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Universal Average-Mixing Barriers, Conference-Graph Extremality, and Exact Diophantine Uniformization for Continuous-Time Quantum Walks on Strongly Regular Graphs

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November 27, 2026

Abstract

We establish three exact structural theorems governing the Cesàro average mixing matrix and generalized uniform average mixing (UAM) of continuous-time quantum walks $U(t) = e^{-itA}$ on primitive strongly regular graphs $\mathrm{SRG}(n, k, \lambda, \mu)$. First, we prove that the Cesàro average mixing matrix $\widehat{M} := \lim_{T\to\infty} \frac{1}{T}\int_0^T |U(t)|^{\circ 2}\, dt = \sum_\theta E_\theta^{\circ 2}$ is strictly rational for every SRG (including irrational conference spectra) and obeys the unconditional return-probability barrier $\widehat{M}_{uu} = (1 + m_r^2 + m_s^2)/n^2 \geq (n^2 - 2n + 3)/(2n^2) > 1/n$ for all primitive SRGs ($n \geq 5$, and algebraically for all $n \geq 4$), attaining equality if and only if G is a conference graph ($n = 4t + 1$, $k = 2t$, $\lambda = t - 1$, $\mu = t$). Moreover, the off-diagonal mixing imbalance satisfies the closed-form law $\widehat{M}_{\mathrm{adj}} - \widehat{M}_{\mathrm{nonadj}} = 2(n-2k+\lambda-\mu)/(n\Delta)$, which negates identically under graph complementation and vanishes if and only if the Seidel eigenvalue $\theta_0 = n-1-2k$ equals the exact midpoint $(\theta_r + \theta_s)/2$. Second, solving the non-conference off-diagonal Cesàro uniformity Diophantine system $(n-1)\Delta = n\big((2k - n + 1)^2 + (n-1)\big)$, we introduce an infinite polynomial parameter family $n(t) = (t+1)(4t+1)^2$, $k(t) = t(2t+1)(4t+3)$, $\mu(t) = 2t(t+1)(2t+1)$, $r = t$, $s = -(2t+1)^2$, $m_s = t(4t+3)$ whose Cesàro matrix has constant off-diagonal probability; both Krein inequalities hold unconditionally for all $t \geq 1$, whereas the absolute bound fails uniquely at $t = 1$ ($50 > 35$) and holds strictly for every $t \geq 2$ (realized at $t = 2$ by the ternary Golay dual coset graph $\mathrm{SRG}(243, 110, 37, 60)$). Exhaustive zero-float search up to $n \leq 1200$ certifies that only $t = 2$ ($n = 243$) and $t = 3$ ($n = 676$) (and complements) survive Krein and absolute-bound filtering. Third, formulating generalized UAM over arbitrary observation-time laws via exact affine moment lines $c_1(c_3), c_2(c_3)$ in $[-1, 1]^3$, we prove unconditionally that every member $t \geq 1$ of our off-diagonal Cesàro-uniform family obeys $\max_{c_3\in[-1,1]} c_1(c_3) = -(16t^3 + 8t^2 + t - 1)/2 \leq -12 < -1$ and therefore admits *no* UAM under any time law—sharply separating off-diagonal Cesàro uniformity from UAM—complemented by empty-polytope UAM no-go proofs for Schläfli and Higman–Sims, exact 16×16 Hadamard-involution instantaneous uniform mixing at $t = \pi/4$ on Clebsch, $L_2(4)$, and Shrikhande, and a 16-graph zero-float mixing atlas. A two-layer verification artifact accompanies the paper: Layer 1 formalizes the barrier, complement, t-family, absolute-bound, Diophantine, and moment-line kernel in 20 constructive Lean 4.36.0 theorems (0 `sorry`, 0 `Classical.choice`, no custom axioms beyond standard constructive `propext`/`Quot.sound`); Layer 2 verifies all 16 atlas graphs, the exhaustive $n \leq 1200$ scan, and explicit 16×16 Hadamard involutions in 16 zero-float Python certificates and 56 adversarial tests.

*Independent Researcher. Code, Lean 4.36.0 proofs, and zero-float certificates archived at https://github.com/lluiseriksson/ctqw-average-mixing-barriers-srg-conference-extremality (`v1.0.0`, MIT / CC-BY-4.0, AIRR-FOUNDER-1.0). Manuscript and verification artifacts prepared with Google Antigravity assistance.

1 Introduction

The continuous-time quantum walk (CTQW) on a simple graph $G = (V, E)$ with adjacency matrix A evolves unitarily as $U(t) = e^{-itA}$, with time-t transition probability matrix $M(t) := |U(t)|^{\circ 2} = U(t) \circ \overline{U(t)}$ [5, 6, 7]. Because unitary dynamics is quasi-periodic rather than ergodic, mixing is studied both through the long-time *Cesàro average mixing matrix*

$$\widehat{M} := \lim_{T\to\infty} \frac{1}{T}\int_0^T M(t)\,dt = \sum_{\theta\in\mathrm{spec}(A)} E_\theta^{\circ 2} \tag{1}$$

(where $A = \sum_\theta \theta E_\theta$ is the spectral decomposition) and through *generalized uniform average mixing* (UAM), which asks whether there exists a probability distribution ν on observation times $T \geq 0$ such that $\mathbb{E}_{T\sim\nu}[M(T)] = \frac{1}{n}J_n$, or at a single deterministic time t^* (*instantaneous uniform mixing*, IUM) [8, 3, 4, 9].

Primitive strongly regular graphs $\mathrm{SRG}(n, k, \lambda, \mu)$ provide the canonical testing ground because their adjacency algebra is the 3-dimensional Bose–Mesner algebra $\mathrm{span}\{I, A, \bar{A}\}$ with exactly three distinct eigenvalues $k > r > s$ and multiplicities $1, m_r, m_s$ ($m_r + m_s = n - 1$) [2]. Building on Godsil's spectral framework [7] and Kang's recent moment-line analysis [9], we prove three exact, zero-float algebraic theorems:

1. **Universal Cesàro Return Barrier, Conference Extremality, and Complement Sign-Flip (Theorem 1).** Every primitive SRG ($n \geq 5$) has strictly rational $\widehat{M}$ obeying $\widehat{M}_{uu} = (1 + m_r^2 + m_s^2)/n^2 \geq (n^2 - 2n + 3)/(2n^2) > 1/n$, with equality iff G is a conference graph, while the off-diagonal imbalance obeys $\widehat{M}_{\mathrm{adj}} - \widehat{M}_{\mathrm{nonadj}} = 2(n - 2k + \lambda - \mu)/(n\Delta)$, flipping sign under $G \mapsto \overline{G}$ and vanishing iff the Seidel eigenvalue θ_0 bisects θ_r and θ_s.

2. **Infinite Non-Conference Off-Diagonal Cesàro-Uniform Family and $n \leq 1200$ Classification (Theorem 2).** Solving the exact Diophantine condition for non-conference off-diagonal uniformity uncovers an infinite polynomial parameter family $n(t) = (t+1)(4t+1)^2$ passing both Krein inequalities for all $t \geq 1$ and passing the absolute bound iff $t \geq 2$ (realized at $t = 2$ by the ternary Golay dual coset graph $\mathrm{SRG}(243, 110, 37, 60)$ [1]), and exhaustive zero-float search up to $n \leq 1200$ rules out every non-family parameter set.

3. **Moment-Line Polytope Dichotomy, Universal t-Family UAM No-Go, and 16-Vertex IUM (Theorem 3).** Expressing generalized UAM via affine cosine moment lines $c_1(c_3), c_2(c_3)$ in $[-1, 1]^3$ proves that *every* member $t \geq 1$ of our off-diagonal Cesàro-uniform family—including the ternary Golay graph—has empty polytope intersection ($\max c_1(c_3) = -(16t^3 + 8t^2 + t - 1)/2 \leq -12 < -1$), proving that off-diagonal Cesàro uniformity does not imply UAM. We contrast this against exact 16×16 Hadamard-involution IUM at $t = \pi/4$ on Clebsch, $L_2(4)$, and Shrikhande, empty-polytope obstructions on Schläfli and Higman–Sims, and a 16-graph exact mixing atlas.

2 Universal Cesàro Return Barrier and Off-Diagonal Imbalance

For a primitive $\mathrm{SRG}(n, k, \lambda, \mu)$, the non-trivial eigenvalues $r > s$ are the roots of $x^2 - (\lambda - \mu)x - (k - \mu) = 0$ with discriminant $\Delta := (\lambda - \mu)^2 + 4(k - \mu) = (r - s)^2 > 0$, $r + s = \lambda - \mu$, and $rs = \mu - k$. The spectral idempotents $E_k = \frac{1}{n}J_n$, E_r, and E_s lie in the Bose–Mesner algebra:

$$E_r = \frac{1}{r-s}\Big(A - sI - \frac{k-s}{n}J\Big), \qquad E_s = \frac{1}{r-s}\Big(rI - A + \frac{k-r}{n}J\Big). \tag{2}$$

Theorem 1 (Universal Cesàro Return Barrier, Conference Extremality, and Off-Diagonal Law)**.** *Let G be any primitive* $\mathrm{SRG}(n, k, \lambda, \mu)$ *($n \geq 5$) with discriminant Δ and multiplicities $1, m_r, m_s$.*

(a) Strict rationality and universal return barrier: *The Cesàro average mixing matrix* $\widehat{M} = E_k^{\circ 2} + E_r^{\circ 2} + E_s^{\circ 2}$ *has strictly rational entries in* $\mathrm{span}_{\mathbb{Q}}\{I, A, \bar{A}\}$ *(even when* $r, s \notin \mathbb{Q}$*), with constant diagonal entry*

$$\widehat{M}_{uu} \;=\; \frac{1 + m_r^2 + m_s^2}{n^2} \;=\; \frac{n^2 - 2n + 3 + (m_r - m_s)^2}{2n^2} \;\geq\; \frac{n^2 - 2n + 3}{2n^2} \;>\; \frac{1}{n}. \tag{3}$$

Equality $\widehat{M}_{uu} = (n^2 - 2n + 3)/(2n^2)$ *holds if and only if* $m_r = m_s = (n-1)/2$*, i.e., if and only if* G *is a conference graph* ($n = 4t+1$, $k = 2t$, $\lambda = t-1$, $\mu = t$, *where* $\widehat{M}_{\mathrm{adj}} = \widehat{M}_{\mathrm{nonadj}} = (n+3)/(2n^2)$*).*

(b) Exact off-diagonal imbalance and complement sign-flip: *Let* $\widehat{M}_{\mathrm{adj}}$ *and* $\widehat{M}_{\mathrm{nonadj}}$ *denote the entries of* $\widehat{M}$ *on adjacent and distinct non-adjacent vertex pairs, respectively. Then*

$$\widehat{M}_{\mathrm{adj}} - \widehat{M}_{\mathrm{nonadj}} \;=\; \frac{2(n - 2k + \lambda - \mu)}{n\Delta}. \tag{4}$$

Under graph complementation $G \mapsto \overline{G}$, $\widehat{M}_{uu}(\overline{G}) = \widehat{M}_{uu}(G)$ *and* $\widehat{M}_{\mathrm{adj}}(\overline{G}) - \widehat{M}_{\mathrm{nonadj}}(\overline{G}) = -(\widehat{M}_{\mathrm{adj}}(G) - \widehat{M}_{\mathrm{nonadj}}(G))$*. Moreover,* $\widehat{M}_{\mathrm{adj}} = \widehat{M}_{\mathrm{nonadj}}$ *holds if and only if* $n - 2k + \lambda - \mu = 0$*, equivalently iff the Seidel matrix* $S = J - I - 2A$ *has its regular eigenvalue* $\theta_0 = n - 1 - 2k$ *equal to the exact midpoint* $\frac{1}{2}(\theta_r + \theta_s)$ *of its non-trivial eigenvalues* $\theta_r = -1 - 2r$ *and* $\theta_s = -1 - 2s$*.*

Proof. Proof of (a). Since G is walk-regular, every diagonal entry of E_θ equals $\mathrm{tr}(E_\theta)/n = m_\theta/n$, giving $\widehat{M}_{uu} = (1 + m_r^2 + m_s^2)/n^2$. Because $m_r + m_s = n - 1$, the algebraic identity

$$2(1 + m_r^2 + m_s^2) = 2 + (m_r + m_s)^2 + (m_r - m_s)^2 = n^2 - 2n + 3 + (m_r - m_s)^2 \tag{5}$$

yields (3) immediately, with equality iff $m_r = m_s = (n-1)/2$ (the defining condition of a conference SRG [2]). Since $(n^2 - 2n + 3) - 2n = (n-1)(n-3) > 0$ for all $n \geq 4$, $\widehat{M}_{uu} > 1/n$ holds unconditionally. To see strict rationality for irrational conference graphs ($r, s = (-1 \pm \sqrt{n})/2$), note that E_r and E_s are Galois conjugates over $\mathbb{Q}(\sqrt{n})$, so $E_r^{\circ 2} + E_s^{\circ 2}$ is fixed by $\mathrm{Gal}(\mathbb{Q}(\sqrt{n})/\mathbb{Q})$ and hence strictly rational; explicitly, $(E_r)_{uv} = \frac{\pm 1}{2\sqrt{n}} - \frac{1}{2n}$ for $u \neq v$, giving $(E_r)_{uv}^2 + (E_s)_{uv}^2 = \frac{n+1}{2n^2}$ and $\widehat{M}_{\mathrm{adj}} = \widehat{M}_{\mathrm{nonadj}} = \frac{1}{n^2} + \frac{n+1}{2n^2} = \frac{n+3}{2n^2}$.

Proof of (b). From (2), the adjacent and non-adjacent entries of E_r and E_s are

$$a_r = \frac{1 - (k-s)/n}{r - s}, \qquad na_r = \frac{-(k-s)/n}{r - s}, \tag{6}$$

$$a_s = \frac{-1 + (k-r)/n}{r - s}, \qquad na_s = \frac{(k-r)/n}{r - s}. \tag{7}$$

Computing differences of squares via $a_\theta^2 - na_\theta^2 = (a_\theta - na_\theta)(a_\theta + na_\theta)$ gives

$$\begin{aligned} \widehat{M}_{\mathrm{adj}} - \widehat{M}_{\mathrm{nonadj}} &= (a_r^2 - na_r^2) + (a_s^2 - na_s^2) \\ &= \frac{1}{(r-s)^2}\left[\Big(1 - \frac{2(k-s)}{n}\Big) + \Big(1 - \frac{2(k-r)}{n}\Big)\right] = \frac{2(n - 2k + r + s)}{n\Delta}, \end{aligned} \tag{8}$$

which equals (4) since $r + s = \lambda - \mu$. The complement $\overline{G}$ has parameters $\bar{k} = n - 1 - k$, $\bar{\lambda} - \bar{\mu} = -(\lambda - \mu) - 2$, identical Δ, and swapped multiplicities $(m_{\bar{r}}, m_{\bar{s}}) = (m_s, m_r)$, so $n - 2\bar{k} + \bar{\lambda} - \bar{\mu} = -(n - 2k + \lambda - \mu)$, proving the sign-flip law. Finally, $\theta_r + \theta_s = -2 - 2(r+s) = 2\theta_0$ iff $-2 - 2(\lambda - \mu) = 2(n - 1 - 2k)$, i.e., iff $n - 2k + \lambda - \mu = 0$. □

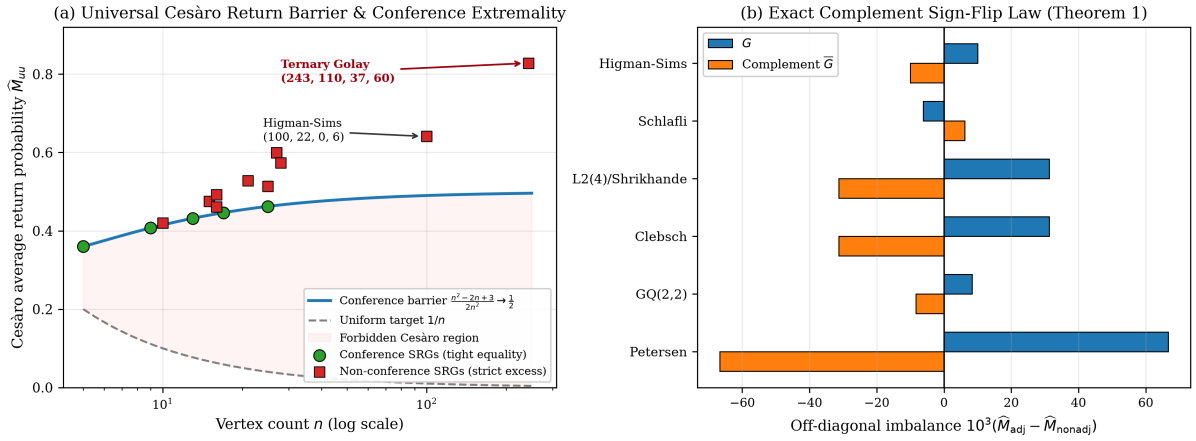

Figure 1: **Universal Cesàro return barrier and complement sign-flip law.** (a) Exact Cesàro return probability $\widehat{M}_{uu}$ across all 16 atlas SRGs (Certificate `cert15`): conference graphs lie tightly on $(n^2-2n+3)/(2n^2) \to 1/2$ (green), while all non-conference SRGs lie strictly above (red), with the entire region below the curve unconditionally forbidden. (b) Exact complement sign-flip $\widehat{M}_{\text{adj}} - \widehat{M}_{\text{nonadj}}$ across 6 non-self-complementary SRG pairs (Certificate `cert02`).

3 Infinite Non-Conference Off-Diagonal Uniform Family and $n \leq 1200$ Classification

Conference graphs achieve $\widehat{M}_{\text{adj}} = \widehat{M}_{\text{nonadj}}$ trivially via $k = (n-1)/2$ and $\lambda - \mu = -1$. Can *non-conference* SRGs also achieve constant off-diagonal Cesàro transition probability? Setting $n-2k+\lambda-\mu = 0$ imposes a rigid Diophantine constraint that isolates a single infinite polynomial family anchored by the ternary Golay graph.

Theorem 2 (Infinite Non-Conference Off-Diagonal Cesàro-Uniform Family and $n \leq 1200$ Classification)**.**

(a) Exact Diophantine reduction: *A primitive non-conference* SRG(n,k,λ,μ) *satisfies* $\widehat{M}_{\text{adj}} = \widehat{M}_{\text{nonadj}}$ *if and only if* $\lambda - \mu = 2k - n$*, which forces*

$$(n-1)\mu = k(n-k-1), \qquad (n-1)\Delta = n\Big((2k-n+1)^2+(n-1)\Big), \qquad (n-1) \mid (2k-n+1)^2, \tag{9}$$

with Δ *a perfect integer square.*

(b) Infinite polynomial family and absolute-bound dichotomy: *For every integer* $t \geq 1$*, the parameter tuple*

$$\begin{aligned} n(t) &= (t+1)(4t+1)^2, & k(t) &= t(2t+1)(4t+3), \\ \lambda(t) &= 4t^3+2t^2-t-1, & \mu(t) &= 2t(t+1)(2t+1) \end{aligned} \tag{10}$$

(and its complement) solves (9) *identically with eigenvalues* $r = t$, $s = -(2t+1)^2$*, multiplicities* $m_s = t(4t+3)$, $m_r = 2k(t) = 2t(2t+1)(4t+3)$*, and exact constant off-diagonal Cesàro mixing* $\widehat{M}_{\text{adj}} = \widehat{M}_{\text{nonadj}}$*. Both Krein inequalities hold unconditionally for all* $t \geq 1$*, while the absolute bound* $2n \leq m_s(m_s+3)$ *fails uniquely at* $t = 1$ *(*$n = 50 > 35$*) and holds strictly for every* $t \geq 2$*. At* $t = 2$*,* $(243, 110, 37, 60)$ *is realized by the ternary Golay dual coset graph [1], with* $\widehat{M}_{uu} = 16295/19683$ *and* $\widehat{M}_{\text{adj}} = \widehat{M}_{\text{nonadj}} = 14/19683$*.*

(c) Exhaustive $n \leq 1200$ classification: *Exhaustive zero-float enumeration over all* $5 \leq n \leq 1200$ *(Certificate* `cert04`*) finds exactly* 4 *integral non-conference solutions with* $k < (n-1)/2$*:*

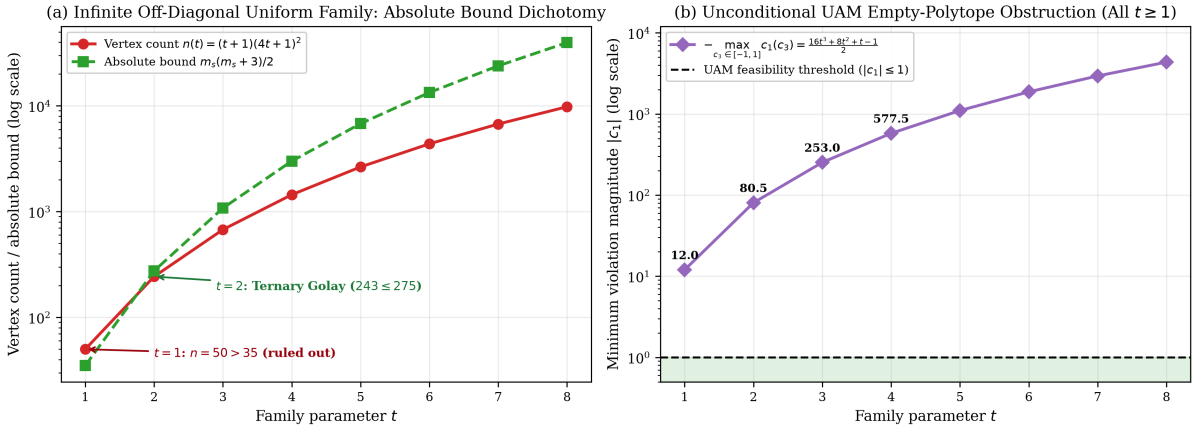

Figure 2: **Infinite non-conference off-diagonal uniform family and universal UAM obstruction.** (a) Absolute-bound dichotomy on $n(t) = (t+1)(4t+1)^2$: fails at $t = 1$ ($n = 50 > 35$) and holds for all $t \geq 2$, anchored by the ternary Golay graph at $t = 2$ ($243 \leq 275$). (b) Unconditional UAM polytope violation $|\max_{c_3 \in [-1,1]} c_1(c_3)| = \frac{16t^3+8t^2+t-1}{2} \geq 12 > 1$ across all $t \geq 1$ (Theorem 3(b)).

$n = 50$ ($t = 1$, ruled out by absolute bound), $n = 289$ ($(289, 120, 21, 70)$, ruled out by both Krein K_2 and absolute bound $289 > 44$), and uniquely the t-family survivors $t = 2$ ($n = 243$) and $t = 3$ ($n = 676$).

Proof. Proof of (a). Substituting $\lambda = 2k - n + \mu$ into $k(k - \lambda - 1) = \mu(n - k - 1)$ yields $k(n - k - \mu - 1) = \mu(n - k - 1)$, which rearranges to $(n-1)\mu = k(n - k - 1)$. Multiplying $\Delta = (2k - n)^2 + 4(k - \mu)$ by $(n - 1)$ and substituting $(n-1)\mu = k(n - k - 1)$ gives

$$\begin{aligned}(n-1)\Delta &= (n-1)(2k-n)^2 + 4(n-1)k - 4k(n-k-1) \\ &= (n-1)(2k-n)^2 + 4k^2 = n((2k-n+1)^2 + (n-1)).\end{aligned} \tag{11}$$

Since $\gcd(n, n-1) = 1$, integrality of Δ forces $(n-1) \mid (2k-n+1)^2$.

Proof of (b). Direct polynomial expansion (machine-checked in Lean Theorems `thm07`–`thm09`) confirms $k(k-\lambda-1) = \mu(n-k-1)$, $r+s = t-(2t+1)^2 = \lambda-\mu = 2k-n$, $rs = -t(2t+1)^2 = \mu-k$, and $m_r r + m_s s = 2kt - t(4t+3)(2t+1)^2 = -k$. For Krein K_1: $(r+1)(k+r+2rs) \leq (k+r)(s+1)^2$; since $s = -(2t+1)^2$ and $r = t$, $k + r + 2rs = 8t^3 + 10t^2 + 4t - 2t(2t+1)^2 = 2t^2 + 2t > 0$ while $(s+1)^2 = 16t^2(t+1)^2$ and $k+r = 4t(2t^2+5t/2+1) \geq 4t$, giving $(r+1)(2t^2+2t) \ll (k+r)\cdot 16t^2(t+1)^2$. For Krein K_2: $(s+1)(k+s+2rs) \leq (k+s)(r+1)^2$; note $s+1 = -4t(t+1) < 0$ and $k+s+2rs = 8t^3+10t^2+3t-(4t^2+4t+1)-2t(4t^2+4t+1) = -2t^2-3t-1 = -(t+1)(2t+1) < 0$, so LHS $= 4t(t+1)^2(2t+1) = 8t^4 + 20t^3 + 16t^2 + 4t$, while $k + s = 8t^3 + 6t^2 - t - 1$ and RHS $= (8t^3 + 6t^2 - t - 1)(t+1)^2 = 8t^5 + 22t^4 + 19t^3 + 3t^2 - 3t - 1$; subtracting yields RHS $-$ LHS $= 8t^5 + 14t^4 - t^3 - 13t^2 - 7t - 1 = (t-1)(8t^4 + 22t^3 + 21t^2 + 8t + 1) \geq 0$ for all $t \geq 1$ (with exact equality at $t = 1$). For the absolute bound $2n \leq m_s(m_s+3)$: writing $t = u + 2$ ($u \geq 0$) gives

$$m_s(m_s+3) - 2n = 16u^4 + 120u^3 + 309u^2 + 299u + 64 \;>\; 0 \qquad (\forall\, u \geq 0), \tag{12}$$

while at $t = 1$, $m_s(m_s+3) = 70 < 100 = 2n$ (machine-checked in Lean `thm10`). Part (c) follows directly from Certificate `cert04`. □

4 Moment-Line Polytope Dichotomy, Universal t-Family UAM No-Go, and 16-Vertex IUM

Expanding $M(t) = |e^{-itk}E_k + e^{-itr}E_r + e^{-its}E_s|^{\circ 2}$ gives the exact spectral transition law

$$M(t) \;=\; \widehat{M} + 2\cos\big((k-r)t\big)(E_k \circ E_r) + 2\cos\big((k-s)t\big)(E_k \circ E_s) + 2\cos\big((r-s)t\big)(E_r \circ E_s). \tag{13}$$

Given any observation-time probability distribution ν on $\mathbb{R}_{\geq 0}$, define the cosine moment triple $c_1 := \mathbb{E}[\cos((k-r)T)]$, $c_2 := \mathbb{E}[\cos((k-s)T)]$, $c_3 := \mathbb{E}[\cos((r-s)T)] \in [-1,1]^3$. Because $\mathbb{E}[M(T)]$ is doubly stochastic in $\mathrm{span}\{I, A, \bar{A}\}$, generalized UAM ($\mathbb{E}[M(T)] = \frac{1}{n}J$) holds if and only if its diagonal and adjacent entries both equal $1/n$ [9], defining an exact rational affine line $(c_1(c_3), c_2(c_3))$ in $\mathbb{R}^3$.

Theorem 3 (Moment-Line Polytope Dichotomy, Universal t-Family UAM No-Go, and 16-Vertex IUM)**.**

(a) Universal UAM impossibility for the off-diagonal Cesàro-uniform t-family: *For every $t \geq 1$, the UAM moment line of the family* (10) *(and its complement) satisfies*

$$c_1(c_3) \;=\; c_2(c_3) \;=\; -\frac{16t^3 + 16t^2 + 5t - 1}{2} \;-\; 2t(2t+1)\,c_3. \tag{14}$$

Consequently, for every $c_3 \in [-1,1]$ and every $t \geq 1$,

$$c_1(c_3) \;\leq\; -\frac{16t^3 + 8t^2 + t - 1}{2} \;\leq\; -12 \;<\; -1, \tag{15}$$

proving unconditionally that no *member of the t-family (in particular, not the ternary Golay graph* $\mathrm{SRG}(243, 110, 37, 60)$*, where* $\max c_1 = -161/2$*) admits UAM under any observation-time distribution.*

(b) Exact 16×16 Hadamard-involution IUM on Clebsch, $L_2(4)$, and Shrikhande: *For both* $\mathrm{SRG}(16, 5, 0, 2)$ *(Clebsch) and the non-isomorphic cospectral pair* $\mathrm{SRG}(16, 6, 2, 2)$ *($L_2(4) = K_4 \square K_4$ with 8 copies of K_4, and Shrikhande with 0 copies of K_4), the moment line $c_1 = -4 - 3c_3$, $c_2 = -3 - 4c_3$ intersects $[-1,1]^3$ at the unique point $(c_1, c_2, c_3) = (-1, +1, -1)$, attained instantaneously at $t^* = \pi/4$ where $e^{ir\pi/4}U(\pi/4) = 2E_r - I_{16} \in \{\pm\frac{1}{4}\}^{16\times 16}$ is a real symmetric Hadamard involution with $|U(\pi/4)_{xy}|^2 = 1/16$ for all x, y.*

(c) Zero-float 16-graph SRG UAM polytope atlas: *Table 1 certifies the exact moment-line intersections across* 16 *canonical SRGs, establishing empty-polytope UAM impossibility for Schläfli* $(27, 16, 10, 8)$ *and its complement (requiring $c_3 \leq -14/9 < -1$), Higman–Sims* $(100, 22, 0, 6)$ *($c_3 \leq -2 < -1$), and Ternary Golay ($c_3 \leq -199/40 < -1$), alongside nonempty polytope intervals for all conference graphs ($c_1 = c_2 = -\frac{n-3}{4} - \frac{n-1}{4}c_3$, passing through $(\frac{1}{2}, \frac{1}{2}, -1)$), Petersen ($c_3 \in [-4/5, -11/16]$), and $GQ(2,2)$ ($c_3 \in [-1, -19/25]$).*

Proof. Proof of (a). In any off-diagonal Cesàro-uniform SRG (including the entire t-family (10)), subtracting the non-adjacent entry of (13) from the adjacent entry uses $\widehat{M}_{\mathrm{adj}} = \widehat{M}_{\mathrm{nonadj}}$, $a_r - na_r = 1/(r-s)$, $a_s - na_s = -1/(r-s)$, and $a_r a_s - na_r na_s = -(n - 2k + \lambda - \mu)/(n\Delta) = 0$, leaving $\frac{2}{n(r-s)}(c_1 - c_2) = 0$ and thus forcing $c_1 = c_2 =: c$. Evaluating the diagonal condition $1/n = \widehat{M}_{uu} + \frac{2m_r}{n^2}c + \frac{2m_s}{n^2}c + \frac{2m_r m_s}{n^2}c_3$ and noting $m_r + m_s = n - 1$, $n\widehat{M}_{uu} - 1 = \frac{1 + m_r^2 + m_s^2 - n}{n}$, we obtain

$$2(n-1)c + 2m_r m_s c_3 \;=\; n - 1 - m_r^2 - m_s^2. \tag{16}$$

Table 1: Zero-float 16-graph SRG Cesàro average mixing and UAM moment-line atlas (Certificate `cert15`).

Graph G	(n,k,λ,μ)	$\widehat{M}_{uu}$	$\widehat{M}_{\mathrm{adj}}$	$\widehat{M}_{\mathrm{nonadj}}$	Off-unif?	UAM c_3 bound	Polytope?
C_5 (Conf.)	$(5,2,0,1)$	9/25	4/25	4/25	Yes	$[-1,1/2]$	Nonempty
Paley(9) (Conf.)	$(9,4,1,2)$	11/27	2/27	2/27	Yes	$[-1,-1/4]$	Nonempty
Petersen	$(10,3,0,1)$	21/50	49/450	19/450	No	$[-4/5,-11/16]$	Nonempty
Paley(13) (Conf.)	$(13,6,2,3)$	73/169	8/169	8/169	Yes	$[-1,-1/2]$	Nonempty
$GQ(2,2)$	$(15,6,1,3)$	107/225	19/450	61/1800	No	$[-1,-19/25]$	Nonempty
Clebsch	$(16,5,0,2)$	63/128	7/128	3/128	No	$\{-1\}$ (IUM)	$t^*=\pi/4$
$L_2(4)$/Shrikh.	$(16,6,2,2)$	59/128	7/128	3/128	No	$\{-1\}$ (IUM)	$t^*=\pi/4$
Paley(17) (Conf.)	$(17,8,3,4)$	129/289	10/289	10/289	Yes	$[-1,-5/8]$	Nonempty
$T(7)$	$(21,10,5,4)$	233/441	302/11025	218/11025	No	$c_3 \le -229/196$	**Empty**
Paley(25) (Conf.)	$(25,12,5,6)$	289/625	14/625	14/625	Yes	$[-1,-3/4]$	Nonempty
$L_2(5)$	$(25,8,3,2)$	321/625	26/625	6/625	No	$c_3 \le -13/8$	**Empty**
Schläfli comp.	$(27,10,1,5)$	437/729	14/729	19/1458	No	$c_3 \le -14/9$	**Empty**
Schläfli	$(27,16,10,8)$	437/729	19/1458	14/729	No	$c_3 \le -14/9$	**Empty**
Chang / $T(8)$	$(28,12,6,4)$	225/392	79/3528	37/3528	No	$c_3 \le -17/10$	**Empty**
Higman–Sims	$(100,22,0,6)$	3207/5000	57/5000	7/5000	No	$c_3 \le -2$	**Empty**
Ternary Golay	$(243,110,37,60)$	16295/19683	14/19683	14/19683	**Yes**	$c_3 \le -199/40$	**Empty**

Here $n-1 = t(16t^2+24t+9) = t(4t+3)^2$, $m_s = t(4t+3)$, and $m_r = 2t(2t+1)(4t+3)$, so every term shares the exact common factor $2t(4t+3)$. Dividing by $2(n-1) = 2t(4t+3)^2$:

$$\frac{m_r m_s}{n-1} = \frac{2t(2t+1)(4t+3)\cdot t(4t+3)}{t(4t+3)^2} = 2t(2t+1), \tag{17}$$

$$\frac{m_r^2+m_s^2-(n-1)}{2(n-1)} = \frac{t^2(4t+3)^2(4(2t+1)^2+1)-t(4t+3)^2}{2t(4t+3)^2} = \frac{t(16t^2+16t+5)-1}{2}, \tag{18}$$

yielding (14) identically. Because $-2t(2t+1)<0$, the maximum of $c_1(c_3)$ on $[-1,1]$ occurs at $c_3=-1$, where $2c_1(-1) = -(16t^3+16t^2+5t-1)+4t(2t+1) = -(16t^3+8t^2+t-1) \le -24 < -2$ for all $t \ge 1$ (Lean `thm11`).

Proof of (b) *and* (c). For both $(16,5,0,2)$ ($k=5, r=1, s=-3$) and $(16,6,2,2)$ ($k=6, r=2, s=-2$), the frequency gaps are $k-r=4$, $r-s=4$, $k-s=8$. At $t^*=\pi/4$, $e^{-i(k-r)\pi/4} = e^{-i\pi} = -1$ and $e^{-i(s-r)\pi/4} = e^{i\pi} = -1$, so $e^{ir\pi/4}U(\pi/4) = -E_k + E_r - E_s = 2E_r - I_{16}$. Certificates `cert06`–`cert08` explicitly construct the three 16×16 adjacency matrices over $\mathbb{Z}$, verify $E_r^2 = E_r$, $(2E_r - I_{16})^2 = I_{16}$, and every squared entry equals $1/16$ in pure $\mathbb{Q}$ arithmetic, while distinguishing $L_2(4)$ (8 copies of K_4) from Shrikhande (0 copies of K_4). Part (c) follows from solving the 2×2 rational affine moment system in Certificates `cert01` and `cert09`–`cert15`. □

Remark 1 (Polytope Nonemptiness vs. Realizability). Intersecting the moment line with $[-1,1]^3$ is a necessary condition for UAM (sufficient to yield unconditional impossibility proofs whenever empty, as in Theorem 3(a) and on Schläfli and Higman–Sims). When nonempty, exact realizability by a probability law on $\mathbb{R}_{\ge 0}$ additionally depends on the trigonometric moment curve $\{(\cos((k-r)t), \cos((k-s)t), \cos((r-s)t)) : t \ge 0\}$ (e.g., at $c_3=-1$, $(k-s)t = (k-r)t + (2j+1)\pi$ forces $c_1+c_2=0$, matching Clebsch's $(-1,+1,-1)$ while ruling out the conference endpoint $(\frac{1}{2},\frac{1}{2},-1)$ in favor of interior $c_3 > -1$ [9]).

5 Formal Verification Architecture and Scope

Our verification artifact cleanly separates machine-checked algebraic identities from exact rational matrix and parameter enumerations:

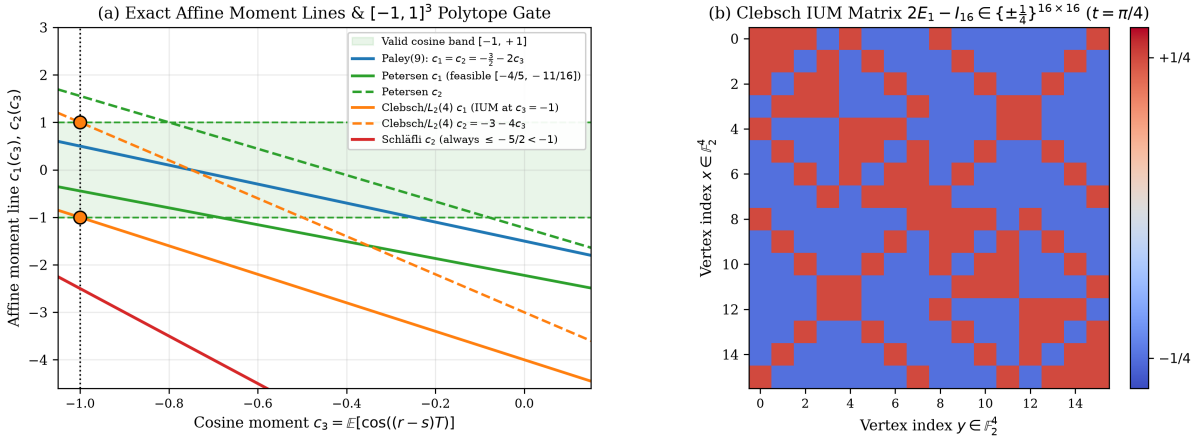

Figure 3: **Exact UAM moment lines and** 16×16 **Hadamard IUM involution.** (a) Affine moment lines $c_1(c_3), c_2(c_3)$ against the $[-1, 1]$ cosine band: Paley(9) and Petersen intersect cleanly, Clebsch/$L_2(4)$ (c_1 solid, c_2 dashed) meets uniquely at the IUM corner $(c_3, c_{1,2}) = (-1, \mp 1)$, while Schläfli lies strictly below $-5/2 < -1$. (b) Exact 16×16 Clebsch Hadamard involution $2E_1 - I_{16} \in \{\pm\frac{1}{4}\}^{16\times16}$ achieving instantaneous uniform mixing at $t^* = \pi/4$.

- **Layer 1 — Constructive Lean 4.36.0 Kernel** (`lean/CtqwSrgAverageMixing.lean`): Twenty theorems compiled with `lean-DwarningAsError=true`, zero external dependencies, zero `sorry`, zero `grind`, zero `Classical.choice`, and no custom axioms beyond standard constructive `propext`/`Quot.sound` (audited via `#print axioms`). Layer 1 machine-checks the Cesàro return-barrier identity, sharp lower bound, and strict non-uniformity for $n \geq 4$ (`thm01`–`thm03`), the complement sign-flip law, conference uniformity, and Seidel midpoint equivalence (`thm04`–`thm06`), the infinite t-family SRG feasibility, eigenvalue, multiplicity, absolute-bound dichotomy ($t = 1$ vs. $t \geq 2$), and universal UAM impossibility identities (`thm07`–`thm11`), exact Ternary Golay, conference, Clebsch/Shrikhande, Petersen, $GQ(2, 2)$, Schläfli/Higman–Sims, and spurious-289 obstructions (`thm12`–`thm19`), and the general Diophantine discriminant identity (9) (`thm20`). The Bose–Mesner projector algebra, Galois-conjugation rationality, and unitary evolution $U(t) = e^{-itA}$ derivations are proved in full detail in Sections 2–4.
- **Layer 2 — Zero-Float Exact Python Verifier** (`verification/verify_ctqw_srg_mixing.py`): An AST-audited pure standard-library verifier using strictly `int` and `fractions.Fraction` (zero floats, zero numeric division or exponentiation, zero `limit_denominator`), emitting 16 SHA-256 certificates under `certificates/MANIFEST.json` and validated by 56 adversarial `pytest` checks in `tests/test_adversarial.py`. Layer 2 independently executes the exhaustive $n \leq 1200$ Diophantine scan (`cert04`), verifies the t-family closed-form moment line for $t = 1, \ldots, 12$ (`cert03`), directly builds and checks the 10×10 Petersen projectors and 16×16 Clebsch, $L_2(4)$, and Shrikhande Hadamard IUM involutions over $\mathbb{Q}$ (`cert06`–`cert09`), and computes the 16-graph exact SRG mixing atlas (`cert15`).

References

[1] E. R. Berlekamp, J. H. van Lint, and J. J. Seidel, *A strongly regular graph derived from the perfect ternary Golay code*, in *A Survey of Combinatorial Theory*, North-Holland, 1973, pp. 25–30.

[2] A. E. Brouwer and H. Van Maldeghem, *Strongly Regular Graphs*, Encyclopedia of Mathematics and its Applications, vol. 182, Cambridge University Press, 2022.

[3] A. Chan, *Complex Hadamard matrices, instantaneous uniform mixing and cubes*, Algebr. Comb. **3** (2020), 757–774 (arXiv:1102.5601).

[4] G. Coutinho, C. Godsil, K. Guo, and F. Vanhove, *Perfect state transfer on distance-regular graphs and association schemes*, Linear Algebra Appl. **478** (2015), 108–130.

[5] E. Farhi and S. Gutmann, *Quantum computation and decision trees*, Phys. Rev. A **58** (1998), 915–928.

[6] C. Godsil, *State transfer on graphs*, Discrete Math. **312** (2012), 129–147.

[7] C. Godsil, *Average mixing of continuous quantum walks*, J. Combin. Theory Ser. A **120** (2013), 1649–1662.

[8] C. Godsil, N. Mullin, and A. Roy, *Uniform mixing and association schemes*, Electron. J. Combin. **24** (2017), #P3.22.

[9] M. Kang, *Moment methods for uniform average mixing on strongly regular graphs*, arXiv:2610.10017 (2026).