

Six-spin Lorentzian obstructions for clique blowups of independence-stable graphs

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Abstract

Let F be a connected simple graph with at least one edge, and suppose that deleting any edge leaves its independence number unchanged. For every integer $s \geq 2$, we construct a strictly positive rational antiferromagnetic matrix of rank two on six spins for which the partition polynomial of the clique blowup $F[K_s]$ is not Lorentzian. The number of spins is independent of the order and degrees of F and of the blowup size. The matrix itself may depend on F and s . The proof combines the classical equality case of Turan's theorem with a conditioned Hessian identity. Edge-deletion independence stability forces all minimum-edge occupations onto maximum independent sets. The first four active profiles then form a nonsingular block of inertia $(2, 2, 0)$; a two-stage limiting argument realizes this block using six spins. The theorem includes clique blowups of all connected biregular bipartite graphs with at least one edge other than K_2 , and of connected graphs with at least one edge and a unique maximum independent set. Exact checks cover 720 blowups of 360 bases of orders three through seven, and four explicit positive six-spin witnesses. The connected-source classification conjecture remains open.

1 Statement and relation to earlier work

For a finite simple graph $G = (V, E)$ and an entrywise nonnegative symmetric matrix K indexed by q spins, define

$$Z_G(\mathbf{x}; K) = \sum_{\sigma: V \rightarrow \{0, \dots, q-1\}} \prod_{uv \in E} K_{\sigma(u), \sigma(v)} \prod_{v \in V} x_{\sigma(v)}. \quad (1)$$

We use the spectral convention that K is *antiferromagnetic* when it has at most one positive eigenvalue. Lee–Oh–Seo prove that clique partition polynomials are Lorentzian for every such target and propose a converse [2]. We use the connected-source formulation discussed by Lee–Seo [3]: must a connected graph whose partition polynomial is Lorentzian for every antiferromagnetic target be a clique? Connectedness is necessary because partition polynomials factor over components and Lorentzian polynomials are closed under products [1].

For a homogeneous polynomial of degree $n \geq 2$ with strictly positive coefficients on the full degree- n simplex, Lorentzianity is equivalent to every partial derivative of total order $n - 2$ having a Hessian with at most one positive eigenvalue [1, Theorem 2.25]. The partition polynomials for positive targets have this full support. Inertia triples list positive, negative and zero eigenvalue counts, in that order.

Write $\alpha(F)$ for the independence number. A graph is α^- -stable if $\alpha(F - e) = \alpha(F)$ for every edge e . This terminology and its characterization by maximum independent sets are classical [5, Theorem 2.1]. The graph $F[K_s]$ replaces each vertex of F by a clique of order s , and each edge by all edges between the corresponding cliques.

Theorem 1. *Let F be a connected α^- -stable simple graph with at least one edge. For every integer $s \geq 2$, there are distinct rational numbers $a_0, \dots, a_5$ in $(-1, 1)$, with $a_1 = 0$, such that*

$$K = \mathbf{1}\mathbf{1}^\top - aa^\top$$

is strictly positive entrywise and $Z_{F[K_s]}(\mathbf{x}; K)$ is not Lorentzian. The matrix K has inertia $(1, 1, 4)$.

The six-spin bound is uniform. The rational parameters are not asserted to be uniform, and the theorem does not identify the smallest possible number of spins.

The preceding work [4] establishes a positive rank-two obstruction for connected noncliques with no true twins among their maximum-degree vertices, together with several other source classes. Every vertex of $F[K_s]$ has true twins when $s \geq 2$, so that maximum-degree theorem does not apply to these blowups. Some special families overlap earlier results, including particular joins of cliques. The present statement supplies one criterion for all α^- -stable bases and a spin bound independent of graph size. It is not a claim to solve the full conjecture.

For a concrete distinction, $C_6[K_2]$ is covered here. It is 5-regular with true twins, and adjacent pairs have either two or four common neighbours, so it is not edge-regular. The earlier neighbourhood-gap quantities satisfy $b = c = 2$, so that strict-gap criterion also does not apply. In contrast, $P_3[K_s]$ overlaps the earlier join-of-cliques family. The abstract block form is already familiar from [4]; the contribution here is its uniform occurrence at the *first* active profiles, together with suppression of every later profile by six-spin compression.

Ingredient	Role here	Attribution
α^- -stability	Controls maximum independent sets	Classical [5]
Turan equality	Classifies the leading occupations	Classical [7]
Conditioned profiles	Expresses the Hessian	Earlier framework [4]
Initial four-profile block	Produces two positive directions	Present argument
Six-spin compression	Realizes those directions in a positive target	Present argument

2 Conditioned Hessians and their leading term

We recall the identity needed from [4], including its derivation to make the argument self-contained. Choose target nodes $a_0 = \ell$, $a_1 = 0$, and $a_i \in (-1, 1)$, where $\ell \neq 0$, and put $\theta = \ell^2$. Let $n = |V(G)|$ and $0 \leq h \leq n - 2$.

For an h -vertex subset S , define the profile of an outside vertex by

$$r_S(v) = |N_G(v) \cap S|.$$

Let ν_r^S count outside vertices with profile r , and let C_{rt}^S count *ordered* adjacent pairs of distinct outside vertices with respective profiles r, t . Thus C^S is symmetric. Pad ν^S with a zero in coordinate $h + 1$. Let $De_r = e_r - e_{r+1}$, for $0 \leq r \leq h$. Here C^S is $(h + 1) \times (h + 1)$ and D is $(h + 2) \times (h + 1)$. Set

$$Q_h(\theta) = \sum_{|S|=h} (1 - \theta)^{e(G[S])} \left[\theta(\nu^S(\nu^S)^\top - \text{diag } \nu^S) - DC^S D^\top \right]. \quad (2)$$

The matrix has profile coordinates $0, \dots, h + 1$.

Lemma 2. For $U_{ir} = (1 - \ell a_i)^r$,

$$\text{Hess}\left(\partial_0^h \partial_1^{n-h-2} Z_G(\mathbf{x}; K)\right) = \frac{h!(n-h-2)!}{\theta} U Q_h(\theta) U^\top. \quad (3)$$

This identity holds for any number of target states.

Proof. Choose the h vertices differentiated in spin zero; they form S . All but two of the other vertices are differentiated in spin one. Because $a_1 = 0$, all interactions involving those latter vertices have weight one. The internal edges of S contribute $(1 - \theta)^{e(G[S])}$. If the two remaining vertices have profiles r, t and spins i, j , their interactions with S contribute $(1 - \ell a_i)^r (1 - \ell a_j)^t$. Their mutual edge, when present, adds the factor $1 - a_i a_j$. The matrix $\nu \nu^\top - \text{diag } \nu$ counts all ordered distinct pairs, and C counts the ordered edges. Finally

$$(UD)_{ir} = (1 - \ell a_i)^r - (1 - \ell a_i)^{r+1} = \ell a_i (1 - \ell a_i)^r.$$

This yields (3), including the differentiation factorials. $\square$

A Lorentzian polynomial has at most one positive Hessian eigenvalue in every quadratic partial derivative [1]. Thus two positive eigenvalues in any principal restriction of (3) provide an obstruction.

Let

$$e_0 = \min_{|S|=h} e(G[S]).$$

The polynomial identity (2) gives

$$\lim_{\theta \uparrow 1} \frac{Q_h(\theta)}{(1 - \theta)^{e_0}} = L, \quad L = \sum_{\substack{|S|=h \\ e(G[S])=e_0}} \left[\nu^S (\nu^S)^\top - \text{diag } \nu^S - DC^S D^\top \right]. \quad (4)$$

All limits are in fixed finite-dimensional matrix spaces. We call L the leading profile kernel. A profile index is *active* if its row in L contains a nonzero entry. It is a rescaled limit; the unscaled derivative at $\ell = -1$ may vanish.

3 Minimum-edge occupations

We first record the classical stability characterization and the part used in the proof.

Lemma 3 (Classical; see [5]). *The following are equivalent:*

1. $\alpha(F - e) = \alpha(F)$ for every edge e ;
2. every maximum independent set I is 2-dominating: $|N_F(u) \cap I| \geq 2$ for every $u \notin I$.

Under these conditions, two maximum independent sets cannot differ by replacing a single vertex.

Proof. Maximality excludes zero neighbours in I . If u has exactly one neighbour $v \in I$, deleting uv makes $I \cup \{u\}$ independent. Conversely, an independent $(\alpha(F) + 1)$ -set created by deleting uv must contain both endpoints; removing u gives a maximum independent set in which u has just the neighbour v . For the last assertion, a replacement vertex is nonadjacent to all the retained vertices, and can have at most one neighbour in the original maximum independent set. $\square$

Let $\alpha = \alpha(F)$, $G = F[K_s]$, and

$$h = \alpha s - 1, \quad e_0 = (\alpha - 1) \binom{s}{2} + \binom{s-1}{2}. \quad (5)$$

The hypotheses of Theorem 1 imply $\alpha \geq 2$: a connected graph with an edge and $\alpha = 1$ is complete and is not α^- -stable. Also $\alpha < |V(F)|$, so $h \leq |V(G)| - 2$.

Lemma 4. *The minimum number of edges induced by an h -vertex subset of G is e_0 . Every minimizing subset is obtained from a maximum independent set I of F by selecting all vertices of its classes except for one omitted vertex in one class $v \in I$. For each pair (I, v) there are exactly s such subsets.*

Proof. The independence number of every induced subgraph of G is at most α . Turan's theorem and its equality case [7], applied to the complement of an h -vertex induced subgraph, imply the bound (5). Equality requires a disjoint union of α cliques: one of order $s - 1$ and the others of order s . The construction in the statement attains it.

It remains to exclude other equality occupations. Choose one occupied base class from each of these α clique components. Their base vertices form a maximum independent set in F . If a component occupied two or more base classes, replacing its chosen representative by another would give two maximum independent sets differing in one vertex. This contradicts Lemma 3. Therefore each component occupies one class, and its size determines the asserted construction. The omitted vertex has s choices. $\square$

This argument uses the equality case of a classical extremal theorem. It does not assert a new extremal edge bound.

4 The initial four-profile block

Let $\mathcal{I}$ be the set of maximum independent sets of F . For $I \in \mathcal{I}$ and $u \notin I$, put $d_I(u) = |N_F(u) \cap I|$, and define

$$N_d = |\{(I, u) : I \in \mathcal{I}, u \notin I, d_I(u) = d\}|, \quad d_0 = \min\{d : N_d > 0\}. \quad (6)$$

By Lemma 3, $d_0 \geq 2$.

For a minimizing occupation indexed by (I, v) , the single remaining vertex of class v has profile $s - 1$. Each outside class $u \notin I$ has s remaining vertices and profile

$$r_u = s d_I(u) - \mathbf{1}_{\{uv \in E(F)\}} \geq 2s - 1. \quad (7)$$

All other classes in I have no remaining vertices.

Proposition 5. *The first four active profile coordinates of L are*

$$A = (s - 1, s, d_0 s - 1, d_0 s).$$

The corresponding principal block has the form

$$L[A, A] = \begin{pmatrix} 0_{2 \times 2} & B \\ B^\top & H \end{pmatrix}, \quad B = s^2 N_{d_0} \begin{pmatrix} 0 & \alpha \\ d_0 & -d_0 \end{pmatrix}. \quad (8)$$

It has inertia $(2, 2, 0)$, and

$$\det L[A, A] = s^8 \alpha^2 d_0^2 N_{d_0}^4 > 0.$$

Proof. There is exactly one remaining vertex of profile $s - 1$, no remaining vertex of profile s , and no earlier profile. The internal ordered-edge contribution at $s - 1$ is zero. Consequently the leading 2×2 block on $s - 1, s$ is zero.

We compute its couplings to the other coordinates. For a fixed (I, v) and an outside class u , write $d = d_I(u)$. The term $\nu\nu^\top - \text{diag } \nu$ contributes s between the remaining vertex of class v and class u . If uv is an edge, the edge correction cancels this contribution at $ds - 1$ and restores it at ds through the difference operator. If uv is not an edge, its profile is already ds . The row at s is the opposite difference of the two edge endpoints. Summing over $v \in I$ and the s choices of omitted vertex yields

$$L_{s-1,ds} = s^2 \alpha N_d, \quad L_{s-1,ds-1} = 0, \quad (9)$$

$$L_{s,ds-1} = s^2 d N_d, \quad L_{s,ds} = -s^2 d N_d. \quad (10)$$

All other entries in these two rows vanish. Edges between outside classes affect only the remaining block, and need not be counted.

Equation (7) and the one-step difference operator show that no coordinate strictly between s and $d_0 s - 1$ is active. The four displayed coordinates are distinct because $s, d_0 \geq 2$; their couplings show that they are active. This proves (8). Its off-diagonal block has determinant $-s^4 \alpha d_0 N_{d_0}^2 \neq 0$. An invertible congruence removes H , leaving $\begin{pmatrix} 0 & B \\ B^\top & 0 \end{pmatrix}$, which has two eigenvalues of each sign. The determinant is $(\det B)^2$. $\square$

5 Compressing to six spins

The fact that the block in Proposition 5 uses the *first* active profiles is essential to the following argument.

Lemma 6. *Let L be a finite real symmetric matrix indexed by distinct nonnegative integers. Let $b_1 < \dots < b_r$ be its first r active indices, and suppose $L[B, B]$ is nonsingular. Choose distinct positive numbers $z_1, \dots, z_r$, and put $U_\delta(i, j) = (\delta z_i)^j$ on the active columns. For sufficiently small $\delta > 0$, $U_\delta L U_\delta^\top$ has the same inertia as $L[B, B]$.*

Proof. Let $V_\delta = U_\delta[:, B]$. The generalized Vandermonde matrix $V_1 = (z_i^{b_j})$ is invertible. Indeed, a nonzero linear combination of r distinct monomials has at most $r - 1$ distinct positive zeros, by Descartes' rule of signs. Since $V_\delta = V_1 \text{diag}(\delta^{b_j})$, every entry of $V_\delta^{-1} U_\delta[:, t]$ for $t > b_r$ is a constant times δ^{t-b_j} , and tends to zero. Hence

$$V_\delta^{-1} U_\delta L U_\delta^\top V_\delta^{-\top} \longrightarrow L[B, B].$$

Inertia is locally constant at a nonsingular real symmetric matrix, and invariant under invertible congruence. $\square$

Proof of Theorem 1. Apply Lemma 6 with $r = 4$, $z_i = i$, and the block in Proposition 5. Choose a sufficiently small rational $\delta \in (0, 1/4)$ so that $U_\delta L U_\delta^\top$ has inertia $(2, 2, 0)$.

Fix this δ , and choose the six target nodes

$$a_0 = \ell, \quad a_1 = 0, \quad a_{i+1} = -1 + \delta i \quad (1 \leq i \leq 4). \quad (11)$$

As $\ell \downarrow -1$ from above, $1 - \ell a_{i+1} \rightarrow \delta i$. Let W_ℓ consist of rows 2, 3, 4, 5 of U . By (4),

$$\frac{W_\ell Q_h(\ell^2) W_\ell^\top}{(1 - \ell^2)^{e_0}} \longrightarrow U_\delta L U_\delta^\top.$$

It retains two positive eigenvalues for $\ell > -1$ sufficiently close to -1 . Choose such an ℓ rational, and sufficiently close that it is different from all other nodes.

All six nodes now lie in $(-1, 1)$, so every entry of K is strictly positive. The vectors $\mathbf{1}$ and a are independent; the representation

$$K = \begin{pmatrix} \mathbf{1} & a \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \mathbf{1} & a \end{pmatrix}^\top$$

gives inertia $(1, 1, 4)$. The positive scalar in (3) preserves the two positive eigenvalues of the principal Hessian restriction. Therefore the quadratic partial derivative is not Lorentzian, and neither is Z_G . $\square$

The limits are taken in a specified order: first choose δ , then choose ℓ . No interchange or graph-uniform neighbourhood is used. In particular, the argument does not assert that the boundary target with $\ell = -1$ is itself an obstruction.

6 Examples and a finite construction procedure

Corollary 7. *Theorem 1 applies to every connected biregular bipartite base with at least one edge other than K_2 .*

Proof. Let the bipartition sizes be $a \leq b$, with respective degrees $D \geq d > 0$, so $aD = bd$. For an independent set J ,

$$D|J \cap A| + d|J \cap B| \leq |E(F)| = db.$$

It follows that $\alpha(F) = b$. If $D > d$, equality in cardinality forces $J = B$, so the maximum independent set is unique. If $D = d$, equality means every edge has exactly one endpoint in J , so the complement is also independent. Connectedness gives just the two bipartition classes. Except for K_2 , those two classes cannot differ by a single-vertex replacement. In either case a maximum independent set cannot have an outside vertex with just one neighbour in it; apply Lemma 3. $\square$

Corollary 8. *Every connected base with at least one edge and a unique maximum independent set satisfies Theorem 1.*

Proof. An outside vertex with one neighbour would give a different maximum independent set by replacement. The zero-neighbour case contradicts maximality. $\square$

This graph-class inclusion is classical [6, Lemma 2.3]; the conclusion about six-spin obstructions is its application here.

The construction is effective for a finite input, without a polynomial-time complexity claim. Enumerate maximum independent sets and the leading kernel. Search through rational $\delta = 2^{-t}$, $t \geq 3$, until exact congruence certifies inertia $(2, 2, 0)$ for $U_\delta L U_\delta^\top$. Then search through $\ell = -1 + 2^{-u}$, with $u > t + 1$, until the exact quadratic Hessian restriction has two positive eigenvalues. The two limiting arguments prove termination of these unbounded searches. Enumerating all occupations or subsets is generally expensive; a practical search cap is not a mathematical bound.

For a direct check of the second step, sum over occupations c of the base classes, with $\sum c_v = h$. Put $b_v = s - c_v$, $r_v = c_v + \sum_{u \sim v} c_u$, $w(c) = \prod_v \binom{s}{c_v}$, and

$$e(c) = \sum_v \binom{c_v}{2} + \sum_{uv \in E(F)} c_u c_v.$$

For witness spins i, j , let $u_i(v) = (1 - \ell a_i)^{r_v}$ and $v_i = \sum_v b_v u_i(v)$. The normalized Hessian entry is

$$\sum_c w(c)(1 - \ell^2)^{e(c)} \left[v_i v_j - \sum_v b_v u_i(v) u_j(v) - a_i a_j E_{ij}(c) \right], \quad (12)$$

$$E_{ij}(c) = \sum_v b_v (b_v - 1) u_i(v) u_j(v) + \sum_{uv \in E(F)} b_u b_v (u_i(u) u_j(v) + u_i(v) u_j(u)).$$

The omitted factor is $h!(|V(G)| - h - 2)! > 0$. This formula counts outside ordered pairs directly and is suitable for rational verification.

7 Exact checks and reproducibility

All 994 connected graphs of orders three through seven in the NetworkX graph atlas were screened by two equivalent definitions: edge deletion and the maximum-independent-set neighbour condition. Exactly 360 satisfy the hypothesis. For each such base, both $s = 2$ and $s = 3$ were checked, giving 720 blowups. For every case, a separate enumeration of bounded occupation vectors found all minimum-edge patterns. They agreed exactly with Lemma 4, including multiplicities, and the first four-profile block agreed with Proposition 5. These checks support the argument; the general theorem follows from its proof, not an extrapolation of this census.

Four complete rational Hessian witnesses were also constructed using (12). All four used the same nodes

$$(a_0, \dots, a_5) = \frac{1}{32}(-31, 0, -28, -24, -20, -16).$$

This is a statement about the four examples, not a universal choice for the theorem.

Base F	s	h	Occupations	Hessian inertia
P_3	2	3	7	(2, 2, 0)
C_6	2	5	126	(2, 2, 0)
K_4 minus one edge	2	3	16	(2, 2, 0)
Mixed-degree example below	2	5	51	(2, 2, 0)

The last base has vertex set $\{0, 1, 2, 3, 4\}$ and edges $\{03, 13, 04, 14, 24, 34\}$. Its unique maximum independent set is $\{0, 1, 2\}$; outside degrees into that set are two and three.

The supplement includes the graph lists, occupation checks, exact rational matrices and congruence records. The driver `verify_all.py` reproduces both checks and compares all exact records with the archived results. The witness-construction script has finite practical search caps, documented separately from the unbounded terminating construction above. No floating-point eigenvalue calculation is used as a correctness certificate. The mathematical checks are not a formal proof assistant verification.

8 Scope and remaining questions

The result handles uniform clique blowups of α^- -stable bases. It does not assert that every noncomplete base is stable: odd cycles, for example, have critical edges and fall outside the hypothesis. It does not classify all graphs whose partition polynomials are universally Lorentzian. The six-spin construction also leaves open whether fewer spins suffice, whether target parameters can be chosen uniformly for useful subclasses, and whether effective parameter bounds can avoid full enumeration.

The proof uses a general transfer mechanism: a nonsingular initial profile block can be realized by a small number of evaluation nodes, with a separate base spin and pin spin.

Applying that mechanism requires an actual initial block with the needed inertia; an arbitrary principal minor at later profiles does not automatically give the same spin bound.

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References

- [1] P. Brändén and J. Huh. *Lorentzian polynomials*. [arXiv:1902.03719](#).
- [2] J. Lee, J. Oh and J. Seo. *Counting homomorphisms in antiferromagnetic graphs via Lorentzian polynomials*. [arXiv:2506.13659](#), version 2.
- [3] J. Lee and J. Seo. *Antiferromagnetic models are clique-minimizing*. [arXiv:2608.17920](#), version 1.
- [4] L. Eriksson. *Rank-two obstructions to Lorentzian graph partition functions*. AIRR, ARR-2026-7993ZVQWNC9E38P2, version 1, 17 October 2026. [Permanent version record](#). Working paper.
- [5] V. E. Levit and E. Mandrescu. *Matrices and α -Stable Bipartite Graphs*. [arXiv:math/9912120](#), 1999. Theorem 2.1 attributes the stability characterization to T. W. Haynes, L. M. Lawson, R. C. Brigham and R. D. Dutton.
- [6] V. E. Levit and E. Mandrescu. *On the structure of α -stable graphs*. [arXiv:math/9911227](#), 1999.
- [7] P. Turán. *On an extremal problem in graph theory* (in Hungarian). *Matematikai és Fizikai Lapok* **48** (1941), 436–452.