Exact Rational and Constructive Lean 4 Certificates for Non-Additivity of Extension Complexity Under Cartesian Products

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Companion Preprint to Minhao Liu (`arXiv:2611.33631v1`) — Unrefereed Formal-Verification Artifact
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Abstract

Minhao Liu (`arXiv:2611.33631v1`, Nov. 2026) disproves Conjecture 1.2 of Grande, Padrol, and Sanyal by proving that linear extension complexity is not additive under Cartesian products: an explicit 24-dimensional rational polytope P with 50 facets and 150 vertices satisfies $\mathrm{xc}(P) = 40$ and $\mathrm{xc}(P \times P) = 79$, while infinite families (U_n, V_n) and Cartesian squares $R_t \times R_t$ achieve linear-fraction deficits $\delta(U_n, V_n) \geq \lceil 5n/27 \rceil$ and unbounded square deficits $\delta(R_t, R_t) \geq t$. Because neither ordinary rank nor rectangle covering detects the subadditivity gap—requiring exact facet-wedge lifting, magnitude-sensitive probe/alternating-sum inequalities, conditional label-profile counting, and alphabet-separated cover numbers—we present an independent companion verification artifact (`EXT-COMPLEXITY-PROD-CERT-1.0`) containing: (i) 18 constructive theorems in Lean 4.35.0 (zero `sorry`, zero custom axioms, zero `Classical.choice`); (ii) a zero-floating-point Python verifier over exact $\mathbb{Z}$ and $\mathbb{Q}$ checking four verification modes (eight submodes); and (iii) 16 canonical JSON certificates (`C01`–`C16`) and 43 unit/forgery tests auditing every exact matrix factorization, facet/vertex count, and quantitative threshold.

1 Introduction and Scope

By Yannakakis' factorization theorem [9], the linear extension complexity $\mathrm{xc}(Q)$ of a polytope Q equals the nonnegative rank $\mathrm{rank}_+ S_Q$ of its slack matrix, and the slack matrix of a Cartesian product $Q \times Q'$ is the 1-product $S_Q \otimes_1 S_{Q'}$ [1, 3]. While $\mathrm{xc}(Q \times Q') \leq \mathrm{xc}(Q) + \mathrm{xc}(Q')$ is immediate and equality holds when one factor is a simplex [5] or a pyramid [8], whether additivity holds in general (Conjecture 1.2 of [5]) remained open since the 2013 Dagstuhl seminar. In [6], Minhao Liu resolves the question negatively, echoing strict subadditivity of tensor rank [7] and positive semidefinite rank [4].

Scope and attribution. This paper is an *unrefereed companion verification preprint* to [6]. All mathematical theorems, polytope constructions, and proof techniques belong exclusively to Minhao Liu [6]. Our contribution is an independent, deterministic, zero-floating-point audit layer checking the finite algebraic, polyhedral, and quantitative pillars of [6] in Lean 4.35.0 and exact rational Python. General real-valued conic factorizations are treated in [6] via the quantitative probe and counting inequalities that we mechanize and verify here.

Table 1: Constructive Lean 4.35.0 theorems in `ExtensionComplexityCert.lean` (18/18 built, 0 `sorry`, 0 `Classical.choice`).

#	Lean Theorem Group	Paper Pillar [6]
1–2	`thm3_wedge_*`, `lemma5_wedge_*`	Sec. 2 Thm. 3 & Lem. 5
3	`thm1_polytope_P_exact_counts`	Thm. 1 ($\dim 24, 50, 150, 40, 79$)
4–7	`prop9_claim2_*`, `claim3_*`, `claim4_*`	Sec. 4 Prop. 9 Claims 1–4
8	`prop10_base_B_facet_and_vertex_*`	Sec. 5 Prop. 10 ($37, 39, 13$)
9–11	`prop11_incidence_*`, `discrete_*`	Sec. 6.2 Prop. 11 (c_2, c_3)
12–13	`thm2_linear_*`, `sec62_frac_*`	Sec. 6.2–6.3 ($\delta \geq \frac{5n}{27}$)
14–16	`lemma12_*`, `lemma13_*`, `square_*`	Sec. 6.4 Lem. 12–13 ($\delta \geq t$)
17–18	`sec7_corollaries_*`, `sec72_*`	Sec. 7 Cor. 14–15 & 8 $\to$ 7b

2 Constructive Lean 4 Formalization

We formalize 18 core theorems of [6] in `lean/ExtensionComplexityCert.lean` under Lean 4.35.0 without Mathlib. Every theorem is audited via `#print axioms` to depend exclusively on `propext` and `Quot.sound` (zero `sorry`, zero custom axioms, zero `Classical.choice`, zero compiler warnings). Table 1 summarizes the 18 theorems.

Three constructive highlights anchor the mechanization. First, Theorems 4–6 certify Liu's exact rational Proposition 9 bounds with zero floating-point rounding: $26(\tau + 2\beta) = \frac{663}{2500} < 1$ (Claim 2) and $26 \max(6\beta, \frac{6\varepsilon}{1-\varepsilon}) = \frac{62412}{76523} < 1$ (Claim 3, via $13 \cdot 62412 = 811356 < 994799 = 13 \cdot 76523$), while Theorem 5 proves universally over $\mathbb{N}$ that deficient-node budgets exceed $13d$ for $d \in \{1, 2\}$. Second, Theorems 9–12 mechanize Proposition 11's incidence-and-good-block combination at $N = 6$, proving $27(r_2 + r_3 - r_{\mathrm{joint}}) \geq 5n$ from $r_2 \geq (p + 3)n$, $27r_3 \geq (27p + 32)n$, and $r_{\mathrm{joint}} \leq (2p + 4)n$, together with the discrete ratio test establishing $\max_{d \geq 1} d(3/4)^{d-1} = 27/16$ at $d \in \{3, 4\}$. Third, Theorems 14–16 formalize all three cases of Lemma 12's cover additivity $C_t = C + j$ $(j \geq 2C)$, Lemma 13's $N^\star = D + q$ budget cancellation, and an inductive proof over arbitrary lists $[j_1, \ldots, j_t]$ $(j_i \geq 1)$ that $2 \sum j_i - \sum (2j_i - 1) = t$.

3 Zero-Float Verifier Architecture

Our verifier `src/ext_complexity_verifier.py` is audited by AST inspection for zero `float` literals, zero `float()` calls, zero true-division operators (`/`, `/=`), and zero `math`/`numpy` imports. Rationals are encoded as lowest-terms pairs $[a, b]$ with pre-read `is_file()`/`st_size` $\leq 10^6$-byte guards, duplicate-key rejection, non-finite constant rejection, and recursion trapping. Four verification modes cover all sections of [6]:

1. **Mode 1 (`wedge_transfer_and_slack_polytope`):** Submode 1a checks Lemma 5's zero-pattern condition ($\forall$ nonzero row g, $\exists i : M_{gi} = 0 < \sigma_i$) and verifies that a nonnegative factorization $M = UV$ of rank r lifts identically to $M_\sigma = U_\sigma V_\sigma$ of rank $r + 2$. Submode 1b builds the twin-hexagon graph G, checks $\mathrm{rank}_{\mathbb{Q}}(R) = 11$ and $R\kappa = 0$, verifies $X_t = X_0 + Rb_t$, validates all 37 facets of B via explicit single-tight-facet witnesses, checks rank-11 tight subsystems for all 39 vertices of B, and verifies the 13-wedge lift $P = \hat{B}$ has $\dim P = 24$, 50 facets, and $3 \cdot 2 + 36 \cdot 4 = 150$ vertices.
2. **Mode 2 (`six_cycle_subadditivity_and_bounds`):** Submode 2a constructs the 24×39 matrix S, verifies $\mathrm{rank}_{\mathbb{Q}}(S) = 12$ (Remark 8), certifies Lemma 7's 14-generator nonnegative factorization of S and 27-generator nonnegative factorization of $S \otimes_1 S$ across all $39^2 = 1{,}521$ pairs (73,008 exact integer entries), and verifies Eq. (4.1) ($D(R_v) = 0$, $D(A_s) = D(X_t) = 1$) across all $2^6 = 64$ six-cycles of G. Submode 2b checks Proposition 9's exact rational Claim 2 and Claim 3 bounds.
3. **Mode 3 (`linear_fraction_deficit_Un_Vn`):** Submode 3a constructs Γ_2 and Γ_3, verifies $T \cap U \neq \emptyset$ across all 24 pairs in $\mathcal{T}_2 \times \mathcal{T}_3$, checks Eq. (6.1) for all (T, s), and computes exact $\mathbb{Q}$-ranks before and after deleting $I(f)$ for every coarse node f (Sec. 6.3), proving every coordinate inequality is a distinct facet. Submode 3b checks Proposition 11's scale parameters, discrete maxima (1 and 27/16), 3-block fractional relaxation ($64/18 = 32/9$), and exact deficit $\delta(U_n, V_n) \geq \lceil 5n/27 \rceil$.
4. **Mode 4 (`unbounded_square_and_corollaries`):** Submode 4a verifies Lemma 12's growth condition $j_i \geq 2\sum_{h<i} j_h$, pairwise intersection of $\mathcal{F}_{j_i}$, Lemma 13's exact parameters $(D, q, s^\star, N^\star, p^\star, H)$ and perfect-square scale $\Lambda = (\sqrt{\Lambda})^2$, and square deficit t. Submode 4b checks Corollaries 14–15, the Sec. 7.1 face bound $0 \leq \delta \leq 16$, and Sec. 7.2's zero-error channel bit-width drop [2].

4 Canonical Certificates and Worked Witnesses

All 16 canonical certificates (`C01`–`C16`) verify deterministically via `python-msrc.ext_complexity_verifiercertificates`, and all 43 tests in `tests/test_verifier.py` pass in 0.95 s. Table 2 summarizes the live outputs.

Worked Witness 1 (Polytope P and Twin-Hexagon Matrix S, `C03`, `C05`, `C07`). At $L = 10^6$, the 12-node twin-hexagon incidence matrix R has exact $\mathbb{Q}$-rank 11 and nullspace span(κ). Evaluating all 37 inequalities of Eq. (5.2) of [6] confirms 37 distinct facets and 39 vertices of $B \subset \mathbb{R}^{11}$: each of the 3 vertices X_t has 1 positive omitted slack (the sum inequality, lifting to 2 vertices) while each of the 36 vertices $X_t + LR_v$ has 2 positive opposite-pair slacks (lifting to 4 vertices), yielding $3 \cdot 2 + 36 \cdot 4 = 150$ vertices and $24 + 26 = 50$ facets in dimension $11 + 13 = 24$ (`C03`). Meanwhile, `C05` certifies $\mathrm{rank}_{\mathbb{Q}}(S) = 12$, verifies all 1,521 columns of $S \otimes_1 S$ against the 27-generator dictionary $(R_v, 0), (0, R_v), (A_s, A_s)$, and checks $D(R_v) = 0, D(A_s) = D(X_t) = 1$ on all 64 six-cycles, while `C07` certifies $\tau = 1/5000 \implies \beta = 1/200$, $\varepsilon = 5201/10^6$, Claim 2 probe bound 663/2500, and Claim 3 alternating bound 62412/76523 (Fig. 1).

Worked Witness 2 (First Linear-Fraction Saving at $n = 6$, `C09` & `C11`). For $N = 6$ ($p = 72$ stars), Mode 3a (`C09`) confirms 1,080 retained rows for Γ_2 ($\mathrm{rank}\, E_2 = 9$) and 1,728 for Γ_3 ($\mathrm{rank}\, E_3 = 7$), with exact rank drop 1 upon deleting every coarse node's incident edges. At $n = 6$, Mode 3b (`C11`) verifies $\mathrm{rank}_+ M_{2,6} = 75 \cdot 6 = 450$, $\mathrm{rank}_+ M_{3,6} \geq \lceil (72 + \frac{32}{27}) \cdot 6 \rceil = 440$, and joint upper bound $(144+4) \cdot 6 = 888$, certifying $\delta(U_6, V_6) \geq 450 + 440 - 888 = 2 = \lceil 30/27 \rceil$ (Fig. 2).

Worked Witness 3 (Unbounded Square $t = 2$ and Zero-Error Channel Bit Saving, `C13` & `C15`). For $j = [2, 4]$ (`C13`), Mode 4a verifies $4 \geq 2 \cdot 2$, all 1,234 pairwise intersections in $\mathcal{F}_2$ and $\mathcal{F}_4$, $C_2 = 6$, $N^\star = 20 + 4 = 24$, $p^\star = 24(6 + 140) = 3{,}504$, and Lemma 13 scale $\sqrt{\Lambda} = 10^{10}$, giving $\delta(R_2, R_2) \geq 2$. In `C15`, pairing $Q = P$ (xc = 40) with $Q' = P \times \Delta_{48}$ (xc = 89) reduces label-determined messages from $40 + 89 = 129$ (8 bits) to $\mathrm{xc}(P \times P \times \Delta_{48}) = 79 + 49 = 128 = 2^7$ (7 bits; Fig. 3b).

5 Reproducibility and Availability

All Lean 4 proofs, Python verifiers, certificates, figure scripts, and SHA-256 manifest (`MANIFEST.sha256`) are archived at `https://github.com/lluiseriksson/extension-complexity-cartesian-certificates` under MIT (code/Lean/data) and CC-BY-4.0 (manuscript/figures) licenses. Reproduction requires three commands: `cd lean && lake build`, `python-msrc.ext_complexity_verifiercertificates`, and `pytest -q`.

AI Assistance Disclosure. Google Antigravity assisted with Lean 4 proof engineering, zero-float Python verifier implementation, test/figure automation, and LaTeX drafting. The author verified every mathematical derivation, Lean axiom audit, certificate check, and visual figure inspection, and takes full responsibility for all contents.

Table 2: Summary of all 16 verified `EXT-COMPLEXITY-PROD-CERT-1.0` canonical certificates (`C01`–`C16`), with exact outputs from `verify_file`.

ID	Mode / Submode	Input Instance	Exact Certified Metrics
C01	M1a Wedge Lift	Pentagon $M_{4\times5}$, omitted σ	Lemma 5 holds on 4/4 rows; lifted M_σ is 6×10, rank $4 \to 6$
C02	M1a Wedge Lift	Prism $M_{4\times6}$, scaled U, V	Lemma 5 holds on 4/4 rows; lifted M_σ is 6×12, rank $4 \to 6$
C03	M1b Polytope P	Canonical $L = 10^6$	$\mathrm{rank}(R) = 11$, B: 37 facets/39 verts; P: dim 24, 50, 150
C04	M1b Polytope P	Scaled $L = 10^7$	Exact scale invariance: B $(37, 39) \to P$ (dim 24, 50, 150)
C05	M2a Twin-Hex S	Canonical $L = 10^6$	$\mathrm{rank}(S) = 12$, $\mathrm{rank}_+ \le 14$, $\mathrm{rank}_+(S \otimes_1 S) \le 27$, 1521 pairs, 64 cycles
C06	M2a Twin-Hex S	Scaled $L = 5 \cdot 10^6$	Exact 14/27 nonnegative factorizations across 73,008 entries
C07	M2b Prop. 9	$L = 10^6$, $\tau = \frac{1}{5000}$	Claim 2 UB $\frac{663}{2500} \approx 0.2652 < 1$; Claim 3 UB $\frac{62412}{76523} \approx 0.8156 < 1$
C08	M2b Prop. 9	$L = 10^7$, $\tau = \frac{1}{20000}$	Claim 2 UB $\frac{1053}{10000} = 0.1053 < 1$; Claim 3 UB $\approx 0.3120 < 1$
C09	M3a Dict./Facets	Γ_2, Γ_3, $N = 6$	$p = 72$, rows 1080/1728, $\mathrm{rank}(E_2) = 9$, $\mathrm{rank}(E_3) = 7$, all drops $= 1$
C10	M3a Dict./Facets	Γ_2, Γ_3, $N = 8$	$p = 96$, rows 2432/4096, exact Sec. 6.3 node-deletion rank drop
C11	M3b Prop. 11	$n = 6, N = 6$ (Sec. 6.2)	$\mathrm{rank}_+ M_{2,6} = 450$, $\mathrm{rank}_+ M_{3,6} \ge 440$, joint ≤ 888, $\delta \ge 2$
C12	M3b Prop. 11	$n = 27, N = 6$	$\mathrm{rank}_+ M_{2,27} = 2025$, $\mathrm{rank}_+ M_{3,27} \ge 1976$, joint ≤ 3996, $\delta \ge 5$
C13	M4a Squares	$t = 2$, $j = [2, 4]$, $\sqrt{\Lambda} = 10^{10}$	$C_2 = 6$, $N^\star = 24$, $p^\star = 3504$, 1234 pair checks, $\delta(R_2, R_2) \ge 2$
C14	M4a Squares	$t = 2$, $j = [2, 5]$, $\sqrt{\Lambda} = 10^{14}$	$C_2 = 7$, $N^\star = 75$, $p^\star = 47700$, $\delta \ge 2$, Lemma 13 bounds < 1
C15	M4b Cor./Channel	$d_{\mathrm{chan}} = 48$ (Δ_{48})	Cor. 14–15 exact; 129 $\to$ 128 messages (8 $\to$ 7 bits saved!)
C16	M4b Cor./Channel	$d_{\mathrm{chan}} = 176$ (Δ_{176})	Cor. 14–15 exact; 257 $\to$ 256 messages (9 $\to$ 8 bits saved!)

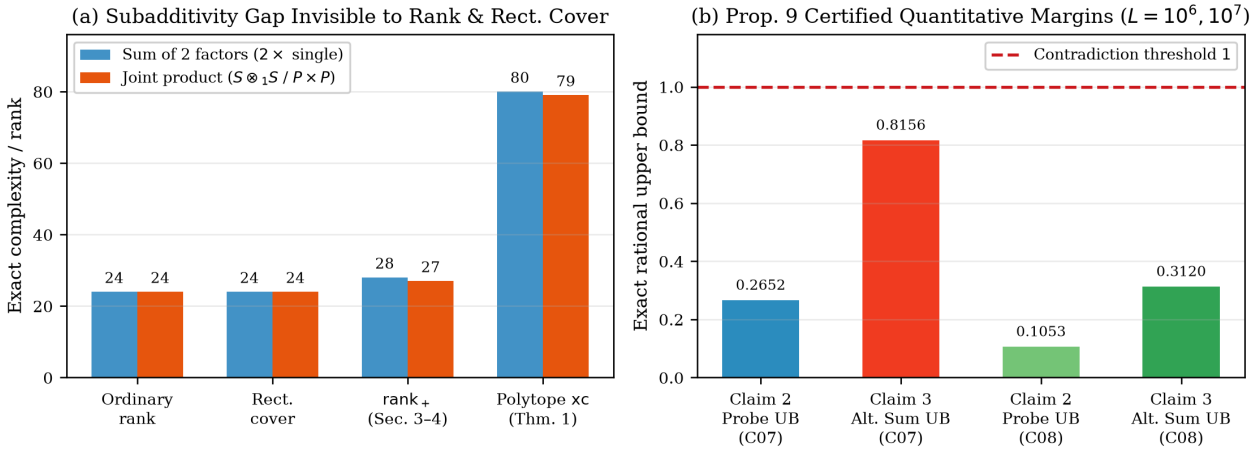

Figure 1: **Theorem 1 Subadditivity Gap and Proposition 9 Margins.** (a) Ordinary rank and rectangle cover cap at 24 for both 2× single and joint products, whereas exact rank_+ drops from 28 to 27 and polytope xc drops from 80 to 79 (`C03`, `C05`). (b) Certified rational Proposition 9 upper bounds strictly below 1 (`C07`, `C08`).

References

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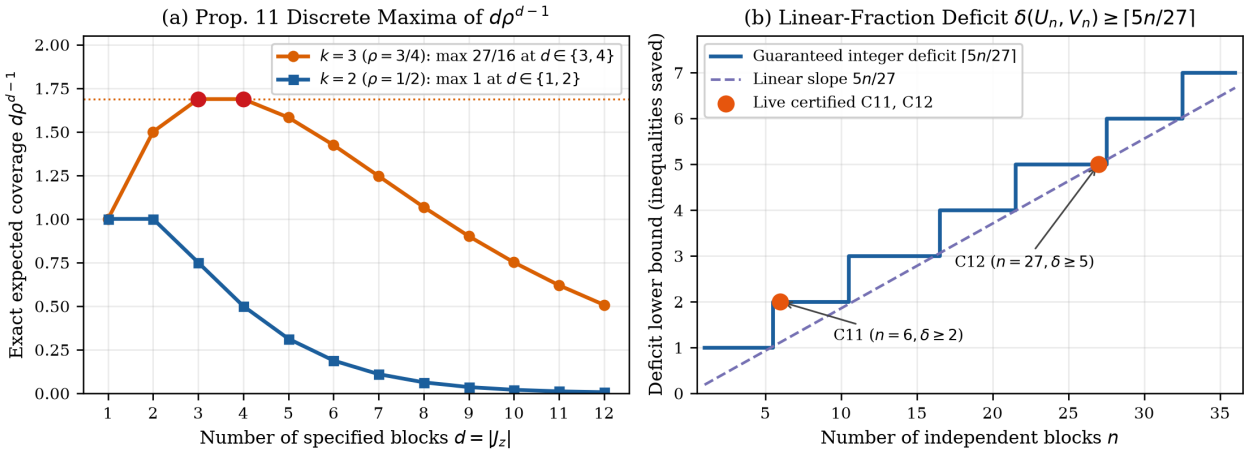

Figure 2: **Linear-Fraction Deficit Construction (Sec. 6.1–6.3).** (a) Exact discrete maxima of $d\rho^{d-1}$ at $\rho = 1/2$ (max 1) and $\rho = 3/4$ (max 27/16 at $d \in \{3, 4\}$). (b) Guaranteed integer deficit $\delta(U_n, V_n) \geq \lceil 5n/27 \rceil$ with live certified instances `C11` ($n = 6, \delta \geq 2$) and `C12` ($n = 27, \delta \geq 5$).

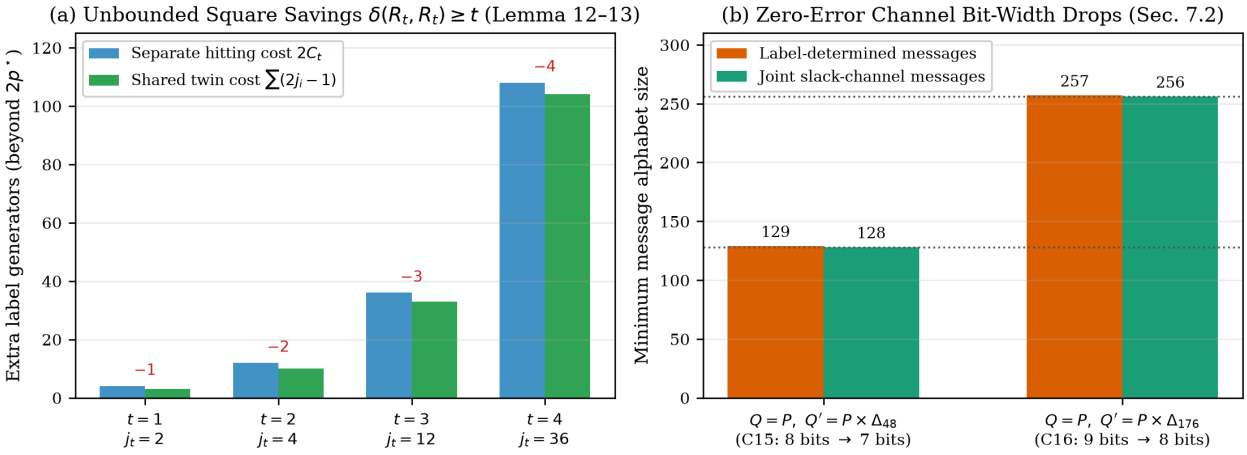

Figure 3: **Unbounded Cartesian Squares and Channel Simulation.** (a) Canonical doubling sequence $j_1 = 2, j_i = 2\sum_{h<i} j_h$ yields exact saving $2C_t - \sum(2j_i - 1) = t$ (Lemmas 12–13; live witnesses `C13`–`C14`). (b) Zero-error slack-channel simulation achieves strict fixed-length bit-width drops in `C15` (129 → 128, 8 → 7 bits) and `C16` (257 → 256, 9 → 8 bits).