthogonal projection $K^{(\theta)} = W_\theta(W_\theta^\top W_\theta)^{-1}W_\theta^\top$ where $W_\theta := e^{\frac{\theta}{2}D_f}U$ (while $r = 0$ gives $K^{(\theta)} = 0$). Moreover,

$$\Lambda_f'(\theta) \;=\; \mathrm{Tr}(D_f K^{(\theta)}), \qquad \Lambda_f''(\theta) \;=\; \mathrm{Var}_{K^{(\theta)}}(X_f) \;=\; f^\top(K^{(\theta)} \circ (I_n - K^{(\theta)}))f. \tag{10}$$

(b) ***Sharp Signed and Unsigned Sub-Gaussian Concentration.*** *For every $t \geq 0$,*

$$\mathbb{P}(|X_f - \mathbb{E}X_f| \geq t) \;\leq\; 2\exp\left(-\frac{t^2}{\|f\|_2^2}\right) \qquad \text{for all } f \in \mathbb{R}^n \setminus \{0\}, \tag{11}$$

$$\mathbb{P}(|X_f - \mathbb{E}X_f| \geq t) \;\leq\; 2\exp\left(-\frac{2t^2}{\|f\|_2^2}\right) \qquad \text{for all } f \in (\mathbb{R}^n_{\geq 0} \cup \mathbb{R}^n_{\leq 0}) \setminus \{0\}. \tag{12}$$

When $K^2 = K$ (projection DPP) and $f \notin \mathbb{R}\mathbf{1}_n$, $X_f - \mathbb{E}X_f$ is invariant under shifting $f \mapsto f - c\mathbf{1}_n$, so $\|f\|_2^2$ in (11) improves automatically to $\|f - \bar{f}\mathbf{1}_n\|_2^2$ ($\bar{f} := \frac{1}{n}\sum_i f_i$).

Proof. **(a)** First assume $0 \prec K \prec I_n$, so $K = L(I_n + L)^{-1}$ with $L = K(I_n - K)^{-1} \succ 0$. For every $S \subseteq [n]$, $e^{\theta \sum_{i\in S} f_i}\det(L_{S,S}) = \det((e^{\frac{\theta}{2}D_f} L e^{\frac{\theta}{2}D_f})_{S,S})$, so $\mathbb{P}_\theta$ is the L-ensemble with $L^{(\theta)} := e^{\frac{\theta}{2}D_f} L e^{\frac{\theta}{2}D_f} \succ 0$, whose marginal kernel is

$$\begin{aligned} K^{(\theta)} &= L^{(\theta)}\big(I_n + L^{(\theta)}\big)^{-1} = e^{\frac{\theta}{2}D_f} K(I_n - K)^{-1} e^{\frac{\theta}{2}D_f}\Big(I_n + e^{\frac{\theta}{2}D_f} K(I_n - K)^{-1} e^{\frac{\theta}{2}D_f}\Big)^{-1} \\ &= e^{\frac{\theta}{2}D_f} K\Big(e^{-\frac{\theta}{2}D_f}(I_n - K) + e^{\frac{\theta}{2}D_f} K\Big)^{-1} = e^{\frac{\theta}{2}D_f} K\big(I_n - K + e^{\theta D_f} K\big)^{-1} e^{\frac{\theta}{2}D_f}, \end{aligned}$$

which is symmetric and satisfies $0 \prec K^{(\theta)} \prec I_n$. For general $0 \preceq K \preceq I_n$ (including boundary eigenvalues 0 and 1), suppose $v \in \mathbb{R}^n$ is a right nullvector of $M_\theta := I_n - K + e^{\theta D_f} K$, so $(I_n - K)v = -e^{\theta D_f} K v$. Taking the inner product with Kv and using $K(I_n - K) \succeq 0$ and $e^{\theta D_f} \succ 0$ gives $0 \leq v^\top K(I_n - K)v = -(Kv)^\top e^{\theta D_f}(Kv) \leq 0$, forcing $Kv = 0$ and hence $(I_n - K)v = 0$, so $v = 0$. Thus M_θ is universally nonsingular for every $0 \preceq K \preceq I_n$. Approximating $K_\varepsilon := (1-\varepsilon)K + \frac{\varepsilon}{2} I_n$ ($\varepsilon \downarrow 0^+$) yields $0 \prec K_\varepsilon \prec I_n$, so every principal minor of $K_\varepsilon^{(\theta)}$ converges to the corresponding tilted inclusion probability $\mathbb{P}_\theta(S \subseteq Y)$ while preserving symmetry and $0 \preceq K^{(\theta)} \preceq I_n$ in the limit. For $K = UU^\top$ ($r \geq 1$), set $V_1 := e^{-\frac{\theta}{2}D_f}\operatorname{col}(U)^\perp$ and $V_2 := e^{\frac{\theta}{2}D_f}\operatorname{col}(U) = \operatorname{col}(W_\theta)$; for any $w \in \operatorname{col}(U)^\perp$ and $u \in \operatorname{col}(U)$, $\langle e^{-\frac{\theta}{2}D_f} w, e^{\frac{\theta}{2}D_f} u\rangle = \langle w, u\rangle = 0$, so $V_1 = V_2^\perp$ and $\mathbb{R}^n = V_1 \oplus V_2$. Direct action of (9) shows that $K^{(\theta)}$ vanishes on V_1 and acts as the identity on V_2, matching the orthogonal projection $W_\theta(W_\theta^\top W_\theta)^{-1}W_\theta^\top$ identically. Standard exponential-family differentiation of $\Lambda_f(\theta) = \log\sum_S e^{\theta X_f(S)}\mathbb{P}(S)$ yields $\Lambda_f'(\theta) = \mathbb{E}_\theta[X_f] = \operatorname{Tr}(D_f K^{(\theta)})$ and $\Lambda_f''(\theta) = \operatorname{Var}_\theta(X_f)$, giving (10) via Theorem 2.1(a).

(b) Because $0 \preceq K^{(\theta)} \preceq I_n$ for every $\theta \in \mathbb{R}$, Theorem 2.1(b) gives $\Lambda_f''(\theta) = \operatorname{Var}_{K^{(\theta)}}(X_f) \leq \sigma^2$ globally for all $\theta \in \mathbb{R}$, with $\sigma^2 = \frac{1}{2}\|f\|_2^2$ for arbitrary $f \in \mathbb{R}^n$ and $\sigma^2 = \frac{1}{4}\|f\|_2^2$ for $f \in \mathbb{R}^n_{\geq 0}$ (or $\mathbb{R}^n_{\leq 0}$). Integrating twice from $\theta = 0$ ($\Lambda_f(0) = 0$, $\Lambda_f'(0) = \mathbb{E}X_f$) yields $\Lambda_f(\theta) - \theta\,\mathbb{E}X_f \leq \frac{1}{2}\sigma^2\theta^2$ for all $\theta \in \mathbb{R}$. Applying the Chernoff bound $\mathbb{P}(X_f - \mathbb{E}X_f \geq t) \leq \inf_{\theta>0}\exp(\frac{1}{2}\sigma^2\theta^2 - \theta t) = \exp(-t^2/(2\sigma^2))$ to both X_f and $-X_f$ proves (11) and (12). When $K^2 = K$, $|Y| = r$ is deterministic, so $X_f - \mathbb{E}X_f = X_{f-c\mathbf{1}_n} - \mathbb{E}X_{f-c\mathbf{1}_n}$ for all $c \in \mathbb{R}$, minimized at $c = \bar{f}$. □

4 Schur-Complement Conditioning and Canonical-Correlation Law

Theorem 4.1 (Schur-Complement Repulsion Trace Drop and Canonical-Correlation Inclusion Law)**.** *Let $Y \sim \mathrm{DPP}(K)$ with $0 \preceq K \preceq I_n$, and let $A, B \subseteq [n]$ be nonempty disjoint subsets with $\mathbb{P}(A \subseteq Y) = \det(K_{A,A}) > 0$.*

(a) ***Schur-Complement Kernel and Exact Repulsion Trace Drop.*** *Conditional on $A \subseteq Y$, the restriction $Y \cap B$ is a symmetric DPP on B with marginal kernel given by the Schur complement*

$$K^A_{B,B} \;:=\; K_{B,B} - K_{B,A}K_{A,A}^{-1}K_{A,B}, \qquad 0 \;\preceq\; K^A_{B,B} \;\preceq\; K_{B,B} \;\preceq\; I_{|B|}, \tag{13}$$

and exact expected-cardinality repulsion drop

$$\mathbb{E}[|Y \cap B|] - \mathbb{E}[|Y \cap B| \mid A \subseteq Y] \;=\; \operatorname{Tr}(K_{B,A}K_{A,A}^{-1}K_{A,B}) \;=\; \|K_{B,A}K_{A,A}^{-1/2}\|_F^2 \;\geq\; 0. \tag{14}$$

(b) ***Exact Canonical-Correlation Inclusion Factorization.*** *Suppose additionally that* $\mathbb{P}(B \subseteq Y) = \det(K_{B,B}) > 0$*, and let* $1 \geq \rho_1 \geq \cdots \geq \rho_r \geq 0$ *(*$r := \min(|A|, |B|)$*) be the singular values (canonical correlations) of the whitened cross-kernel* $C_{A,B} := K_{A,A}^{-1/2} K_{A,B} K_{B,B}^{-1/2}$*. Then*

$$\mathbb{P}(A \cup B \subseteq Y) \;=\; \mathbb{P}(A \subseteq Y)\,\mathbb{P}(B \subseteq Y) \prod_{j=1}^{r} (1 - \rho_j^2) \;\leq\; \mathbb{P}(A \subseteq Y)\,\mathbb{P}(B \subseteq Y). \tag{15}$$

Proof. **(a)** For every subset $S \subseteq B$, Schur's determinant identity applied to the principal submatrix $K_{A\cup S, A\cup S}$ yields

$$\mathbb{P}(S \subseteq Y \cap B \mid A \subseteq Y) \;=\; \frac{\det(K_{A\cup S, A\cup S})}{\det(K_{A,A})} \;=\; \det(K_{S,S} - K_{S,A} K_{A,A}^{-1} K_{A,S}) \;=\; \det((K^A_{B,B})_{S,S}),$$

proving that $Y \cap B \mid (A \subseteq Y) \sim \mathrm{DPP}(K^A_{B,B})$. Since $K_{A\cup B, A\cup B} \succeq 0$ and $K_{A,A} \succ 0$, Schur positivity gives $K^A_{B,B} \succeq 0$, while $K_{B,A} K_{A,A}^{-1} K_{A,B} \succeq 0$ gives $K^A_{B,B} \preceq K_{B,B} \preceq I_{|B|}$. Taking traces yields $\mathbb{E}[|Y \cap B|] - \mathbb{E}[|Y \cap B| \mid A \subseteq Y] = \mathrm{Tr}(K_{B,B} - K^A_{B,B}) = \mathrm{Tr}(K_{B,A} K_{A,A}^{-1} K_{A,B}) = \|K_{B,A} K_{A,A}^{-1/2}\|_F^2$.

(b) Factoring $K^A_{B,B} = K^{1/2}_{B,B}(I_{|B|} - C_{A,B}^\top C_{A,B}) K^{1/2}_{B,B}$ (where $C_{A,B}^\top C_{A,B} \preceq I_{|B|}$ since $K^A_{B,B} \succeq 0$, so all singular values $\rho_j \in [0,1]$) and taking determinants gives

$$\begin{aligned}\mathbb{P}(A \cup B \subseteq Y) &= \det(K_{A,A}) \det(K^A_{B,B}) = \det(K_{A,A}) \det(K_{B,B}) \det(I_{|B|} - C_{A,B}^\top C_{A,B}) \\ &= \mathbb{P}(A \subseteq Y)\mathbb{P}(B \subseteq Y) \prod_{j=1}^{r} (1 - \rho_j^2).\end{aligned}$$

□

5 Trace-Finsler Isometry and L-Ensemble Perturbation Stability

Let $\mathbb{P}_n^{++}$ denote the open cone of real $n \times n$ symmetric positive-definite matrices, equipped with the affine-invariant trace-Finsler (Schatten-1) geodesic distance [1]

$$d_1(L_1, L_2) \;:=\; \big\|\log(L_1^{-1/2} L_2 L_1^{-1/2})\big\|_{S_1} \;=\; \sum_{j=1}^{n} |\log \mu_j|, \tag{16}$$

where $\mu_1 \geq \cdots \geq \mu_n > 0$ are the eigenvalues of $L_1^{-1/2} L_2 L_1^{-1/2}$ (generalized eigenvalues of (L_2, L_1)). For two strictly positive distributions $p, q \in \Delta^\circ_{2^n}$ on $2^{[n]}$, recall Hilbert's projective metric $d_H(p,q) := \max_{S,T\subseteq[n]} \log \frac{p(S)q(T)}{p(T)q(S)}$, Total Variation $\|p - q\|_{\mathrm{TV}} := \frac{1}{2}\sum_S |p(S) - q(S)|$, and Jeffreys (symmetrized KL) divergence $D_J(p,q) := D_{\mathrm{KL}}(p\|q) + D_{\mathrm{KL}}(q\|p)$.

Theorem 5.1 (Trace-Finsler 1-Lipschitz Stability, Loewner Isometry, and Hyperbolic TV/KL Bounds)**.** *For every* $n \geq 1$ *and every pair of strictly positive-definite L-ensembles* $L_1, L_2 \in \mathbb{P}_n^{++}$*, write* $\mathbb{P}_1 := \mathrm{DPP}_{L}(L_1)$ *and* $\mathbb{P}_2 := \mathrm{DPP}_{L}(L_2)$*.*

(a) ***Universal 1-Lipschitz Hilbert Stability and Exact Loewner/Diagonal Isometry.***

$$d_H(\mathbb{P}_1, \mathbb{P}_2) \;\leq\; d_1(L_1, L_2) \;=\; \big\|\log(L_1^{-1/2} L_2 L_1^{-1/2})\big\|_{S_1}. \tag{17}$$

Moreover, exact equality $d_H(\mathbb{P}_1, \mathbb{P}_2) = d_1(L_1, L_2)$ *holds identically whenever (i)* L_1, L_2 *are diagonal in the coordinate basis, or (ii)* L_1 *and* L_2 *are Loewner-ordered (*$L_2 \succeq L_1$ *or* $L_1 \succeq L_2$*), regardless of whether* L_1 *and* L_2 *commute.*

(b) ***Sharp Nonlinear Total Variation and Jeffreys-Divergence Bounds.***

$$\|\mathbb{P}_1 - \mathbb{P}_2\|_{\mathrm{TV}} \leq \tanh\!\left(\frac{d_H(\mathbb{P}_1, \mathbb{P}_2)}{4}\right) \leq \tanh\!\left(\frac{d_1(L_1, L_2)}{4}\right) \leq \frac{1}{4}\, d_1(L_1, L_2), \tag{18}$$

$$D_J(\mathbb{P}_1, \mathbb{P}_2) \leq d_H(\mathbb{P}_1, \mathbb{P}_2)\|\mathbb{P}_1 - \mathbb{P}_2\|_{\mathrm{TV}} \leq d_1(L_1, L_2) \tanh\!\left(\frac{d_1(L_1, L_2)}{4}\right) \leq \frac{1}{4}\, d_1(L_1, L_2)^2. \tag{19}$$

Proof. **(a)** Because the normalizing constants $\det(I_n + L_1)$ and $\det(I_n + L_2)$ cancel in the cross-ratio,

$$d_H(\mathbb{P}_1, \mathbb{P}_2) \;=\; \max_{S\subseteq[n]} \log \frac{\det((L_2)_{S,S})}{\det((L_1)_{S,S})} \;-\; \min_{T\subseteq[n]} \log \frac{\det((L_2)_{T,T})}{\det((L_1)_{T,T})}.$$

Fix any nonempty subset $S \subseteq [n]$ of size $k = |S|$, and let $E_S \in \mathbb{R}^{n\times k}$ be the coordinate inclusion matrix ($E_S^\top L E_S = L_{S,S}$). Define the k-dimensional column-orthonormal matrix $V_S := L_1^{1/2} E_S (L_1)_{S,S}^{-1/2} \in \mathbb{R}^{n\times k}$ (satisfying $V_S^\top V_S = I_k$). Then

$$(L_1)_{S,S}^{-1/2} (L_2)_{S,S} (L_1)_{S,S}^{-1/2} \;=\; V_S^\top (L_1^{-1/2} L_2 L_1^{-1/2}) V_S.$$

By the Poincaré–Cauchy eigenvalue interlacing theorem [1], the eigenvalues $\eta_1 \geq \cdots \geq \eta_k > 0$ of $V_S^\top (L_1^{-1/2} L_2 L_1^{-1/2}) V_S$ satisfy $\mu_{n-k+j} \leq \eta_j \leq \mu_j$ for every $j = 1, \ldots, k$. Taking determinants and logarithms gives

$$\sum_{j:\mu_j<1} \log \mu_j \;\leq\; \sum_{j=n-k+1}^{n} \log \mu_j \;\leq\; \log \frac{\det((L_2)_{S,S})}{\det((L_1)_{S,S})} \;\leq\; \sum_{j=1}^{k} \log \mu_j \;\leq\; \sum_{j:\mu_j>1} \log \mu_j,$$

which also holds trivially for $S = \emptyset$ ($\log 1 = 0$). Subtracting the lower bound from the upper bound yields

$$d_H(\mathbb{P}_1, \mathbb{P}_2) \;\leq\; \sum_{j:\mu_j>1} \log \mu_j - \sum_{j:\mu_j<1} \log \mu_j \;=\; \sum_{j=1}^{n} |\log \mu_j| \;=\; d_1(L_1, L_2),$$

proving (17). For exact equality: (i) if $L_1 = \mathrm{diag}(\ell_1), L_2 = \mathrm{diag}(\ell_2)$, choosing $S_+ = \{j : \ell_{2,j} > \ell_{1,j}\}$ and $S_- = \{j : \ell_{2,j} < \ell_{1,j}\}$ attains both endpoints; (ii) if $L_2 \succeq L_1$, then every $\mu_j \geq 1$, and choosing $S = [n]$ and $T = \emptyset$ yields $\log(\det L_2 / \det L_1) - \log(1/1) = \sum_{j=1}^n \log \mu_j = d_1(L_1, L_2)$ (and symmetrically $S = \emptyset, T = [n]$ when $L_1 \succeq L_2$).

(b) Let $R := d_H(\mathbb{P}_1, \mathbb{P}_2)$, $S_+ := \{S : \mathbb{P}_1(S) \geq \mathbb{P}_2(S)\}$, $a := \mathbb{P}_1(S_+)$, $b := \mathbb{P}_2(S_+)$. If $a = b$ then $\|\mathbb{P}_1 - \mathbb{P}_2\|_{\mathrm{TV}} = 0$; otherwise $0 < b < a < 1$, and averaging over S_+ and S_+^c yields $e^R \geq \frac{a(1-b)}{b(1-a)}$. Setting $s := b/(1-b) > 0$, AM–GM gives $a - b \leq \frac{e^R-1}{e^R s + s^{-1} + e^R + 1} \leq \frac{e^R-1}{(e^{R/2}+1)^2} = \tanh(R/4)$, proving (18) via (17). Finally, letting $c := \frac{1}{2}(\max_S \log \frac{\mathbb{P}_1(S)}{\mathbb{P}_2(S)} + \min_S \log \frac{\mathbb{P}_1(S)}{\mathbb{P}_2(S)})$, we have

$$D_J(\mathbb{P}_1, \mathbb{P}_2) = \sum_{S\subseteq[n]} (\mathbb{P}_1(S) - \mathbb{P}_2(S)) \left(\log \frac{\mathbb{P}_1(S)}{\mathbb{P}_2(S)} - c \right) \leq \frac{R}{2} \sum_{S\subseteq[n]} |\mathbb{P}_1(S) - \mathbb{P}_2(S)| = R\, \|\mathbb{P}_1 - \mathbb{P}_2\|_{\mathrm{TV}},$$

and applying (17) and (18) establishes (19). □

6 Computational Verification and Benchmark Tables

All four theorems were verified programmatically by exact 2^n-subset enumeration in `scripts/verify_all.py` and the automated test suite `tests/test_dpp_geometry.py` (4/4 pytest modules passing in 0.14 s). Results are summarized below, tabulated in Table 1, and plotted in Figures 1 and 2.

- **Theorem 2.1 (45 random DPPs across $n \in \{5, 6, 7\}$, comprising 30 L-ensembles and 15 projection DPPs, seeded by `202611`):** Exact Repulsion-Laplacian variance identity (2) matched full 2^n enumeration to max error 1.78×10^{-15}; projection boundary-cut hyperuniformity (3) matched to 2.22×10^{-16}; zero violations of the three-phase signed ceiling (5) (0.0) and unsigned ceiling (6) (0.0); exact attainment on $K = \frac{1}{2}\mathbf{1}_2\mathbf{1}_2^\top$, $f = (1, -1)^\top$ verified to 0.0.

- **Theorem 3.1 (30 random DPPs across $n \in \{5, 6\}$, including 10 projection DPPs, seeded by `202612`):** Closed-form tilted kernel (9) matched exact tilted subset marginals to 5.55×10^{-16}; exact CGF second derivative (10) matched central differences ($h = 10^{-5}$) to 8.18×10^{-6}; zero violations of signed ($\sigma^2 = \frac{1}{2}\|f\|_2^2$) and unsigned ($\sigma^2 = \frac{1}{4}\|f\|_2^2$) sub-Gaussian CGF bounds (0.0).
- **Theorem 4.1 (50 random L-ensembles across $n \in \{6, 7\}$ with $|A| = 2, |B| = 3$, seeded by `202613`):** Schur-complement conditional marginals (13) matched direct 2^n conditioning to 3.89×10^{-16}; repulsion trace drop (14) matched to 6.66×10^{-16}; canonical-correlation inclusion law (15) matched to 2.08×10^{-17}.
- **Theorem 5.1 (60 random L-ensemble pairs across $n \in \{4, 5, 6\}$, seeded by `202614`: 3 coordinate-diagonal pairs $\ell_i \sim U[0.3, 3.0]$, 30 Loewner-ordered non-commuting pairs $L_1 = \frac{1}{n} G_1 G_1^\top + 0.25 I_n$, $L_2 = L_1 + u \frac{1}{n} G_2 G_2^\top$ with $u \sim U[0.05, 0.8]$, and 27 non-Loewner-ordered non-commuting pairs $L_2 = \frac{1}{n} G_2 G_2^\top + 0.25 I_n$ with i.i.d. standard Gaussian G_1, G_2):** Zero violations of (17)–(19) (0.0); exact Hilbert–trace-Finsler isometry $d_H = d_1$ verified across all coordinate-diagonal pairs (max error 1.33×10^{-15}) and all 30 Loewner-ordered non-commuting pairs (max error $< 10^{-14}$). Table 1 lists the first representative pair from each class per dimension, and Figure 1(b) plots the explicit 2×2 unsigned L-ensemble $L = \left[\begin{smallmatrix} 1.2 & 0.6 \\ 0.6 & 1.2 \end{smallmatrix}\right]$ alongside $K = \frac{1}{2}\mathbf{1}_2\mathbf{1}_2^\top$.

Table 1: Trace-Finsler kernel distance $d_1(L_1, L_2)$ versus exact Hilbert distance d_H, Total Variation $\|\mathbb{P}_1 - \mathbb{P}_2\|_{\mathrm{TV}}$, and Jeffreys divergence D_J across coordinate-diagonal, Loewner-ordered non-commuting (`LoewnerNC`), and non-Loewner-ordered non-commuting (`NonLoewnerNC`) L-ensemble pairs (seeded by `numpy.random.default_rng(202614)`).

Pair class (n)	$d_1(L_1, L_2)$	$d_H(\mathbb{P}_1, \mathbb{P}_2)$	$\|\mathbb{P}_1 - \mathbb{P}_2\|_{\mathrm{TV}}$	$\tanh(d_1/4)$	$D_J(\mathbb{P}_1, \mathbb{P}_2)$	$d_1 \tanh(d_1/4)$
Diagonal ($n = 4$)	1.7301	1.7301	0.3527	0.4074	0.5252	0.7049
LoewnerNC ($n = 4$)	0.4066	0.4066	0.0476	0.1013	0.0119	0.0412
NonLoewnerNC ($n = 4$)	5.8325	2.8396	0.3641	0.8973	0.7393	5.2334
Diagonal ($n = 5$)	5.6866	5.6866	0.5177	0.8899	1.8655	5.0607
LoewnerNC ($n = 5$)	3.6956	3.6956	0.3239	0.7277	0.6762	2.6894
NonLoewnerNC ($n = 5$)	6.4388	4.2080	0.4507	0.9231	1.2367	5.9438
Diagonal ($n = 6$)	2.5278	2.5278	0.3323	0.5594	0.5501	1.4141
LoewnerNC ($n = 6$)	1.9241	1.9241	0.1775	0.4471	0.1861	0.8602
NonLoewnerNC ($n = 6$)	6.5172	3.2623	0.3395	0.9260	0.6872	6.0347

7 Discussion and Open Directions

We have established a unified geometric and concentration theory for finite symmetric Determinantal Point Processes linking repulsion graph Laplacians, exponential-tilting DPP closure, canonical-correlation Schur conditioning, and trace-Finsler stability on $\mathbb{P}_n^{++}$. Two natural open directions emerge:

1. **Non-symmetric DPPs.** Determining whether an analogous repulsion-Laplacian decomposition and signed/unsigned factor-of-2 variance dichotomy hold for non-symmetric DPPs with skew-symmetric interaction components [3].
2. **Trace-Finsler stability of k-DPPs.** Characterizing the exact Hilbert-projective Lipschitz constant of fixed-size k-DPPs ($L \mapsto \mathrm{DPP}_L^{(k)}(L)$) with respect to the k-th Ky Fan logarithmic distance $\sum_{j=1}^{k} \log \mu_j - \sum_{j=n-k+1}^{n} \log \mu_j$.

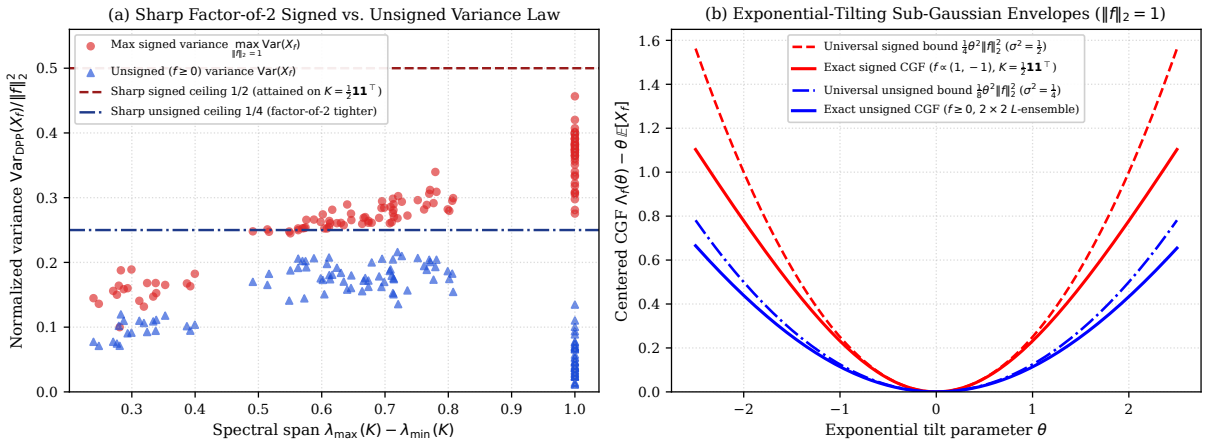


Figure 1: **(a)** Normalized variance $\mathrm{Var}_{\mathrm{DPP}}(X_f)/\|f\|_2^2$ across 120 random DPPs ($n = 8$): worst-case signed directions (red dots) saturate the sharp ceiling $1/2$, while unsigned statistics $f \geq 0$ (blue triangles) stay strictly below $1/4$ (exact factor-of-2 separation). **(b)** Centered CGF $\Lambda_f(\theta) - \theta\,\mathbb{E}[X_f]$ ($\|f\|_2 = 1$) compared with the universal signed ($\frac{1}{4}\theta^2$) and unsigned ($\frac{1}{8}\theta^2$) sub-Gaussian envelopes from Theorem 3.1(b).

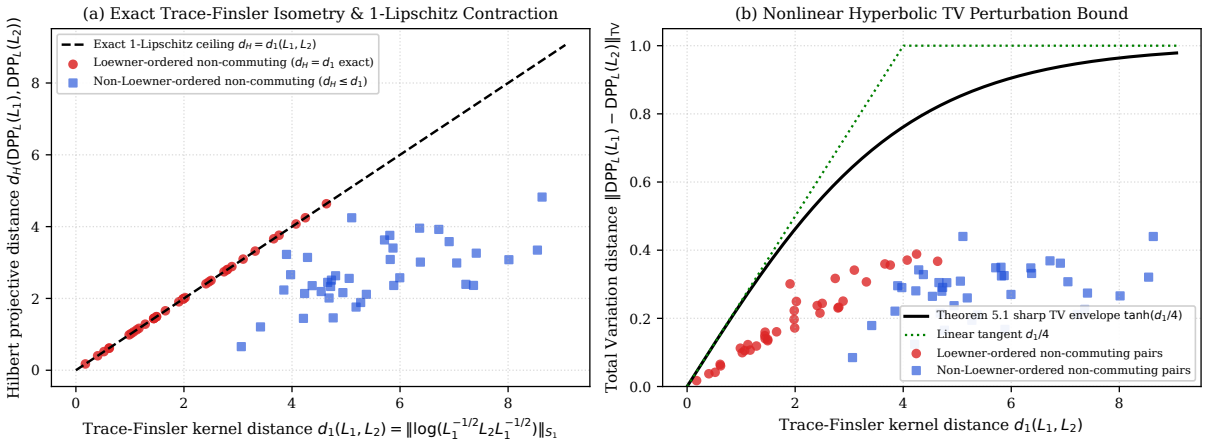


Figure 2: **(a)** Exact Hilbert–trace-Finsler isometry $d_H = d_1$ on Loewner-ordered non-commuting pairs (red circles) versus contraction $d_H \leq d_1$ on non-Loewner-ordered non-commuting pairs (blue squares) across $n \in \{4, 5, 6\}$. **(b)** Sharp nonlinear Total Variation perturbation envelope $\|\mathbb{P}_1 - \mathbb{P}_2\|_{\mathrm{TV}} \leq \tanh(d_1(L_1, L_2)/4)$ versus its linear tangent $d_1/4$.

Reproducibility and Open-Source Code

All exact enumeration benchmarks, tables, figures, and automated tests (4/4 pytest modules) are openly archived under the MIT license at:
https://github.com/lluiseriksson/dpp-repulsion-geometry.

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