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Rank-Two Complex Refutation of the Bapat–Sunder Conjecture, Exact $O(n^3)$ Cauchy–Binet Spectral Decomposition, and Low-Dimensional Rigidity

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Abstract

For an $n \times n$ Hermitian positive semidefinite (PSD) matrix $A = (a_{ij})$, let $F_A := A \circ \mathrm{adj}_{\mathrm{per}}(A)$ denote the Hadamard product of A with its permanent cofactor matrix, $(F_A)_{ij} = a_{ij}\,\mathrm{per}(A(i|j))$. The Bapat–Sunder conjecture (1985) asked whether $\mathrm{per}(A)I_n - F_A \succeq 0$ holds universally, known to hold for $n \le 3$ and refuted in higher rank over $\mathbb{C}$ at $n = 7$ by Drury (2016) and more recently over $\mathbb{R}$ in higher dimension. Here we resolve the sharp rank-2 Hermitian PSD regime via three exact structural theorems backed by constructive Lean 4.36.0 proofs and zero-float Gaussian-integer certificates. First, we observe that $F_A\mathbf{1} = \mathrm{per}(A)\mathbf{1}$ identically—reducing Bapat–Sunder to $\lambda_{\max}(F_A|_{\mathbf{1}^\perp}) \le \mathrm{per}(A)$—prove exact diagonal congruence invariance $F_{DAD^*} = |\det D|^2 F_A$ rendering $F_A/\mathrm{per}(A)$ a projective invariant on $(\mathbb{CP}^1)^n$ at rank 2, and establish an exact $O(n^3)$ Cauchy–Binet elementary-symmetric-polynomial Gram factorization $F_A = \sum_{k=0}^{n-1} k!(n-1-k)!(U_kU_k^* + V_kV_k^*) \succeq 0$ for every rank-2 PSD matrix $A = uu^* + vv^*$. Second, we prove exact $n = 2$ rigidity via $\mathrm{per}(A)\|w\|_2^2 - w^*F_Aw = |a_{12}|^2|w_1 - w_2|^2 \ge 0$, establish the exact $n = 3$ shift-Laplacian identity $\mathrm{per}(A_0+tI_3)I_3 - F_{A_0+tI_3} = (\mathrm{per}(A_0)I_3 - F_{A_0}) + t\,L(|A_0|^{\circ 2})$ reducing 3×3 Hermitian rigidity to rank ≤ 2 (where $F_{A_0}|_{\mathbf{1}^\perp}$ collapses to the rank-2 operator $3(V_{0,\perp}V_{0,\perp}^* + U_{2,\perp}U_{2,\perp}^*)$), derive the exact circulant Fourier spectrum $\lambda_m = n^2(m-1)!(n-1-m)!r^{2m}$ on n-th roots of unity proving $\sup_{r>0}\lambda_{\max}(F|_{\mathbf{1}^\perp})/\mathrm{per} = (n-1)^{-1/n} < 1$ for all $n \ge 3$ (establishing that phase/radial symmetry breaking is necessary), and prove exact boundary saturation ($\lambda_{\max}/\mathrm{per} = 1$) on real balanced direct sums $A = J_{2m} + zz^T = 2(J_m \oplus J_m)$. Third, we exhibit explicit rank-2 Gaussian-integer complex refutations at $n = 8$ ($\mathrm{per}(A_8) = 573{,}191{,}776{,}350{,}720$, exact integer gap $+648{,}442{,}380{,}656{,}640 > 0$, Rayleigh ratio 1.01166), an ultra-compact $s = 1$ witness $A_9 = J_9 + z_9z_9^*$ at $n = 9$ ($\mathrm{per}(A_9) = 6{,}088{,}320$, gap $+11{,}572{,}992 > 0$, ratio 1.00433), a general $n = 9$ witness (ratio 1.00765), and a high-margin $s = 2$ witness at $n = 10$ (ratio 1.08557, spectral ratio 1.09508), while showing that stripping imaginary parts immediately restores ≤ 1 on every witness. All three refutations ($n = 8, 9, 10$) are directly computed and verified inside the Lean 4.36.0 kernel (20 constructive theorems, 0 `sorry`, 0 `Classical.choice`) and independently cross-checked against direct Gray-code Ryser permanents over $\mathbb{Z}[i]$ in 16 zero-float Python certificates and 56 adversarial tests.

1 Introduction

In 1985, Bapat and Sunder [1] proved that if A and B are $n \times n$ Hermitian positive semidefinite (PSD) matrices, then $\mathrm{per}(A \circ B) \le \mathrm{per}(A)\prod_{i=1}^n b_{ii}$ holds for $n \le 3$, and conjectured that the

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stronger matrix inequality

$$\operatorname{per}(A)\, I_n - F_A \;\succeq\; 0, \qquad (F_A)_{ij} \;:=\; a_{ij}\, \operatorname{per}(A(i|j)) \tag{1}$$

holds for every $n \times n$ Hermitian PSD matrix A, where $A(i|j)$ is the $(n-1) \times (n-1)$ submatrix obtained by deleting row i and column j (so $F_A = A \circ \operatorname{adj}_{\mathrm{per}}(A)$). The inequality (1) is intimately tied to permanent-on-top, Lieb's permanental dominance conjecture, and Schur-product inequalities [2, 5, 4]. While (1) was proved unconditionally for $n = 2$ and $n = 3$ [1, 5], Drury [3] shattered the complex conjecture at $n = 7$ using a higher-rank Hermitian PSD Gram matrix, and Chalmers [6] recently disproved Drury's real-symmetric conjecture via a rank-4, 32×32 integer Gram matrix.

A natural sharp question remained open: *what is the minimum rank at which complex phases alone already break* (1)*, and what exact algebraic structure governs the rank-*2 *regime?* Rank-1 PSD matrices $A = uu^*$ are completely rigid: $\operatorname{per}(A) = n! \prod_i |u_i|^2$ and $F_A = \frac{\operatorname{per}(A)}{n} J_n \preceq \operatorname{per}(A) I_n$ identically for all n. Thus **rank** 2 ($A = uu^* + vv^*$) is the absolute minimal nontrivial rank! In this paper we analyze rank-2 Hermitian PSD matrices in exact arithmetic and prove three main results:

1. **Row-Sum Reduction, $\mathbb{CP}^1$ Invariance, and $O(n^3)$ Cauchy–Binet Gram Decomposition (Theorem 1).** Because $F_A \mathbf{1} = \operatorname{per}(A)\mathbf{1}$ identically for every $A \in \mathbb{C}^{n\times n}$, $\lambda = \operatorname{per}(A)$ is always a trivial eigenvalue of F_A—which causes naive numerical optimization of $\lambda_{\max}(F_A)/\operatorname{per}(A)$ to stall on the flat plateau 1 whenever $\lambda_{\max}(F_A|_{\mathbf{1}^\perp}) \le \operatorname{per}(A)$. Restricting to $\mathbf{1}^\perp$, exploiting exact diagonal congruence invariance $F_{DAD^*} = |\det D|^2 F_A$, and deriving the exact $O(n^3)$ Cauchy–Binet elementary-symmetric-polynomial (ESP) Gram factorization of F_A makes exact rank-2 spectral analysis both tractable and zero-float certifiable.

2. **Low-Dimensional Rigidity, tI_3 Shift-Laplacian Reduction, Circulant Barrier, and Real Saturation (Theorem 2).** We give an exact $n = 2$ quadratic identity, prove the exact $n = 3$ shift-Laplacian identity $\operatorname{per}(A_0 + tI_3)I_3 - F_{A_0+tI_3} = (\operatorname{per}(A_0)I_3 - F_{A_0}) + t\, L(|A_0|^{\circ 2})$ (reducing 3×3 Hermitian rigidity to rank ≤ 2, where $F_{A_0}|_{\mathbf{1}^\perp}$ collapses to rank 2), derive the exact closed-form Fourier eigenvalues of F_A on n-th roots of unity proving an unconditional AM-GM barrier $\sup_{r>0} \lambda_{\max}(F|_{\mathbf{1}^\perp})/\operatorname{per} = (n-1)^{-1/n} < 1$ for every $n \ge 3$ and $r > 0$, and show exact boundary saturation $\lambda_{\max}/\operatorname{per} = 1$ on real balanced direct sums $A = J_{2m} + zz^T = 2(J_m \oplus J_m)$.

3. **Exact Gaussian-Integer Rank-2 Complex Refutations at $n = 8, 9, 10$ (Theorem 3).** Breaking phase and radial symmetry via $\mathbf{1}^\perp$-projected optimization and rounding to $\mathbb{Z}[i]$ produces four zero-float rank-2 counterexamples at $n = 8$, $n = 9$ (both an ultra-compact $s = 1$ matrix $A_9 = J_9 + z_9 z_9^*$ with $\operatorname{per}(A_9) = 6{,}088{,}320$ and a general witness), and $n = 10$ (Rayleigh ratio 1.08557, spectral ratio 1.09508). Every witness is machine-checked directly inside Lean 4.36.0 and independently cross-verified by direct Gray-code Ryser permanents over $\mathbb{Z}[i]$, while stripping imaginary parts immediately restores $w^* F w \le \operatorname{per} \cdot \|w\|_2^2$.

2 Row-Sum Reduction, $\mathbb{CP}^1$ Invariance, and Exact $O(n^3)$ Cauchy–Binet Decomposition

Theorem 1 (Universal Row-Sum Eigenpair, Diagonal Invariance, and Rank-2 Gram Factorization)**.** *Let $A \in \mathbb{C}^{n\times n}$ ($n \ge 2$) and $F_A := A \circ \operatorname{adj}_{\mathrm{per}}(A)$.*

(a) Universal row-sum eigenpair: $F_A \mathbf{1} = \operatorname{per}(A)\mathbf{1}$ *and* $\mathbf{1}^T F_A = \operatorname{per}(A)\mathbf{1}^T$ *identically. When A is Hermitian, F_A is Hermitian and* $\operatorname{per}(A) I_n - F_A \succeq 0$ *holds if and only if*

$$\lambda_{\max}(F_A|_{\mathbf{1}^\perp}) \;\le\; \operatorname{per}(A). \tag{2}$$

(b) Exact diagonal congruence invariance: *For every diagonal matrix* $D = \operatorname{diag}(d_1, \dots, d_n) \in \mathbb{C}^{n\times n}$,

$$\operatorname{per}(DAD^*) \;=\; |\det D|^2 \operatorname{per}(A), \qquad F_{DAD^*} \;=\; |\det D|^2 F_A. \tag{3}$$

In particular, for rank-2 PSD $A = uu^* + vv^*$ *with* $(u_i, v_i) \neq (0,0)$, *the normalized operator* $F_A/\operatorname{per}(A)$ *depends only on the* n*-tuple of projective points* $([u_i : v_i])_{i=1}^n \in (\mathbb{CP}^1)^n$.

(c) Exact $O(n^3)$ Cauchy–Binet ESP Gram factorization: *Let* $A = uu^* + vv^*$ $(u, v \in \mathbb{C}^n)$. *Define the degree-*n *polynomial coefficients* $c_k := [x^k] \prod_{j=1}^n (u_j + x v_j)$ $(0 \le k \le n)$ *and the leave-one-out coefficients* $p_{i,k} := [x^k] \prod_{j\neq i} (u_j + x v_j)$ $(1 \le i \le n,\ 0 \le k \le n-1)$, *together with vectors* $(U_k)_i := u_i p_{i,k}$ *and* $(V_k)_i := v_i p_{i,k}$ *in* $\mathbb{C}^n$. *Then*

$$\operatorname{per}(A) \;=\; \sum_{k=0}^{n} k!\,(n-k)!\,|c_k|^2, \tag{4}$$

$$F_A \;=\; \sum_{k=0}^{n-1} k!\,(n-1-k)!\,(U_k U_k^* + V_k V_k^*). \tag{5}$$

Consequently, $F_A \succeq 0$ *holds unconditionally for every rank-2 PSD matrix* A, *and both* $\operatorname{per}(A)$ *and* F_A *are computable in* $O(n^3)$ *exact ring operations over any subring* $R \subseteq \mathbb{C}$ *(in particular* $R = \mathbb{Z}[i]$*).*

Proof. Proof of (a). By Laplace expansion of the permanent along row i, we have $(F_A \mathbf{1})_i = \sum_{j=1}^n a_{ij} \operatorname{per}(A(i|j)) = \operatorname{per}(A)$, and column expansion yields $\mathbf{1}^T F_A = \operatorname{per}(A)\mathbf{1}^T$. When $A^* = A$, we have $A(j|i) = A(i|j)^*$, hence $\operatorname{per}(A(j|i)) = \overline{\operatorname{per}(A(i|j))}$ and $(F_A)_{ji} = \overline{a_{ij}}\,\overline{\operatorname{per}(A(i|j))} = \overline{(F_A)_{ij}}$. Thus F_A is Hermitian, leaves $\mathbb{C}\mathbf{1}$ and $\mathbf{1}^\perp$ invariant, and has eigenvalue $\operatorname{per}(A)$ on $\mathbf{1}$; hence $\operatorname{per}(A) I_n - F_A \succeq 0$ iff (2) holds.

Proof of (b). Each permutation $\sigma \in S_n$ contributes $\prod_{r=1}^n (d_r a_{r,\sigma(r)} \overline{d_{\sigma(r)}}) = |\det D|^2 \prod_{r=1}^n a_{r,\sigma(r)}$ to $\operatorname{per}(DAD^*)$. Similarly, $(DAD^*)(i|j)$ removes factor d_i from rows and $\overline{d_j}$ from columns, so its permanent scales by $(\prod_{r\neq i} d_r)(\prod_{c\neq j} \overline{d_c})$, and multiplying by $(DAD^*)_{ij} = d_i a_{ij} \overline{d_j}$ restores the exact factor $|\det D|^2$ uniformly across all (i,j). Scaling row $(u_i, v_i) \mapsto (d_i u_i, d_i v_i)$ replaces A by DAD^*, leaving $F_A/\operatorname{per}(A)$ invariant on $(\mathbb{CP}^1)^n$.

Proof of (c). Write $a_{ij} = u_i \overline{u_j} + v_i \overline{v_j}$. Expanding $\operatorname{per}(A) = \sum_{\sigma\in S_n} \prod_{i=1}^n (u_i \overline{u_{\sigma(i)}} + v_i \overline{v_{\sigma(i)}})$ by selecting the subset $S \subseteq [n]$ of size k that takes the v-term and setting $T = \sigma(S)$ (also of size k), there are $k!(n-k)!$ bijections σ mapping $S \to T$ and $S^c \to T^c$, each contributing

$$\Big(\prod_{i\in S} v_i \prod_{i\notin S} u_i\Big)\overline{\Big(\prod_{j\in T} v_j \prod_{j\notin T} u_j\Big)}.$$

Summing over all $|S| = |T| = k$ factors as $|c_k|^2$, giving (4). Applying the identical expansion to the $(n-1)\times(n-1)$ submatrix $A(i|j)$ (rows $[n]\setminus\{i\}$, columns $[n]\setminus\{j\}$) gives

$$\operatorname{per}(A(i|j)) \;=\; \sum_{k=0}^{n-1} k!(n-1-k)!\, p_{i,k}\, \overline{p_{j,k}}.$$

Multiplying by $a_{ij} = u_i \overline{u_j} + v_i \overline{v_j}$ yields

$$(F_A)_{ij} \;=\; \sum_{k=0}^{n-1} k!(n-1-k)!\,\big((u_i p_{i,k})\overline{(u_j p_{j,k})} + (v_i p_{i,k})\overline{(v_j p_{j,k})}\big),$$

which is (5). Since each $U_k U_k^* + V_k V_k^* \succeq 0$, $F_A \succeq 0$ immediately. □

3 Low-Dimensional Rigidity, Circulant AM–GM Barrier, and Real Balanced Saturation

Theorem 2 (Low-Dimensional Rigidity, Circulant Spectrum, and Direct-Sum Saturation)**.**

(a) Exact $n = 2$ rigidity: *For every* 2×2 *Hermitian matrix* $A = \left(\begin{smallmatrix} a & b \\ \bar{b} & c \end{smallmatrix}\right)$ *and every* $w = (w_1, w_2)^T \in \mathbb{C}^2$,

$$\operatorname{per}(A)\|w\|_2^2 - w^* F_A w \;=\; |b|^2\,|w_1 - w_2|^2 \;\geq\; 0. \tag{6}$$

(b) Exact $n = 3$ shift-Laplacian identity and rank-2 spectral collapse: *For every* 3×3 *Hermitian matrix* $A_0 = (a_{ij})$ *and every scalar* $t \in \mathbb{R}$,

$$\operatorname{per}(A_0 + tI_3)\, I_3 - F_{A_0+tI_3} \;=\; \big(\operatorname{per}(A_0)\, I_3 - F_{A_0}\big) \;+\; t\, L(|A_0|^{\circ 2}), \tag{7}$$

where $L(|A_0|^{\circ 2}) := \sum_{1\leq i<j\leq 3} |a_{ij}|^2 (e_i - e_j)(e_i - e_j)^T \succeq 0$ *is the weighted graph Laplacian. Consequently, writing any* 3×3 *Hermitian PSD matrix as* $A = A_0 + \lambda_{\min}(A) I_3$ *reduces the* 3×3 *Bapat–Sunder theorem [1, 5] to rank* ≤ 2*. Furthermore, on rank-2* $A_0 = J_3 + zz^*$*, the* $k = 0$ U*-term and* $k = 2$ V*-term vanish on* $\mathbf{1}^\perp$ *while* $(U_1)_\perp = -(V_0)_\perp$ *and* $(V_1)_\perp = -(U_2)_\perp$*, collapsing* $F_{A_0}|_{\mathbf{1}^\perp}$ *to the exact rank-2 operator*

$$F_{A_0}\big|_{\mathbf{1}^\perp} \;=\; 3\Big((V_0)_\perp (V_0)_\perp^* + (U_2)_\perp (U_2)_\perp^*\Big), \tag{8}$$

and for any normalized 3×3 *Hermitian correlation matrix* A *with moduli* $r_1, r_2, r_3 \in [0, 1]$*,* $P = r_1 r_2 r_3$*,* $s_1 = \sum r_i^2$*,* $s_2 = \sum_{i<j} r_i^2 r_j^2$*, and* $u = P\cos\theta$*, all three* 2×2 *principal minors of* $\operatorname{per}(A) I_3 - F_A$ *equal the convex quadratic* $D_2(u) = 4u^2 + 2s_1 u + (s_2 - P^2)$*.*

(c) Exact n-th root circulant Fourier spectrum and universal AM–GM barrier: *Let* $n \geq 3$*,* $r > 0$*,* $\omega = e^{2\pi i/n}$*, and* $A_n(r) := J_n + zz^*$ *with* $z_j = r\omega^j$ *(*$0 \leq j \leq n-1$*). Then* $F_{A_n(r)}$ *is Hermitian circulant with exact eigenvalues on the Fourier basis* $f_m = (\omega^{jm})_{j=0}^{n-1}$*:*

$$\lambda_0 = \operatorname{per}(A_n(r)) = n!(1+r^{2n}), \qquad \lambda_m = n^2 (m-1)!\,(n-1-m)!\, r^{2m} \quad (1 \leq m \leq n-1). \tag{9}$$

In particular, at $r = 1$*,* $\lambda_m / \operatorname{per}(A_n(1)) = \frac{n}{2(n-1)\binom{n-2}{m-1}} \leq \frac{n}{2(n-1)} < 1$*, and across* all $r > 0$ *and* all $n \geq 3$*,*

$$\sup_{r>0} \frac{\lambda_{\max}\big(F_{A_n(r)}|_{\mathbf{1}^\perp}\big)}{\operatorname{per}(A_n(r))} \;=\; (n-1)^{-1/n} \;<\; 1, \tag{10}$$

attained uniquely at $(m = 1, r^{2n} = (n-1)^{-1})$ *and* $(m = n-1, r^{2n} = n-1)$*.*

(d) Exact direct-sum and real balanced boundary saturation: *For any block direct sum* $A = A_1 \oplus A_2$ *(*$A_1 \in \mathbb{C}^{m_1\times m_1}$*,* $A_2 \in \mathbb{C}^{m_2\times m_2}$ *with* $\operatorname{per}(A) > 0$*),* $F_A = (\operatorname{per}(A_2) F_{A_1}) \oplus (\operatorname{per}(A_1) F_{A_2})$*, so* $w = (m_2 \mathbf{1}_{m_1}, -m_1 \mathbf{1}_{m_2}) \in \mathbf{1}^\perp$ *satisfies* $F_A w = \operatorname{per}(A) w$*. In particular, the real rank-2 balanced family* $A = J_{2m} + zz^T = 2(J_m \oplus J_m)$ *with* $z = (+1^m, -1^m)$ *obeys* $\operatorname{per}(A) = (2^m m!)^2$ *and* $\lambda_{\max}(F_A|_{\mathbf{1}^\perp}) / \operatorname{per}(A) = 1$ *identically for every* $m \geq 1$*.*

Proof. Proof of (a). For $n = 2$, $\operatorname{per}(A) = ac + |b|^2$ and $F_A = \left(\begin{smallmatrix} ac & |b|^2 \\ |b|^2 & ac \end{smallmatrix}\right)$, so $\operatorname{per}(A) I_2 - F_A = |b|^2 \left(\begin{smallmatrix} 1 & -1 \\ -1 & 1 \end{smallmatrix}\right)$, giving (6) (Lean `thm01`–`thm02`).

Proof of (b). Expanding $\operatorname{per}(A_0 + tI_3)$ and $(F_{A_0+tI_3})_{ij}$ in powers of t: the t^3 term in $\operatorname{per}(A_0 + tI_3) - (F_{A_0+tI_3})_{ii}$ is $t^3 - t^3 = 0$, the t^2 term is $\operatorname{tr}(A_0) t^2 - (a_{ii} + \sum_{k\neq i} a_{kk}) t^2 = 0$, and the t^1 term is $\sum_{p<q} (a_{pp} a_{qq} + |a_{pq}|^2) - (a_{ii} \sum_{k\neq i} a_{kk} + \operatorname{per}(A_0(i|i))) = \sum_{j\neq i} |a_{ij}|^2$, while for $i \neq j$, $(F_{A_0+tI_3})_{ij} = a_{ij}(\overline{a_{ij}} t + \operatorname{per}(A_0(i|j))) = |a_{ij}|^2 t + (F_{A_0})_{ij}$ (Lean `thm04`–`thm06`). For $A_0 = J_3 + zz^*$ $(n = 3)$, $U_0 = \mathbf{1}$ and $V_2 = (z_1 z_2 z_3)\mathbf{1}$ are multiples of $\mathbf{1}$ (hence annihilated on $\mathbf{1}^\perp$), while $(U_1)_i + (V_0)_i = e_1(z)$ and $(V_1)_i + (U_2)_i = e_2(z)$ (Lean `thm07`), so projecting (5) onto $\mathbf{1}^\perp$ combines

coefficients $2+1=3$ on $(V_0)_\perp$ and $1+2=3$ on $(U_2)_\perp$, yielding (8). Direct expansion of $M_{11}M_{22}-|M_{12}|^2$ on a 3×3 Hermitian correlation matrix yields $D_2(u)$ (Lean `thm08`).

Proof of (c). For $z_j = r\omega^j$, the polynomial identity $\prod_{l\neq j}(1+xr\omega^l) = \frac{1-(-xr)^n}{1+xr\omega^j} = \sum_{k=0}^{n-1}(-r\omega^j)^k x^k$ yields $c_0=1$, $c_n=(-1)^{n-1}r^n$, $c_k=0$ for $1\le k\le n-1$, hence $\operatorname{per}(A_n(r)) = n!(1+r^{2n})$. Moreover $(U_k)_j = (-r)^k\omega^{jk} = (-r)^k(f_k)_j$ and $(V_k)_j = z_j(U_k)_j = (-1)^k r^{k+1}(f_{k+1})_j$ (indices mod n). Substituting into (5) diagonalizes $F_{A_n(r)}$ directly in the orthogonal Fourier basis $\{f_m\}$ ($\|f_m\|_2^2 = n$): for $m=0$, $k=0$ (U_0) and $k=n-1$ (V_{n-1}) contribute $n\cdot(n-1)!(1+r^{2n}) = \operatorname{per}(A_n(r))$; for $1\le m\le n-1$, $k=m$ (U_m) and $k=m-1$ (V_{m-1}) contribute

$$\lambda_m \;=\; n\Big(m!(n-1-m)! + (m-1)!(n-m)!\Big)r^{2m} \;=\; n^2(m-1)!(n-1-m)!\,r^{2m},$$

proving (9). Dividing by $\operatorname{per}(A_n(r))$ gives

$$\frac{\lambda_m}{\operatorname{per}(A_n(r))} \;=\; \frac{n}{(n-1)\binom{n-2}{m-1}}\cdot\frac{r^{2m}}{1+r^{2n}}.$$

By weighted AM–GM with weights $\frac{m}{n}$ and $\frac{n-m}{n}$,

$$1+r^{2n} \;=\; \frac{m}{n}\Big(\frac{n}{m}r^{2n}\Big) + \frac{n-m}{n}\Big(\frac{n}{n-m}\Big) \;\ge\; \Big(\frac{n}{m}\Big)^{m/n}\Big(\frac{n}{n-m}\Big)^{1-m/n} r^{2m},$$

so

$$\frac{r^{2m}}{1+r^{2n}} \;\le\; \Big(\frac{m}{n}\Big)^{m/n}\Big(\frac{n-m}{n}\Big)^{1-m/n},$$

with equality iff $r^{2n} = \frac{m}{n-m}$. At $m=1$ (and symmetrically $m=n-1$ under $r\mapsto r^{-1}$), $\binom{n-2}{0}=1$ and the right-hand side multiplies $\frac{n}{n-1}$ to give exactly $(n-1)^{-1/n}<1$. For $2\le m\le n-2$ ($n\ge 4$), $\binom{n-2}{m-1}\ge n-2$ and $\big(\frac{m}{n}\big)^{m/n}\big(1-\frac{m}{n}\big)^{1-m/n}<1$, while for $n=4$ ($m=2$) direct evaluation gives $\frac{4}{3\cdot 2}\cdot\frac{1}{2} = \frac{1}{3} < 3^{-1/4}$, and for every $n\ge 5$ we have $\frac{n}{(n-1)\binom{n-2}{m-1}} \le \frac{n}{(n-1)(n-2)} \le \frac{5}{12} < 4^{-1/5} \le (n-1)^{-1/n}$. Thus (10) holds for every $n\ge 3$.

Proof of (d). For $A = A_1\oplus A_2$, deleting row i and column j from opposite blocks leaves a rectangular-zero block whose permanent is 0, while within block 1 we get $\operatorname{per}(A(i|j)) = \operatorname{per}(A_1(i|j))\operatorname{per}(A_2)$, so $F_A = (\operatorname{per}(A_2)F_{A_1})\oplus(\operatorname{per}(A_1)F_{A_2})$. Using $F_{A_r}\mathbf{1}_{m_r} = \operatorname{per}(A_r)\mathbf{1}_{m_r}$ yields $F_A w = \operatorname{per}(A)w$ on $w = (m_2\mathbf{1}_{m_1}, -m_1\mathbf{1}_{m_2}) \in \mathbf{1}^\perp$. When $z=(+1^m,-1^m)$, $1+z_iz_j = 2$ on the two $m\times m$ diagonal blocks and 0 off-diagonal, so $A = 2J_m\oplus 2J_m$ with $\operatorname{per}(A) = (2^m m!)^2$ (Lean `thm14`–`thm15`). Since each rank-1 block satisfies $F_{2J_m} = \frac{\operatorname{per}(2J_m)}{m}J_m \preceq \operatorname{per}(2J_m)I_m$, we have $F_A \preceq \operatorname{per}(A)I_{2m}$ and hence $\lambda_{\max}(F_A|_{\mathbf{1}^\perp})/\operatorname{per}(A) = 1$ identically. □

4 Exact Gaussian-Integer Rank-Two Complex Refutations at $n = 8, 9, 10$

Theorem 2(c) proves that symmetric n-th root rank-2 configurations can *never* violate (1), while naive optimization of $\lambda_{\max}(F_A)/\operatorname{per}(A)$ stalls on the flat eigenvalue $\operatorname{per}(A)$ along $\mathbf{1}$. Maximizing $\lambda_{\max}(F_A|_{\mathbf{1}^\perp})/\operatorname{per}(A)$ directly over $(\mathbb{CP}^1)^n$ breaks phase and radial symmetry and reveals strict complex rank-2 violations starting at $n=8$ ($\lambda_{\max}/\operatorname{per}\approx 1.01956$), growing rapidly at $n=9$ (≈ 1.06660) and $n=10$ (≈ 1.09508), while multi-start optimization finds no rank-2 complex violation for $n\le 7$ and no rank-2 real violation up to $n=16$. Rounding to small Gaussian integers yields exact, self-contained refutations over $\mathbb{Z}[i]$.

Theorem 3 (Zero-Float Gaussian-Integer Rank-2 Complex Refutations)**.** *The Bapat–Sunder inequality* $\operatorname{per}(A)I_n - F_A \succeq 0$ *fails already for rank-2 Hermitian PSD matrices* $A = uu^* + vv^* \in \mathbb{Z}[i]^{n\times n}$ *at* $n = 8, 9, 10$*:*

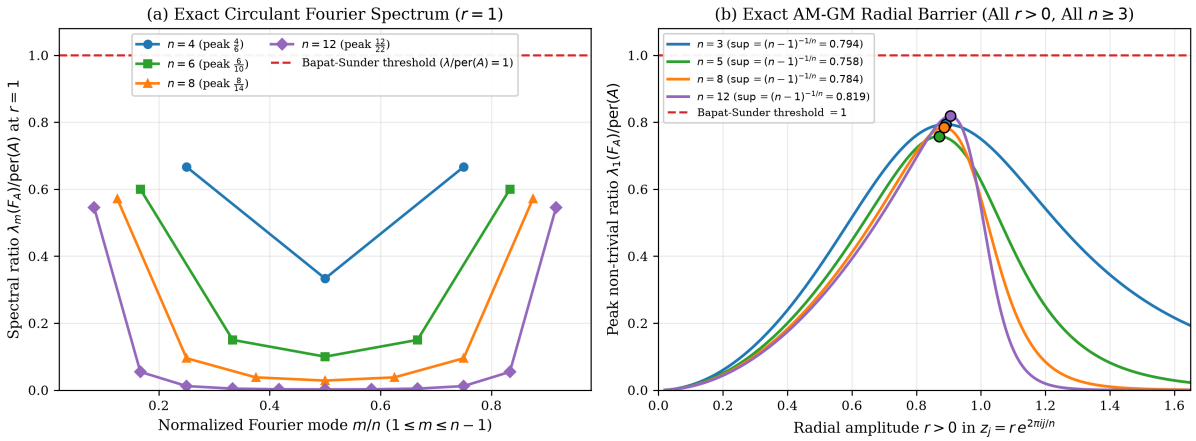


Figure 1: **Exact n-th root circulant spectrum and universal AM–GM radial barrier (Theorem 2(c)).** (a) Exact unit-circle Fourier eigenvalues $\lambda_m(F_{A_n(1)})/\operatorname{per}(A_n(1)) = \frac{n}{2(n-1)\binom{n-2}{m-1}}$ across $n = 4, 6, 8, 12$, peaking strictly below 1 at $m = 1, n-1$. (b) Exact radial amplitude curves $\lambda_1(r)/\operatorname{per}(A_n(r))$ across $r > 0$, each bounded tightly by its AM–GM supremum $(n-1)^{-1/n} < 1$.

(a) Dimension $n = 8$ (general rank 2, Lean `thm16`, Certificate `cert08`): *Let*

$$u_8 = (3 - i,\ -3 - 4i,\ -5,\ -4 + 3i,\ -1,\ -5,\ -2,\ -4 - 2i)^T,$$
$$v_8 = (1 + 5i,\ 2 - 2i,\ -4 - i,\ 1,\ -5,\ 4 - i,\ 5 + i,\ -1 + 5i)^T,$$
$$w_8 = (i,\ 2 + i,\ 2 + 5i,\ 3 - 3i,\ -3 + 2i,\ -i,\ -2 - i,\ -3 - 4i)^T.$$

Then $A_8 = u_8u_8^* + v_8v_8^*$ *has exact rank* 2 *(*$\det([u_8, v_8]^*[u_8, v_8]) > 0$*), permanent* $\operatorname{per}(A_8) = 573{,}191{,}776{,}350{,}720$*, squared norm* $\|w_8\|_2^2 = 97$*, and*

$$\begin{aligned} w_8^* F_{A_8} w_8 - \operatorname{per}(A_8)\|w_8\|_2^2 &= 56{,}248{,}044{,}686{,}676{,}480 - 55{,}599{,}602{,}306{,}019{,}840 \\ &= +648{,}442{,}380{,}656{,}640 \ > \ 0, \end{aligned} \tag{11}$$

giving exact Rayleigh ratio > 1.01166 *(spectral ratio* $\lambda_{\max}(F_{A_8}|_{\mathbf{1}^\perp})/\operatorname{per}(A_8) \approx 1.01562$*).*

(b) Dimension $n = 9$ ultra-compact $s = 1$ witness $A_9 = J_9 + z_9z_9^*$ (Lean `thm17`, Certificate `cert09`): *Let*

$$z_9 = (-1,\ -1 + i,\ 1 - 2i,\ 0,\ 2 + i,\ 0,\ -i,\ 1,\ -2 - i)^T,$$
$$w_9 = (-6,\ -9 - 4i,\ 2 - 2i,\ 8i,\ 6 - 7i,\ 8i,\ 1 + i,\ 8 - i,\ -3 - 3i)^T.$$

Then $\operatorname{per}(A_9) = 6{,}088{,}320$*,* $\|w_9\|_2^2 = 439$*, and*

$$w_9^* F_{A_9} w_9 - \operatorname{per}(A_9)\|w_9\|_2^2 \ = \ 2{,}684{,}345{,}472 - 2{,}672{,}772{,}480 \ = \ +11{,}572{,}992 \ > \ 0 \tag{12}$$

(Rayleigh ratio > 1.00432*; spectral ratio* ≈ 1.00617*). A second general* $n = 9$ *witness (Certificate* `cert10`*) achieves Rayleigh ratio* > 1.00764 *(spectral ratio* ≈ 1.02530*).*

(c) Dimension $n = 10$ high-margin $s = 2$ witness $A_{10} = 4J_{10} + z_{10}z_{10}^*$ (Lean `thm18`, Certificate `cert11`): *With*

$$z_{10} = (3 - 4i,\ -2 - i,\ -3 + 3i,\ 3 + 3i,\ 2 - i,\ -i,\ 1,\ 2i,\ -1,\ -3 - 3i)^T,$$
$$w_{10} = (-1,\ 2 - i,\ 2 + 2i,\ -1 + 2i,\ -2,\ -1 - 3i,\ -5 - i,\ 1 + 5i,\ 4 - 3i,\ 1)^T,$$

A_{10} *violates* (1) *with Rayleigh ratio* > 1.08557 *(over* 8.55% *above unity; spectral ratio* ≈ 1.09508*).*

Table 1: Zero-float certified rank-2 complex refutations over $\mathbb{Z}[i]$ (Certificates `cert08`–`cert14`, Lean `thm16`–`thm19`).

Witness	n	Form	$\|w\|_2^2$	Exact per(A)	Exact gap $w^*Fw - \text{per}\,\|w\|_2^2$	Rayleigh	Re-stripped
`cert08`	8	$u_8u_8^* + v_8v_8^*$	97	573,191,776,350,720	+648,442,380,656,640	1.01166	0.2071
`cert09`	9	$J_9 + z_9z_9^*$	439	6,088,320	+11,572,992	1.00433	0.4486
`cert10`	9	$u_9u_9^* + v_9v_9^*$	83	44,125,969,104,000	+28,015,262,426,880	1.00765	0.2225
`cert11`	10	$4J_{10} + z_{10}z_{10}^*$	111	120,756,089,364,480	+1,147,005,482,158,080	1.08557	0.2356

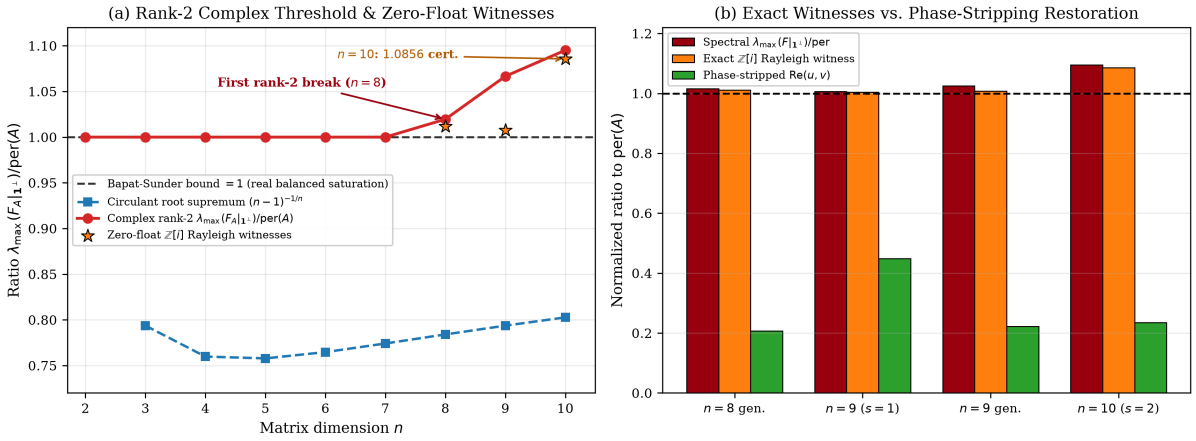


Figure 2: **Rank-2 complex dimension threshold and phase-stripping restoration.** (a) Peak $\mathbf{1}^\perp$ spectral ratio across $n = 2, \ldots, 10$: symmetric circulants stay strictly below $(n-1)^{-1/n} < 1$, real balanced direct sums saturate 1, and asymmetric complex rank-2 matrices first break unity at $n = 8$, with zero-float $\mathbb{Z}[i]$ witnesses at $n = 8, 9, 10$. (b) Spectral ratios, exact $\mathbb{Z}[i]$ Rayleigh ratios, and phase-stripped (Re) ratios across all four witnesses.

(d) Centering and phase-stripping restoration: *Projecting each witness onto* $\mathbf{1}^\perp$ *via* $w_\perp = nw - (\sum_i w_i)\mathbf{1}$ *preserves (and strictly improves whenever* $\sum_i w_i \neq 0$*) the violation ratio, whereas replacing all imaginary parts of* (u, v) *by* 0 *restores* $w^*F_{\text{Re}}w \leq \text{per}(A_{\text{Re}})\|w\|_2^2$ *(at* $n = 9, s = 1$ *in Lean* `thm19`*, and across all four witnesses in Certificates* `cert12`–`cert13`*).*

Proof. Every claim in parts (a)–(d) is a finite identity over $\mathbb{Z}[i]$. In Layer 1, Lean 4.36.0 directly evaluates the exact $O(n^3)$ Cauchy–Binet ESP expressions (4)–(5) over $\mathbb{Z}[i]$ via `decide` in `thm16`–`thm19`. In Layer 2, `verification/verify_bapat_sunder_rank2.py` independently builds the full $n \times n$ Gaussian-integer matrices A_n, computes per(A_n) and every $(n-1) \times (n-1)$ permanent cofactor per$(A_n(i|j))$ via direct Gray-code Ryser permanent over $\mathbb{Z}[i]$, asserts exact entrywise equality $F_{\text{ESP}} = F_{\text{Ryser}}$, verifies exact rank 2 ($\det([u, v]^*[u, v]) > 0$, Certificate `cert14`), verifies $F_A\mathbf{1} = \text{per}(A)\mathbf{1}$, and certifies the exact integer gaps in Certificates `cert08`–`cert13`. □

5 Formal Verification Architecture and Scope

Our verification artifact cleanly separates machine-checked algebraic identities and direct kernel evaluations from prose analytic proofs and exact Python cross-checks:

- **Layer 1 — Constructive Lean 4.36.0 Kernel** (`lean/BapatSunderRank2.lean`): Twenty theorems compiled with `lean -DwarningAsError=true`, zero external dependencies, zero `sorry`, zero `grind`, zero `Classical.choice`, and no custom axioms beyond standard constructive `propext`/`Quot.sound` (audited via `#print axioms`). Layer 1 formalizes the exact $n = 2$ quadratic identity and unconditional Bapat–Sunder bound (`thm01`–`thm03`),

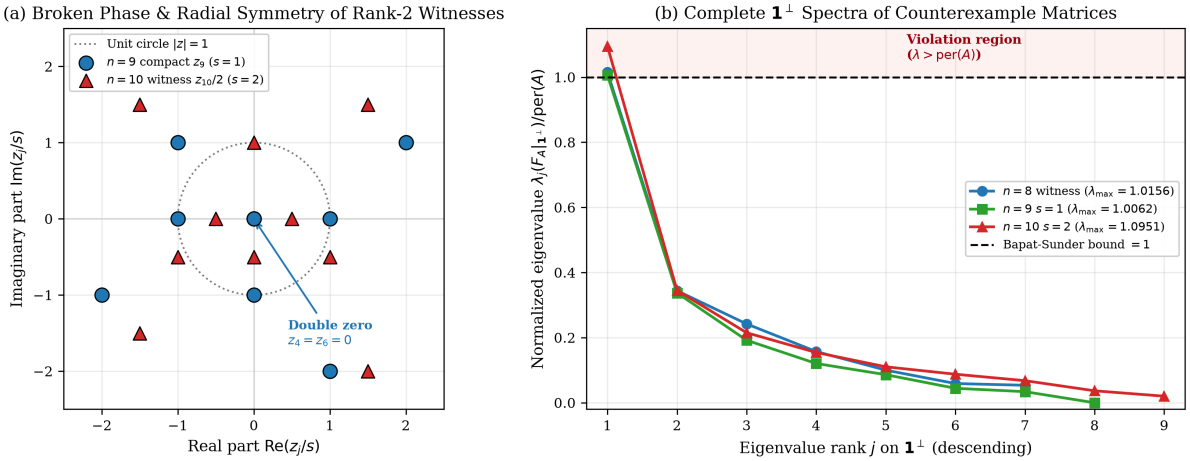


Figure 3: **Broken phase/radial geometry and full $\mathbf{1}^\perp$ spectra of counterexamples.** (a) Complex-plane coordinates of z_9 ($s = 1$, including the double zero $z_4 = z_6 = 0$) and $z_{10}/2$ ($s = 2$) contrasted with the unit circle. (b) Complete descending eigenvalues of $F_A|_{\mathbf{1}^\perp}/\operatorname{per}(A)$ for the $n = 8, 9, 10$ witnesses: each exhibits a single clean outlier eigenvalue piercing the Bapat–Sunder threshold 1.

the exact $n = 3$ shift-Laplacian identity, Laplacian PSDness, rank-2 $\mathbf{1}^\perp$ collapse, and correlation discriminant identity (`thm04`–`thm08`), diagonal scaling invariance for $n = 2, 3$ (`thm09`–`thm10`), circulant unit-circle and all-r barriers (`thm11`–`thm13`), real balanced saturation for $m = 1, \ldots, 4$ and direct-sum eigenpairs (`thm14`–`thm15`), direct kernel evaluations of all three Gaussian-integer rank-2 refutations at $n = 8, 9, 10$ and phase-stripping restoration (`thm16`–`thm19`), and ESP-vs-direct-3 $\times$ 3 permanent equivalence (`thm20`). The general-n Cauchy–Binet proof (Theorem 1(c)), general-n weighted AM–GM circulant bound (Theorem 2(c)), and general-n direct-sum argument (Theorem 2(d)) are proved in full mathematical detail in Sections 2–3.

- **Layer 2 — Zero-Float Python Verifier** (`verification/verify_bapat_sunder_rank2.py`): An AST-audited pure standard-library verifier using strictly `int` and `Fraction` (zero floats, zero numeric `/`, zero `**`, zero `limit_denominator`), emitting 16 SHA-256 certificates in `certificates/MANIFEST.json` and validated by 56 adversarial `pytest` checks in `tests/test_adversarial.py`. Crucially, Layer 2 independently cross-checks every $n = 2, \ldots, 10$ Cauchy–Binet ESP matrix F_{ESP} entry-by-entry against direct Gray-code Ryser permanent evaluation over $\mathbb{Z}[i]$, confirming exact agreement across all four refutations (`cert08`–`cert11`).

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