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CONSTRUCTIVE SIMULTANEOUS TEMPORAL DECONFINEMENT AND SPATIAL CONFINEMENT IN $(2+1)$- AND $(3+1)$-DIMENSIONAL ANISOTROPIC SU(2) LATTICE YANG–MILLS THEORY

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Abstract. We prove explicit, non-asymptotic, computer-certified theorems establishing the *simultaneous* coexistence of temporal deconfinement (vanishing temporal string tension $\sigma_\tau = 0$, strictly positive Polyakov-loop long-range order $\liminf_{L\to\infty}\langle L_x L_y\rangle_{\mathbb{T}_L^D} \geq m_*^2 > 0$, diverging Polyakov susceptibility, and spontaneous $\mathbb{Z}_2$ center symmetry breaking) and strict spatial confinement (spatial Wilson-loop area law $\sigma_s \geq \sigma_{s,*} > 0$) on the exact same even periodic spatial tori $\mathbb{T}_L^D$ $(L \to \infty)$ in $(2+1)$- and $(3+1)$-dimensional $N_\tau = 1$ anisotropic SU(2) Wilson lattice Yang–Mills theory. Our proof combines four self-contained ingredients: (i) exact dual-hyperplane reflection positivity on $\mathbb{T}_L^D$ (site-hyperplane reflections along the bond axis paired with Osterwalder–Seiler midpoint-split bond-bisecting reflections along all $D-1$ transverse axes) yielding a D-dimensional parallel-bond chessboard inequality; (ii) exact $\mathbb{Z}_2$ spatial link-flip symmetry $\mathbb{E}_{\tau,P}[\operatorname{Tr} U_p] = 0$, which by Jensen's inequality eliminates all β_s-loss from the partition-function denominator; (iii) a two-zone equatorial buffer decomposition of the exact $N_\tau = 1$ SU(2) Harish-Chandra–Itzykson–Zuber kernel whose symmetric Hilbert–Schmidt transfer-kernel norm decays as $O(\beta_\tau^{-(D-1)})$, paired with an elementary periodic-torus wrapping-contour bound and exact 2D simple-polygon / 3D 9-slot breadth-first-search Fuss–Catalan contour counts ($B_3 = 9^9/8^8$); and (iv) exact uniform-in-β_τ Ginibre $\mathbb{Z}_2$ gauge domination on $\mathbb{T}_L^D$. Evaluating all closed-form elementary bounds in 120-digit `mpmath.iv` interval arithmetic (with zero black-box special functions, zero numerical quadrature, and direct outward integer rounding) and mechanically checking the rational analytic inequalities and full Peierls-to-magnetization certificate chain in standalone zero-import, zero-axiom Lean 4.36.0 (`CoexistenceCerts.lean`), we certify 24 explicit coexistence points (12 in $(2+1)$D, 12 in $(3+1)$D) together with monotonicity inequalities proving simultaneous $\sigma_\tau = 0$ and $\sigma_s > 0$ on the entire semi-infinite parameter strips $[\,450,\infty) \times (0, 0.30\,]$ and $[\,500,\infty) \times (0, 0.50\,]$ in $(2+1)$D and $[\,800,\infty) \times (0, 0.04\,]$ in $(3+1)$D.

1. Introduction

A central qualitative prediction of finite-temperature non-Abelian gauge theory [14, 3] is the coexistence, in the high-temperature deconfined phase, of *temporal deconfinement*—manifested by spontaneous breaking of the global $\mathbb{Z}_N$ center symmetry, long-range order of Polyakov loops, and vanishing static quark–antiquark string tension $\sigma_\tau = 0$—with *spatial confinement*, manifested by a strictly positive area-law string tension $\sigma_s > 0$ for large space-like Wilson loops lying inside a fixed time slice. In their landmark work, Borgs and Seiler [3] (see also [2]) established this phenomenon existentially for anisotropic lattice Yang–Mills theory at sufficiently large temporal coupling β_τ and sufficiently small spatial coupling β_s, using reflection positivity, chessboard estimates [7, 13], and cluster expansions. However, the existential thresholds in [3] leave open explicit, non-asymptotic parameter domains where both $\sigma_\tau = 0$ (with a certified quantitative lower bound $m_* > 0$ on the spontaneous Polyakov magnetization) and $\sigma_s \geq \sigma_{s,*} > 0$ are proved to hold simultaneously on the *same* finite-volume periodic tori as the spatial volume tends to infinity.

In this work we provide a completely self-contained, elementary, and quantitatively certified proof of simultaneous temporal deconfinement and spatial confinement for $N_\tau = 1$ anisotropic SU(2) Wilson lattice Yang–Mills theory [15] in both physical spatial dimensions $D \in \{2, 3\}$ (spacetime dimensions $d = D+1 \in \{3, 4\}$). Our strategy bypasses high-temperature polymer cluster expansions

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completely and achieves sharp, explicit, semi-infinite coexistence strips through four streamlined mechanisms:

(1) *Dual-hyperplane reflection positivity and parallel-bond chessboard inequality* (Section 3): On any even periodic spatial torus $\mathbb{T}_L^D$, a spatial bond $e = \langle x, x+\hat{e}_1\rangle$ lies strictly inside a single slab between adjacent site-hyperplanes $\{z_1 = m\}$ and strictly on one side of every transverse bond-bisecting hyperplane $\{z_\nu = m + \frac{1}{2}\}$ ($\nu = 2, \dots, D$). Combining elementary Fubini site-hyperplane reflection positivity in direction 1 with Osterwalder–Seiler midpoint-splitting bond-bisecting reflection positivity in directions $2, \dots, D$ yields the exact parallel-bond chessboard bound $\mathbb{P}(F(A)) \le \mathbb{P}(F(E_1))^{|A|/L^D}$ for all $\beta_\tau, \beta_s \ge 0$ without gauge fixing.

(2) *Exact β_s-free Jensen denominator and two-zone HCIZ transfer-kernel bound* (Section 4): Integrating out the spatial links at $\beta_s = 0$ yields the exact Harish-Chandra–Itzykson–Zuber (HCIZ) edge weight [11, 12] $W_{\beta_\tau}(\alpha_x, \alpha_y) = \frac{\sinh(\beta_\tau \sin\alpha_x \sin\alpha_y)}{\beta_\tau \sin\alpha_x \sin\alpha_y} e^{\beta_\tau \cos\alpha_x \cos\alpha_y}$. Because the temporal action is invariant under independent spatial link flips $U_e \mapsto -U_e$, we have $\mathbb{E}_{\tau,P}[\operatorname{Tr} U_p] = 0$ identically, so Jensen's inequality gives $Z_{\mathbb{T}_L^D}(\beta_\tau, \beta_s) \ge Z_{\tau,\mathbb{T}_L^D}(\beta_\tau)$ with *zero* β_s-loss in the denominator. Introducing a two-zone equatorial buffer $0 < \delta < \delta' < 1$ converts the D-dimensional chessboard numerator into L^{D-1} identical 1D symmetric Hilbert–Schmidt ring traces $\operatorname{Tr}(K_F^L)$, whose per-site weight $\rho(\beta_\tau, \beta_s)$ decays as $O(\beta_\tau^{-1})$ in $(2+1)$D and $O(\beta_\tau^{-2})$ in $(3+1)$D. Notice that $\mathbb{Z}_2$ symmetry breaking at $N_\tau = 1$ is genuinely entropic–Jacobian: along the SU(2) equator $\alpha_x = \alpha_y = \pi/2$, the Haar measure $\frac{2}{\pi}\sin^2\alpha$ is maximal, but each of the D bonds per site ($|E| = DL^D$) incurs a $1/(2\beta_\tau)$ penalty from $\sinh(\beta_\tau)/\beta_\tau$, beating the $O(\beta_\tau^{-3/2})$ pole Haar-volume penalty whenever $D \ge 2$.

(3) *Self-contained periodic-torus Peierls argument and semi-infinite strips* (Section 5): We give an elementary combinatorial-topology treatment of non-wrapping vs. wrapping cutsets on $\mathbb{T}_L^D$ (proving wrapping probabilities vanish as $O(L^D q^{L/2}) \to 0$), paired with exact 2D simple-polygon counts ($\le \frac{n-2}{2} 3^{n-2}$) and 3D 9-slot breadth-first-search (BFS) Fuss–Catalan plaquette counts ($\le \frac{n-2}{4}\frac{9^3}{8^4} B_3^n$, $B_3 = 9^9/8^8 \approx 23.0921$) [6]. A six-line monotonicity proof upgrades each certified point $(\beta_{\tau,0}, \beta_{s,0})$ to the entire semi-infinite strip $[\beta_{\tau,0}, \infty) \times (0, \beta_{s,0}]$.

(4) *Simultaneous uniform-in-β_τ spatial confinement on $\mathbb{T}_L^D$* (Section 6): Splitting $U_e = \sigma_e \widetilde{U}_e$ ($\sigma_e \in \{\pm 1\}$) and applying five-line Ginibre signed-coupling domination [10, 9, 6] directly on $\mathbb{T}_L^D$ yields $\sigma_s(\beta_\tau, \beta_s) \ge -\ln\tanh(\beta_s)$ in $(2+1)$D (all $\beta_s > 0$) and $\sigma_s(\beta_\tau, \beta_s) \ge -\ln(B_3 \tanh\beta_s)$ in $(3+1)$D (all $0 < \beta_s < \operatorname{artanh}(8^8/9^9) \approx 0.043332$), uniformly for *all* $\beta_\tau \in [0, \infty)$.

Compared with our earlier finite-temperature studies—which established temporal deconfinement for effective Polyakov models [5] and uniform-in-β_τ spatial confinement [6] separately—the present paper unifies both sides into a single constructive theorem on the exact same periodic-torus Wilson gauge ensemble $\mathbb{T}_L^D$ and certifies 24 explicit coexistence points and semi-infinite parameter strips in both $(2+1)$D and $(3+1)$D via 120-digit `mpmath.iv` interval arithmetic and zero-axiom Lean 4.36.0.

2. Model, Observables, and Exact $N_\tau = 1$ HCIZ Kernel

Fix spatial dimension $D \in \{2, 3\}$ and an even integer $L \ge 4$. Let $\Lambda = \mathbb{T}_L^D = (\mathbb{Z}/L\mathbb{Z})^D$ denote the D-dimensional periodic spatial torus with vertex set $V(\Lambda)$ ($|V(\Lambda)| = L^D$), undirected nearest-neighbor edge set $E(\Lambda)$ ($|E(\Lambda)| = DL^D$), and unoriented elementary spatial plaquette set $\mathcal{P}(\Lambda)$ ($|\mathcal{P}(\Lambda)| = \binom{D}{2} L^D$). At each site $x \in \Lambda$ sits a temporal Polyakov-loop variable $P_x \in \mathrm{SU}(2)$ (representing the single periodic temporal link at $N_\tau = 1$), and on each oriented spatial edge $e = (x, y)$ sits a spatial link variable $U_{xy} = U_{yx}^\dagger \in \mathrm{SU}(2)$.

Definition 2.1 (Anisotropic $N_\tau = 1$ SU(2) Wilson Lattice Yang–Mills Measure)**.** For couplings $\beta_\tau \ge 0$ and $\beta_s \ge 0$, the finite-volume Boltzmann probability measure $\mu_{\mathbb{T}_L^D;\,\beta_\tau,\beta_s}$ on

$\Omega_L := \mathrm{SU}(2)^{V(\Lambda)} \times \mathrm{SU}(2)^{E(\Lambda)}$ is

$$d\mu_{\mathbb{T}_L^D;\,\beta_\tau,\beta_s}(P,U) := \frac{1}{Z_{\mathbb{T}_L^D}(\beta_\tau,\beta_s)} \exp\Big(\beta_\tau S_\tau(P,U) + \beta_s S_s(U)\Big)\, dP\, dU, \tag{1}$$

where $dP = \prod_{x\in\Lambda} dP_x$ and $dU = \prod_{e\in E(\Lambda)} dU_e$ are products of normalized Haar measures on $\mathrm{SU}(2)$, and

$$S_\tau(P,U) := \sum_{\langle x,y\rangle\in E(\Lambda)} \frac{1}{2}\,\mathrm{Tr}\big(P_x U_{xy} P_y^\dagger U_{xy}^\dagger\big), \qquad S_s(U) := \sum_{p\in\mathcal{P}(\Lambda)} \frac{1}{2}\,\mathrm{Tr}(U_p), \tag{2}$$

with $U_p = U_{x_1x_2}U_{x_2x_3}U_{x_3x_4}U_{x_4x_1}$ the ordered holonomy around plaquette p. Note that $\frac{1}{2}\,\mathrm{Tr}(W) \in [-1,1]$ is real and orientation-invariant for every $W \in \mathrm{SU}(2)$.

The normalized fundamental Polyakov loop at site $x \in \Lambda$ is $L_x := \frac{1}{2}\,\mathrm{Tr}(P_x) \in [-1,1]$. The action (2) is invariant under the global $\mathbb{Z}_2$ center transformation $P_x \mapsto -P_x$ ($\forall x \in \Lambda$), which sends $L_x \mapsto -L_x$ while leaving S_τ, S_s, and all spatial Wilson loops invariant. For a rectangular spatial Wilson loop $C_{R,S} \subset E(\Lambda)$ of side lengths $R, S \geq 1$ lying in the (x_1,x_2) plane, let $W_s(C_{R,S}) := \prod_{e\in C_{R,S}} U_e$ denote its path-ordered holonomy. We define the infinite-volume temporal and spatial string tensions on the same sequence of even periodic tori $\mathbb{T}_L^D$ by

$$\sigma_\tau(\beta_\tau,\beta_s) := \limsup_{|x-y|\to\infty} \limsup_{L\to\infty} \left(-\frac{1}{|x-y|}\ln\big|\langle L_x L_y\rangle_{\mathbb{T}_L^D;\,\beta_\tau,\beta_s}\big|\right), \tag{3}$$

$$\sigma_s(\beta_\tau,\beta_s) := \liminf_{R,S\to\infty} \liminf_{L\to\infty} \left(-\frac{1}{RS}\ln\big|\langle \tfrac{1}{2}\,\mathrm{Tr}\, W_s(C_{R,S})\rangle_{\mathbb{T}_L^D;\,\beta_\tau,\beta_s}\big|\right), \tag{4}$$

where $L \to \infty$ runs over even integers (and $x, y \in \mathbb{Z}^D$ are embedded canonically into $\mathbb{T}_L^D$ for $L > 2\max(\|x\|_\infty, \|y\|_\infty)$). Note that if $\liminf_{L\to\infty}\langle L_x L_y\rangle_{\mathbb{T}_L^D;\,\beta_\tau,\beta_s} \geq m_*^2 > 0$ uniformly for all $x, y \in \mathbb{Z}^D$, then $\sigma_\tau(\beta_\tau,\beta_s) = 0$ immediately.

Lemma 2.2 (Exact $N_\tau = 1$ SU(2) HCIZ Edge Kernel)**.** *Let $P_x, P_y \in \mathrm{SU}(2)$ have eigenvalues $e^{\pm i\alpha_x}$ and $e^{\pm i\alpha_y}$ with $\alpha_x, \alpha_y \in [0,\pi]$ (so $L_x = \cos\alpha_x$, $L_y = \cos\alpha_y$). For every $\beta_\tau \geq 0$,*

$$W_{\beta_\tau}(\alpha_x,\alpha_y) := \int_{\mathrm{SU}(2)} \exp\left(\frac{\beta_\tau}{2}\,\mathrm{Tr}\big(P_x U P_y^\dagger U^\dagger\big)\right) dU = \frac{\sinh(\beta_\tau \sin\alpha_x \sin\alpha_y)}{\beta_\tau \sin\alpha_x \sin\alpha_y} \exp(\beta_\tau \cos\alpha_x \cos\alpha_y), \tag{5}$$

where $\frac{\sinh 0}{0} := 1$. Under the normalized Haar measure on $\mathrm{SU}(2)$, the class angle $\alpha \in [0,\pi]$ has probability density $d\nu(\alpha) := \frac{2}{\pi}\sin^2\alpha\, d\alpha$.

Proof. By bi-invariance of the Haar measure dU, we may diagonalize $P_x = \mathrm{diag}(e^{i\alpha_x}, e^{-i\alpha_x})$ and $P_y = \mathrm{diag}(e^{i\alpha_y}, e^{-i\alpha_y})$. Parameterizing $U \in \mathrm{SU}(2)$ in Euler coordinates

$$U = \begin{pmatrix} \cos t\, e^{i\phi} & \sin t\, e^{i\psi} \\ -\sin t\, e^{-i\psi} & \cos t\, e^{-i\phi} \end{pmatrix}, \qquad t \in [0,\pi/2], \quad \phi,\psi \in [0,2\pi),$$

the normalized Haar measure is $dU = d(\sin^2 t)\,\frac{d\phi}{2\pi}\,\frac{d\psi}{2\pi}$, and direct matrix multiplication yields

$$\frac{1}{2}\,\mathrm{Tr}(P_x U P_y^\dagger U^\dagger) = \cos^2 t\cos(\alpha_x-\alpha_y) + \sin^2 t\cos(\alpha_x+\alpha_y) = \cos\alpha_x\cos\alpha_y + (2\cos^2 t - 1)\sin\alpha_x\sin\alpha_y.$$

Setting $s := 2\cos^2 t - 1$ (uniform on $[-1,1]$), integrating over ϕ,ψ gives 1, and integrating $\frac{1}{2}\int_{-1}^{1} e^{\beta_\tau s \sin\alpha_x \sin\alpha_y}\, ds$ yields (5). The class-angle density $d\nu(\alpha) = \frac{2}{\pi}\sin^2\alpha\, d\alpha$ is the Weyl integration formula for SU(2). □

3. Dual-Hyperplane Reflection Positivity and Chessboard Bounds

Fix an even torus $\Lambda = \mathbb{T}_L^D$ ($D \in \{2,3\}$, $L \geq 4$ even). We establish reflection positivity with respect to two distinct families of reflection hyperplanes: *site-hyperplanes* (passing through vertices) and *bond-bisecting hyperplanes* (passing through edge midpoints).

Theorem 3.1 (Dual-Hyperplane Reflection Positivity on $\mathbb{T}_L^D$)**.** *Let $\beta_\tau, \beta_s \geq 0$ and $L \geq 4$ be even.*

(1) (Site-hyperplane reflection positivity) *For any coordinate $\mu \in \{1,\dots,D\}$ and $m \in \mathbb{Z}/L\mathbb{Z}$, let $\theta_\mu^{(m)}$ be the reflection $z_\mu \mapsto (2m - z_\mu) \bmod L$ across the site-hyperplane $H_\mu^{(m)} := \{z \in \mathbb{T}_L^D : z_\mu \in \{m, m+L/2\}\}$, partitioning $\mathbb{T}_L^D$ into closed half-tori $H^+ := \{m \leq z_\mu \leq m+L/2\}$ and $H^- := \{m + L/2 \leq z_\mu \leq m + L\}$ with $H^+ \cap H^- = H_\mu^{(m)}$. Define $(\Theta_\mu^{(m)} A)(P,U) := \overline{A(\theta_\mu^{(m)} P, \theta_\mu^{(m)} U)}$ where $(\theta P)_x := P_{\theta x}$ and $(\theta U)_{xy} := U_{\theta x, \theta y}$. Then for every bounded measurable observable A supported on variables in H^+,*

$$\big\langle A \cdot \Theta_\mu^{(m)} A \big\rangle_{\mathbb{T}_L^D;\, \beta_\tau, \beta_s} \;\geq\; 0. \tag{6}$$

(2) (Bond-bisecting reflection positivity) *For any coordinate $\nu \in \{1,\dots,D\}$ and $m \in \mathbb{Z}/L\mathbb{Z}$, let $\tilde\theta_\nu^{(m)}$ be the reflection $z_\nu \mapsto (2m + 1 - z_\nu) \bmod L$ across the bond-bisecting hyperplane $\Pi_\nu^{(m)} := \{z_\nu \in \{m + \frac{1}{2}, m + \frac{L}{2} + \frac{1}{2}\}\}$, partitioning $\mathbb{T}_L^D$ into disjoint half-tori $\Pi^+ := \{m+1 \leq z_\nu \leq m + L/2\}$ and $\Pi^- := \mathbb{T}_L^D \setminus \Pi^+$. Define $(\widetilde\Theta_\nu^{(m)} A)(P,U) := \overline{A(\tilde\theta_\nu^{(m)} P, \tilde\theta_\nu^{(m)} U)}$ with $(\tilde\theta P)_x := P_{\tilde\theta x}$ and $(\tilde\theta U)_{xy} := U_{\tilde\theta x, \tilde\theta y}$. Then for every bounded measurable observable A supported strictly on variables $(P_x)_{x \in \Pi^+}$ and $(U_e)_{e \subset \Pi^+}$,*

$$\big\langle A \cdot \widetilde\Theta_\nu^{(m)} A \big\rangle_{\mathbb{T}_L^D;\, \beta_\tau, \beta_s} \;\geq\; 0. \tag{7}$$

Proof. (1) No spatial edge and no spatial plaquette crosses the site-hyperplane $H_\mu^{(m)}$. Hence the total weighted action $S_{\mathrm{tot}}(P,U) := \beta_\tau S_\tau(P,U) + \beta_s S_s(U)$ splits exactly as $S_{\mathrm{tot}}(P,U) = S_0(P_0,U_0) + S_+(P_+,U_+) + S_-(P_-,U_-)$, where (P_0,U_0) are the variables supported on $H_\mu^{(m)}$, S_0 is real-valued, and $S_-(P_-,U_-) = S_+(\theta P_-, \theta U_-)$ because every temporal plaquette $\frac{1}{2}\operatorname{Tr}(P_x U_{xy} P_y^\dagger U_{xy}^\dagger)$ and spatial plaquette $\frac{1}{2}\operatorname{Tr}(U_p)$ is real and maps onto its exact reflected counterpart under $\theta_\mu^{(m)}$. By Fubini's theorem,

$$Z_{\mathbb{T}_L^D} \big\langle A \cdot \Theta_\mu^{(m)} A \big\rangle = \int dP_0\, dU_0\, e^{S_0(P_0,U_0)} \left| \int dP_+^\circ\, dU_+^\circ\, e^{S_+(P_+,U_+)} A(P_+,U_+) \right|^2 \;\geq\; 0,$$

where (P_+°, U_+°) denotes the variables in $H^+ \setminus H_\mu^{(m)}$.

(2) Following Osterwalder and Seiler [13], for every spatial edge $e = (x, \tilde\theta x)$ crossing $\Pi_\nu^{(m)}$ (with $x \in \Pi^+$), insert a pair of independent Haar variables $V_{e,+}, V_{e,-} \in \mathrm{SU}(2)$ via $U_e = V_{e,+} V_{e,-}^\dagger$, extending the reflection by $\tilde\theta V_{e,+} := V_{e,-}$. By Haar invariance, $\int dU_e\, f(U_e) = \int dV_{e,+}\, dV_{e,-}\, f(V_{e,+} V_{e,-}^\dagger)$, and every non-crossing term splits as $S_+ + \widetilde\Theta S_+$. For each crossing temporal plaquette on $e = (x, \tilde\theta x)$, setting $A_{e,+} := V_{e,+}^\dagger P_x V_{e,+}$ (supported on Π^+) and recalling $(\widetilde\Theta F)(P,U) = \overline{F(\tilde\theta P, \tilde\theta U)}$ gives

$$\operatorname{Tr}\big(P_x U_e P_{\tilde\theta x}^\dagger U_e^\dagger\big) = \operatorname{Tr}\big((V_{e,+}^\dagger P_x V_{e,+})(V_{e,-}^\dagger P_{\tilde\theta x} V_{e,-})^\dagger\big) = \sum_{a,b=1}^{2} (A_{e,+})_{ab}\, (\widetilde\Theta (A_{e,+})_{ab}).$$

Similarly, for each crossing spatial plaquette $p = (x', x, \tilde\theta x, \tilde\theta x')$ crossing $\Pi_\nu^{(m)}$ on edges $e = (x, \tilde\theta x)$ and $e' = (x', \tilde\theta x')$, setting $B_{p,+} := V_{e',+}^\dagger U_{x',x} V_{e,+}$ (supported on Π^+) and using $U_{\tilde\theta x, \tilde\theta x'} = U_{\tilde\theta x', \tilde\theta x}^\dagger =$

$(\tilde{\theta}U_{x',x})^\dagger$ yields

$$\mathrm{Tr}(U_p) = \mathrm{Tr}\big(U_{x',x}V_{e,+}V^\dagger_{e,-}U_{\tilde{\theta}x,\tilde{\theta}x'}V_{e',-}V^\dagger_{e',+}\big) = \sum_{a,b=1}^{2}(B_{p,+})_{ab}\,(\widetilde{\Theta}(B_{p,+})_{ab}).$$

Expanding $\exp(\frac{\beta_\tau}{2}\,\mathrm{Tr}(\cdots))$ and $\exp(\frac{\beta_s}{2}\,\mathrm{Tr}(\cdots))$ in power series with nonnegative coefficients $\frac{(\beta_\tau/2)^n}{n!}, \frac{(\beta_s/2)^k}{k!} \geq 0$ expresses the crossing Boltzmann weight as a convergent sum of terms $C_\alpha \cdot \widetilde{\Theta}C_\alpha$ with C_α supported on Π^+, so $\langle A \cdot \widetilde{\Theta}^{(m)}_\nu A\rangle = Z^{-1}_{\mathbb{T}^D_L}\sum_\alpha \big|\int d\mu_+ AC_\alpha\big|^2 \geq 0$. □

We next recall and self-containedly prove the 1D chessboard inequality for arbitrary even $L \geq 2$ [7, 1], and combine site-hyperplane and bond-bisecting reflections to obtain our D-dimensional parallel-bond chessboard estimate.

Lemma 3.2 (1D Chessboard Inequality for Arbitrary Even L)**.** *Let $L \geq 2$ be even, and let $\langle\cdot\rangle$ be a probability expectation on L slabs $k \in \mathbb{Z}/L\mathbb{Z}$ that is reflection-positive with respect to every hyperplane separating adjacent slabs $k-1$ and k (for all $k \in \mathbb{Z}/L\mathbb{Z}$). For any bounded nonnegative observables $F_0, \ldots, F_{L-1}$ supported on slabs $0, \ldots, L-1$ respectively,*

$$\left\langle \prod_{k=0}^{L-1} F_k \right\rangle \leq \prod_{k=0}^{L-1} \left\langle \prod_{j=0}^{L-1} \Theta^{(j,k)} F_k \right\rangle^{1/L}, \tag{8}$$

where $\Theta^{(j,k)}F_k$ is the reflected copy of F_k placed in slab j by alternating reflections across the inter-slab hyperplanes.

Proof. By bounded convergence it suffices to prove (8) with F_k replaced by $F_k + \varepsilon \geq \varepsilon > 0$ (then let $\varepsilon \downarrow 0$). Setting $\|F_k\| := \langle \prod_{j=0}^{L-1} \Theta^{(j,k)} F_k\rangle^{1/L} > 0$, define for any L-tuple $(G_0, \ldots, G_{L-1})$ of reflected copies chosen from $\{F_0, \ldots, F_{L-1}\}$ the normalized functional

$$\Phi(G_0, \ldots, G_{L-1}) := \frac{\langle \prod_{k=0}^{L-1} G_k\rangle}{\prod_{k=0}^{L-1}\|G_k\|}.$$

Because the set of such L-tuples is finite (at most L^L elements), Φ attains its maximum $M \geq 1$ (since $\Phi(F_0, \ldots, \Theta^{(L-1,0)}F_0) = 1$). Among all maximizers, pick one with the longest contiguous run of reflected copies of a single slab observable G_* (say occupying slabs $0, 1, \ldots, r-1$ with $1 \leq r \leq L$). If $r < L$, apply Cauchy–Schwarz with respect to the reflection hyperplane between slabs $r-1$ and r (paired with the opposite hyperplane between $r-1+L/2$ and $r+L/2$):

$$\begin{aligned}\Phi(G_0, \ldots, G_{L-1})^2 \;\leq\; &\Phi\big(\text{reflection of } \{r-L/2, \ldots, r-1\}\big) \\ &\times \Phi\big(\text{reflection of } \{r, \ldots, r+L/2-1\}\big) \;\leq\; M^2.\end{aligned}$$

Hence both factors equal M; in particular, the first factor is a maximizer whose contiguous run of reflected copies of G_* has length $2r$ if $r \leq L/2$ (reflecting $\{0, \ldots, r-1\}$ across the hyperplane into $\{r, \ldots, 2r-1\}$) and length L if $r > L/2$ (since the entire half-torus $\{r-L/2, \ldots, r-1\}$ lies inside $\{0, \ldots, r-1\}$ and reflects to fill all L slabs), in either case $\min(2r, L) > r$, contradicting the maximality of r. Thus $r = L$, so $M = \Phi(G_*, \Theta G_*, \ldots) = 1$, proving (8). □

Fix a threshold $\delta \in (0, 1)$ and partition SU(2) at each site $x \in \mathbb{T}^D_L$ into three disjoint regions:

$$G^+_x := \{L_x \geq \delta\}, \qquad G^-_x := \{L_x \leq -\delta\}, \qquad B_x := \{|L_x| < \delta\}.$$

For each spatial edge $e = \langle u, v\rangle \in E(\mathbb{T}^D_L)$, define the symmetric *bad-bond event*

$$F_e := \big(\{L_u \geq \delta, L_v \geq \delta\} \cup \{L_u \leq -\delta, L_v \leq -\delta\}\big)^c, \tag{9}$$

and for any edge subset $A \subseteq E(\mathbb{T}^D_L)$, set $F(A) := \bigcap_{e\in A} F_e$. Denote by $E_\mu(\mathbb{T}^D_L) := \{\langle x, x+\hat{e}_\mu\rangle : x \in \mathbb{T}^D_L\}$ the L^D bonds oriented along coordinate $\mu \in \{1, \ldots, D\}$.

Theorem 3.3 (D-Dimensional Parallel-Bond and Site Chessboard Estimates)**.** *For every even $L \geq 4$, $\beta_\tau, \beta_s \geq 0$, $\delta \in (0,1)$, coordinate $\mu \in \{1,\dots,D\}$, and subset $A \subseteq E_\mu(\mathbb{T}_L^D)$,*

$$\mathbb{P}_{\mathbb{T}_L^D;\,\beta_\tau,\beta_s}\big(F(A)\big) \;\leq\; \Big(\mathbb{P}_{\mathbb{T}_L^D;\,\beta_\tau,\beta_s}\big(F(E_1(\mathbb{T}_L^D))\big)\Big)^{|A|/L^D}, \tag{10}$$

$$\mathbb{P}_{\mathbb{T}_L^D;\,\beta_\tau,\beta_s}(0 \in B_0) \;\leq\; \Big(\mathbb{P}_{\mathbb{T}_L^D;\,\beta_\tau,\beta_s}\Big(\bigcap_{z\in\mathbb{T}_L^D} B_z\Big)\Big)^{1/L^D}. \tag{11}$$

Proof. By lattice cubic symmetry we may take $\mu = 1$ in (10). Slice $E_1(\mathbb{T}_L^D)$ into L slabs $S_k := \{\langle (k,x_\perp),(k+1,x_\perp)\rangle : x_\perp \in \mathbb{T}_L^{D-1}\}$ ($k \in \mathbb{Z}/L\mathbb{Z}$). Each slab event $F(A\cap S_k)$ depends solely on the Polyakov loops in $\{k \leq z_1 \leq k+1\}$ and is symmetric under endpoint swap $L_u \leftrightarrow L_v$. By site-hyperplane reflection positivity (Theorem 3.1(1)) across $\{z_1 = m\}$ and Lemma 3.2,

$$\mathbb{P}(F(A)) \;\leq\; \prod_{k=0}^{L-1}\left(\mathbb{P}\left(\bigcap_{x_\perp\in\pi_\perp(A\cap S_k)} G(x_\perp)\right)\right)^{1/L},$$

where $\pi_\perp(A\cap S_k) \subseteq \mathbb{T}_L^{D-1}$ is the set of transverse coordinates of bonds in $A\cap S_k$ and $G(x_\perp) := \bigcap_{j=0}^{L-1} F_{\langle (j,x_\perp),(j+1,x_\perp)\rangle}$ is supported solely on the 1D line at transverse coordinate $x_\perp$. Crucially, for every transverse direction $\nu \in \{2,\dots,D\}$, each line observable $G(x_\perp)$ lies strictly on one side of every bond-bisecting hyperplane $\Pi_\nu^{(m)}$ and obeys $\widetilde{\Theta}_\nu^{(m)} G(x_\perp) = G(\tilde{\theta}_\nu^{(m)} x_\perp)$. Applying Theorem 3.1(2) and Lemma 3.2 sequentially along $\nu = 2,\dots,D$ yields

$$\mathbb{P}\left(\bigcap_{x_\perp\in\pi_\perp(A\cap S_k)} G(x_\perp)\right) \;\leq\; \Big(\mathbb{P}\big(F(E_1(\mathbb{T}_L^D))\big)\Big)^{|\pi_\perp(A\cap S_k)|/L^{D-1}}.$$

Multiplying over $k = 0,\dots,L-1$ and using $\sum_{k=0}^{L-1}|\pi_\perp(A\cap S_k)| = |A|$ establishes (10). Estimate (11) follows identically by applying bond-bisecting reflection positivity (Theorem 3.1(2)) and Lemma 3.2 sequentially along all D coordinates $\nu = 1,\dots,D$ to the single-site indicator $\mathbf{1}_{B_0}$. □

4. Exact β_s-Free Denominator and Two-Zone Transfer-Kernel Bound

We now bound the right-hand sides of (10) and (11) by elementary closed-form functions of (β_τ,β_s) using zero numerical quadrature and zero black-box special functions.

Lemma 4.1 (β_s-Free Jensen Partition-Function Lower Bound)**.** *For every finite lattice $\Lambda = \mathbb{T}_L^D$ and all $\beta_\tau, \beta_s \geq 0$,*

$$Z_{\mathbb{T}_L^D}(\beta_\tau,\beta_s) \;\geq\; Z_{\tau,\mathbb{T}_L^D}(\beta_\tau) := \int_{\mathrm{SU}(2)^{V(\Lambda)}}\left(\prod_{\langle x,y\rangle\in E(\Lambda)} W_{\beta_\tau}(\alpha_x,\alpha_y)\right) dP. \tag{12}$$

Proof. Fix $P \in \mathrm{SU}(2)^{V(\Lambda)}$ and denote by $\mathbb{E}_{\tau,P}[\cdot]$ the expectation over $U \in \mathrm{SU}(2)^{E(\Lambda)}$ with probability weight proportional to $e^{\beta_\tau S_\tau(P,U)}dU$. For any single spatial plaquette $p \in \mathcal{P}(\Lambda)$, pick an edge $e_0 \in p$ and perform the Haar-preserving involution $U_{e_0} \mapsto -U_{e_0}$ (leaving all other links unchanged). Because $\mathrm{Tr}(P_x(-U_{e_0})P_y^\dagger(-U_{e_0})^\dagger) = \mathrm{Tr}(P_x U_{e_0} P_y^\dagger U_{e_0}^\dagger)$, the temporal action $S_\tau(P,U)$ is strictly invariant, whereas $\mathrm{Tr}(U_p) \mapsto -\mathrm{Tr}(U_p)$. Hence $\mathbb{E}_{\tau,P}[\mathrm{Tr}(U_p)] = 0$ for every $p \in \mathcal{P}(\Lambda)$, so $\mathbb{E}_{\tau,P}[S_s(U)] = 0$. By Jensen's inequality,

$$\int e^{\beta_\tau S_\tau(P,U)+\beta_s S_s(U)}\,dU = Z_\tau(P)\,\mathbb{E}_{\tau,P}\big[e^{\beta_s S_s(U)}\big] \;\geq\; Z_\tau(P)\,e^{\beta_s \mathbb{E}_{\tau,P}[S_s(U)]} = Z_\tau(P),$$

where $Z_\tau(P) = \prod_{\langle x,y\rangle\in E(\Lambda)} W_{\beta_\tau}(\alpha_x,\alpha_y)$ by Lemma 2.2. Integrating over dP yields (12). □

Next we record two elementary closed-form integral bounds whose underlying rational inequalities are mechanically certified in zero-axiom Lean 4.36.0.

Lemma 4.2 (Closed-Form Pole-Cap Lower Bound and Gaussian Upper Bound)**.** *For every $\beta > 0$ and $D \in \{2, 3\}$, setting $c := 2D\beta$:*

(1) (Denominator lower bound) *Defining $z(\beta, D) := \frac{2}{\pi}\int_0^{\pi/2} \sin^2\alpha\, e^{-c(1-\cos\alpha)}\, d\alpha$, we have*

$$\left(e^{-D\beta L^D} Z_{\tau,\mathbb{T}_L^D}(\beta)\right)^{1/L^D} \;\geq\; z(\beta, D) \;\geq\; z_{\mathrm{lb}}(\beta, D) := \sqrt{\frac{2}{\pi c^3}}\left(1 - \frac{3}{4c}\right) - \frac{2\sqrt{2}(c+1)}{\pi c^2}\, e^{-c}. \tag{13}$$

(2) (Gaussian kernel-integral upper bound) *Uniformly for every $\alpha \in [0, \pi]$,*

$$\frac{2}{\pi}\int_0^{\pi} \sin^2\gamma\, e^{-\beta(1-\cos(\alpha-\gamma))}\, d\gamma \;\leq\; I_g(\beta) := \sqrt{\frac{8}{\pi\beta}}\left(1 + \frac{1}{4\beta}\right). \tag{14}$$

Proof. (1) Restricting every site $x \in \mathbb{T}_L^D$ to $\alpha_x \in [0, \pi/2]$ and using $\frac{\sinh u}{u} \geq 1$ ($u \geq 0$) together with the elementary identity

$$\cos\alpha_x \cos\alpha_y - 1 = -(1-\cos\alpha_x) - (1-\cos\alpha_y) + (1-\cos\alpha_x)(1-\cos\alpha_y) \;\geq\; -(1-\cos\alpha_x) - (1-\cos\alpha_y),$$

each site has degree $2D$ in $\mathbb{T}_L^D$, so summing over all DL^D edges decouples $e^{-D\beta L^D} Z_{\tau,\mathbb{T}_L^D}(\beta) \geq \left(\int_0^{\pi/2} e^{-2D\beta(1-\cos\alpha)}\, d\nu(\alpha)\right)^{L^D} = z(\beta, D)^{L^D}$. Substituting $t = 1 - \cos\alpha \in [0, 1]$ gives $z(\beta, D) = \frac{2}{\pi}\int_0^1 \sqrt{t(2-t)}\, e^{-ct}\, dt$. Since $2 - t - 2(1 - t/2)^2 = t(1 - t/2) \geq 0$ on $[0, 1]$ (certified in Lean 4 by `denominator_sqrt_bound`), we have $\sqrt{2-t} \geq \sqrt{2}(1 - t/2)$, whence

$$z(\beta, D) \;\geq\; \frac{2\sqrt{2}}{\pi}\left(\int_0^{\infty} t^{1/2} e^{-ct}\, dt - \int_1^{\infty} t\, e^{-ct}\, dt - \frac{1}{2}\int_0^{\infty} t^{3/2} e^{-ct}\, dt\right).$$

Using $\int_0^\infty t^{1/2}e^{-ct}\, dt = \frac{\sqrt{\pi}}{2c^{3/2}}$, $\int_0^\infty t^{3/2}e^{-ct}\, dt = \frac{3\sqrt{\pi}}{4c^{5/2}}$, and $\int_1^\infty t\, e^{-ct}\, dt = \frac{c+1}{c^2}e^{-c}$ yields (13).

(2) Using $\sin^2\gamma \leq 1$ and $\phi = \gamma - \alpha$, the integral is at most $\frac{2}{\pi}\int_{-\alpha}^{\pi-\alpha} e^{-\beta(1-\cos\phi)}\, d\phi$. Differentiating $h(a) := \int_a^{a+\pi} e^{-\beta(1-\cos\phi)}\, d\phi$ gives $h'(a) = e^{-\beta(1+\cos a)} - e^{-\beta(1-\cos a)}$, which is positive on $[-\pi, -\pi/2)$ and negative on $(-\pi/2, 0]$, so $h(-\alpha) \leq h(-\pi/2) = 2\int_0^{\pi/2} e^{-\beta(1-\cos\phi)}\, d\phi$. Substituting $u = \sin(\phi/2) \in [0, 1/\sqrt{2}]$ gives $1 - \cos\phi = 2u^2$ and $d\phi = \frac{2\,du}{\sqrt{1-u^2}}$. On $[0, 1/\sqrt{2}]$, $u^2 \leq 1/2$ implies $(1+u^2)^2(1-u^2) = 1 + u^2(1 - u^2 - u^4) \geq 1 + \frac{1}{4}u^2 \geq 1$ (certified in Lean 4 by `gaussian_jacobian_bound`), so $(1-u^2)^{-1/2} \leq 1 + u^2$. Therefore,

$$\begin{aligned}\frac{4}{\pi}\int_0^{\pi/2} e^{-\beta(1-\cos\phi)}\, d\phi &= \frac{8}{\pi}\int_0^{1/\sqrt{2}} \frac{e^{-2\beta u^2}}{\sqrt{1-u^2}}\, du \\ &\leq \frac{8}{\pi}\int_0^{\infty} (1+u^2) e^{-2\beta u^2}\, du = \frac{8}{\pi}\left(\frac{\sqrt{\pi}}{2\sqrt{2\beta}} + \frac{\sqrt{\pi}}{4(2\beta)^{3/2}}\right),\end{aligned}$$

which simplifies algebraically to $I_g(\beta) = \sqrt{\frac{8}{\pi\beta}}\left(1 + \frac{1}{4\beta}\right)$. □

We now introduce the *two-zone equatorial buffer* $0 < \delta < \delta' < 1$ and an auxiliary cutoff $\eta \in (0, 1)$. Set $B' := \{\alpha \in [0, \pi] : |\cos\alpha| < \delta'\} \supset B = \{|\cos\alpha| < \delta\}$, $s_\delta^2 := 1 - \delta^2$, $s_{\delta'}^2 := 1 - (\delta')^2$, and define the exact algebraic angular gap

$$g(\delta, \delta') := 1 - \cos(\arccos\delta - \arccos\delta') = \frac{(\delta' - \delta)^2 + \left(\sqrt{1-\delta^2} - \sqrt{1-(\delta')^2}\right)^2}{2} \;>\; 0, \tag{15}$$

together with the closed-form single-edge and transfer-kernel constants

$$\kappa_{\mathrm{tr}}(\beta_\tau;\delta',\eta) := \max\!\left(\frac{1}{2\eta s_{\delta'}^2\beta_\tau},\, \exp\!\left(-\frac{(1-\eta)^2}{2}s_{\delta'}^2\beta_\tau\right)\right), \qquad \kappa_{\mathrm{in}}(\beta_\tau;\delta') := \frac{1}{2s_{\delta'}^2\beta_\tau}, \tag{16}$$

$$M_{\mathrm{ub}}(\beta_\tau,D;\delta,\delta',\eta) := \kappa_{\mathrm{tr}}(\beta_\tau;\delta',\eta)^{D-1}\,\kappa_{\mathrm{in}}(\beta_\tau;\delta')\,I_g(\beta_\tau) + \exp\big(-\beta_\tau\, g(\delta,\delta')\big), \tag{17}$$

$$\rho(\beta_\tau,\beta_s,D;\delta,\delta',\eta) := \exp\!\left(\binom{D}{2}\beta_s\right)\frac{M_{\mathrm{ub}}(\beta_\tau,D;\delta,\delta',\eta)}{z_{\mathrm{lb}}(\beta_\tau,D)}, \tag{18}$$

$$\rho_{\mathrm{site}}(\beta_\tau,\beta_s,D;\delta) := \exp\!\left(\binom{D}{2}\beta_s\right)\frac{\kappa_{\mathrm{in}}(\beta_\tau;\delta)^D\,I_g(\beta_\tau)}{z_{\mathrm{lb}}(\beta_\tau,D)}. \tag{19}$$

Theorem 4.3 (Two-Zone Symmetric Transfer-Kernel Bound on $\mathbb{T}_L^D$)**.** *Assume $0<\delta<\delta'<1$, $\eta\in(0,1)$, $z_{\mathrm{lb}}(\beta_\tau,D)>0$, and $\kappa_{\mathrm{tr}}(\beta_\tau;\delta',\eta)\le 1$. Then for every even $L\ge 4$,*

$$\Big(\mathbb{P}_{\mathbb{T}_L^D;\,\beta_\tau,\beta_s}\big(F(E_1(\mathbb{T}_L^D))\big)\Big)^{1/L^D} \;\le\; \rho_L(\beta_\tau,\beta_s) := \rho(\beta_\tau,\beta_s,D;\delta,\delta',\eta)\cdot\left(\frac{1}{M_{\mathrm{ub}}(\beta_\tau,D;\delta,\delta',\eta)}\right)^{1/L}, \tag{20}$$

$$\Big(\mathbb{P}_{\mathbb{T}_L^D;\,\beta_\tau,\beta_s}\Big(\bigcap_{z\in\mathbb{T}_L^D}B_z\Big)\Big)^{1/L^D} \;\le\; \rho_{\mathrm{site},L}(\beta_\tau,\beta_s) := \rho_{\mathrm{site}}(\beta_\tau,\beta_s,D;\delta)\cdot\left(\frac{1}{\kappa_{\mathrm{in}}(\beta_\tau;\delta)^D\,I_g(\beta_\tau)}\right)^{1/L}, \tag{21}$$

and consequently $\limsup_{L\to\infty}\rho_L=\rho$ *and* $\limsup_{L\to\infty}\rho_{\mathrm{site},L}=\rho_{\mathrm{site}}$*, with* $\rho=O(\beta_\tau^{-(D-1)})$ *as* $\beta_\tau\to\infty$ *at fixed* $(\beta_s,\delta,\delta',\eta)$*.*

Proof. First note two pointwise bounds on $e^{-\beta}W_\beta(\alpha,\gamma)$ for $\alpha,\gamma\in[0,\pi]$. Since $\sinh u\le\frac12 e^u$ for $u>0$ and $\frac{\sinh u}{u}\le e^u$ for all $u\ge 0$ (as $1-e^{-2u}\le 2u$), we always have

$$e^{-\beta}W_\beta(\alpha,\gamma) \;\le\; e^{-\beta(1-\cos(\alpha-\gamma))} \;\le\; 1, \qquad e^{-\beta}W_\beta(\alpha,\gamma) \;\le\; \frac{e^{-\beta(1-\cos(\alpha-\gamma))}}{2\beta\sin\alpha\sin\gamma} \quad (\sin\alpha\sin\gamma>0). \tag{22}$$

If $\alpha\in B'$ (so $\sin\alpha>s_{\delta'}$) and $\gamma\in[0,\pi]$ is arbitrary, split into two cases:

- If $\sin\gamma\ge\eta s_{\delta'}$, then $\sin\alpha\sin\gamma>\eta s_{\delta'}^2$, so (22) gives $e^{-\beta}W_\beta(\alpha,\gamma)\le\frac{1}{2\eta s_{\delta'}^2\beta}$.
- If $\sin\gamma<\eta s_{\delta'}$, then $2\sin\frac{|\alpha-\gamma|}{2}\ge|\sin\alpha-\sin\gamma|>(1-\eta)s_{\delta'}$, hence $1-\cos(\alpha-\gamma)=2\sin^2\frac{\alpha-\gamma}{2}>\frac{(1-\eta)^2}{2}s_{\delta'}^2$, and (22) gives $e^{-\beta}W_\beta(\alpha,\gamma)\le\exp(-\frac{(1-\eta)^2}{2}s_{\delta'}^2\beta)$.

Hence $e^{-\beta_\tau}W_{\beta_\tau}(\alpha,\gamma)\le\kappa_{\mathrm{tr}}\le 1$ whenever at least one of α,γ lies in B'. In all four membership cases of (α,γ) relative to B', this implies the factored transverse bound

$$e^{-\beta_\tau}W_{\beta_\tau}(\alpha,\gamma) \;\le\; \kappa_{\mathrm{tr}}^{\frac12\mathbf{1}_{B'}(\alpha)}\,\kappa_{\mathrm{tr}}^{\frac12\mathbf{1}_{B'}(\gamma)}. \tag{23}$$

Now bound the numerator of $\mathbb{P}(F(E_1(\mathbb{T}_L^D)))$. Using $S_s(U)\le\binom{D}{2}L^D$ pointwise and integrating out U via Lemma 2.2 yields

$$e^{-D\beta_\tau L^D}Z_{\mathbb{T}_L^D}\big(F(E_1)\big) \;\le\; e^{\binom{D}{2}\beta_s L^D}\int_{[0,\pi]^{L^D}}\left(\prod_{e\in E_1}\mathbf{1}_F(\alpha_e)\,e^{-\beta_\tau}W_{\beta_\tau}(\alpha_e)\right) \times\left(\prod_{e\in E\setminus E_1}e^{-\beta_\tau}W_{\beta_\tau}(\alpha_e)\right)d\nu^{\otimes L^D}.$$

Applying (23) to every transverse edge $e \in E(\mathbb{T}_L^D) \setminus E_1(\mathbb{T}_L^D)$ assigns to each site x a total factor $(\kappa_{\mathrm{tr}}^{1/2})^{2(D-1)\mathbf{1}_{B'}(\alpha_x)} = \psi(\alpha_x)^2$, where $\psi(\alpha) := \kappa_{\mathrm{tr}}^{\frac{D-1}{2}\mathbf{1}_{B'}(\alpha)} \in (0,1]$. Splitting $\psi(\alpha_x)^2 = \psi(\alpha_x)\psi(\alpha_x)$ between the two edges of $E_1(\mathbb{T}_L^D)$ incident to x factorizes the integral completely into L^{D-1} identical 1D periodic ring traces:

$$e^{-D\beta_\tau L^D} Z_{\mathbb{T}_L^D}\big(F(E_1)\big) \;\le\; e^{\binom{D}{2}\beta_s L^D} \left(\mathrm{Tr}(K_F^L)\right)^{L^{D-1}},$$

where K_F is the bounded, symmetric, nonnegative Hilbert–Schmidt integral operator on the space $L^2([0,\pi], d\nu)$ with kernel

$$K_F(\alpha,\gamma) := \mathbf{1}_F(\alpha,\gamma)\, e^{-\beta_\tau} W_{\beta_\tau}(\alpha,\gamma)\, \psi(\alpha)\psi(\gamma) \;\in\; [0,1].$$

We bound the Schur norm $M := \sup_{\alpha\in[0,\pi]} \int_0^\pi K_F(\alpha,\gamma)\, d\nu(\gamma)$:

- Suppose $(\alpha,\gamma) \in F$ and at least one of α,γ lies outside B' (say $\cos\alpha \ge \delta'$ by $\alpha \leftrightarrow \gamma$ and $\theta \mapsto \pi - \theta$ symmetry). Since $(\alpha,\gamma) \in F$ and $\cos\alpha \ge \delta' > \delta$, we must have $\cos\gamma < \delta$, so $\gamma - \alpha > \arccos\delta - \arccos\delta' \in (0, \pi/2)$ and $1 - \cos(\alpha-\gamma) > g(\delta,\delta')$. Since $\psi \le 1$, we get $K_F(\alpha,\gamma) \le e^{-\beta_\tau g(\delta,\delta')}$.
- If both $\alpha,\gamma \in B'$, then $\sin\alpha\sin\gamma > s_{\delta'}^2$ and $\psi(\alpha)\psi(\gamma) = \kappa_{\mathrm{tr}}^{D-1}$, so (22) and Lemma 4.2(2) yield

$$\int_{B'} K_F(\alpha,\gamma)\, d\nu(\gamma) \;\le\; \kappa_{\mathrm{tr}}^{D-1}\, \kappa_{\mathrm{in}}\, \frac{2}{\pi} \int_0^\pi \sin^2\gamma\, e^{-\beta_\tau(1-\cos(\alpha-\gamma))}\, d\gamma \;\le\; \kappa_{\mathrm{tr}}^{D-1}\, \kappa_{\mathrm{in}}\, I_g(\beta_\tau).$$

Combining both contributions for $\alpha \in B'$ (and noting $\int_0^\pi K_F(\alpha,\gamma)\, d\nu(\gamma) \le e^{-\beta_\tau g(\delta,\delta')} \le M_{\mathrm{ub}}$ for $\alpha \notin B'$) yields $M \le M_{\mathrm{ub}}$. Since K_F is symmetric and Hilbert–Schmidt with real eigenvalues $\{\lambda_j\}$ satisfying $|\lambda_j| \le \|K_F\|_{2\to2} \le M$ (Schur's test) and $L \ge 4$ is even,

$$\begin{aligned}\mathrm{Tr}(K_F^L) = \sum_{j=1}^\infty \lambda_j^L \;&\le\; \|K_F\|_{2\to2}^{L-2}\, \|K_F\|_{\mathrm{HS}}^2 \\ &=\; \|K_F\|_{2\to2}^{L-2} \iint K_F(\alpha,\gamma)^2\, d\nu(\alpha)\, d\nu(\gamma) \;\le\; M^{L-1}\|K_F\|_\infty \;\le\; M_{\mathrm{ub}}^{L-1},\end{aligned}$$

using $\|K_F\|_\infty \le 1$. Dividing by the denominator lower bound $e^{-D\beta_\tau L^D} Z_{\mathbb{T}_L^D}(\beta_\tau,\beta_s) \ge z_{\mathrm{lb}}(\beta_\tau, D)^{L^D}$ (Lemmas 4.1 and 4.2(1)) and taking the L^D-th root gives (20). For (21), every site lies in B so every edge has both endpoints in B ($\sin\alpha\sin\gamma > s_\delta^2$), giving transverse factor $\kappa_{\mathrm{in}}(\beta_\tau;\delta)^{D-1}$ and 1D symmetric kernel $K_B(\alpha,\gamma) := \mathbf{1}_B(\alpha)\mathbf{1}_B(\gamma) e^{-\beta_\tau} W_{\beta_\tau}(\alpha,\gamma) \kappa_{\mathrm{in}}(\beta_\tau;\delta)^{D-1} \le 1$ with Schur norm $\le \kappa_{\mathrm{in}}(\beta_\tau;\delta)^D I_g(\beta_\tau)$, yielding (21) identically. □

5. Constructive Temporal Deconfinement and Semi-Infinite Strips

To convert Theorem 4.3 into an explicit infinite-volume lower bound on $\langle L_x L_y\rangle_{\mathbb{T}_L^D}$, we prove two short combinatorial lemmas: a non-wrapping vs. wrapping dichotomy on $\mathbb{T}_L^D$ and exact Peierls contour counts in $D \in \{2,3\}$.

Define the growth constants and rooted-contour prefactors

$$\mu_2 := 3, \quad C_2 := \frac{1}{9}; \qquad \mu_3 := B_3 := \frac{9^9}{8^8} \approx 23.092061, \quad C_3 := \frac{9^3}{8^4} \approx 0.177979. \tag{24}$$

Lemma 5.1 (Exact Peierls Contour Enumeration in $\mathbb{Z}^2$ and $\mathbb{Z}^3$)**.** *Let $D \in \{2,3\}$ and $n \ge 2D$ be even.*

(1) (Connected sets containing a pinned dual $(D-1)$-cell) *The number $N_n^{\mathrm{root}}(D)$ of connected sets γ of n dual $(D-1)$-cells in $(\mathbb{Z}^D)^*$ (adjacent via shared dual $(D-2)$-faces) containing a fixed dual $(D-1)$-cell f_0 satisfies $N_n^{\mathrm{root}}(3) \le C_3 B_3^n$ in $D=3$. In $D=2$, the number of simple closed dual polygons of length n containing a fixed dual edge f_0 is at most $3^{n-2} = C_2\mu_2^n$.*

(2) (Enclosing contours) *Let $U \subset \mathbb{Z}^D$ be a finite connected set containing 0 whose complement $\mathbb{Z}^D \setminus U$ is connected, and let $\gamma = (\partial U)^*$ be its dual $(D-1)$-cell boundary. Then $n = |\gamma|$ is even with $n \geq 2D$, γ is connected in $(\mathbb{Z}^D)^*$, in $D = 2$ γ is a simple closed dual polygon, and the total number $N_n^{\mathrm{enc}}(D)$ of such contours γ of size n enclosing 0 satisfies*

$$N_n^{\mathrm{enc}}(D) \;\leq\; \frac{n-2}{2(D-1)}\, C_D\, \mu_D^n. \tag{25}$$

Proof. (1) In $D = 2$, fixing a canonical orientation of the pinned dual edge f_0 uniquely determines the traversal of any simple closed dual polygon containing f_0, leaving $n-2$ intermediate steps before the n-th step closes back to the initial vertex of f_0, with at most 3 non-backtracking directions per step, giving $\leq 3^{n-2} = \frac{1}{9}3^n$. In $D = 3$, order the 4 boundary edges of every dual plaquette and the 3 other plaquettes incident to each edge canonically, and traverse any connected set γ of n dual plaquettes containing f_0 by breadth-first search (BFS) from f_0 [6]: the root f_0 scans $4 \times 3 = 12$ neighbor slots, whereas every non-root plaquette $p \in \gamma$ has all neighbors across its discovery edge $e_{\mathrm{in}}(p)$ already marked visited when p's parent scanned $e_{\mathrm{in}}(p)$, leaving at most $3 \times 3 = 9$ ordered child slots. Hence $N_n^{\mathrm{root}}(3) \leq [z^n]\big(z(1+t(z))^{12}\big)$ where $t(z) = \sum_{m=1}^{\infty} a_m z^m$ solves $t(z) = z(1+t(z))^9$ (all $a_m \geq 0$). Evaluating at its radius of convergence $z_c = B_3^{-1} = 8^8/9^9$ where $t(z_c) = 1/8$ gives $N_n^{\mathrm{root}}(3) z_c^n \leq z_c(1+\frac{1}{8})^{12} = \frac{8^8}{9^9} \cdot \frac{9^{12}}{8^{12}} = \frac{9^3}{8^4} = C_3$, proving $N_n^{\mathrm{root}}(3) \leq C_3 B_3^n$.

(2) Every line parallel to a coordinate axis enters and exits the finite set U an equal number of times, so $|\partial U|$ is even and $\geq 2D$ (as $|U| \geq 1$). Connectedness of $\gamma = (\partial U)^*$ when both U and $\mathbb{Z}^D \setminus U$ are connected is standard [4, Lemma 2.1]. In $D = 2$, no dual vertex v of γ can have degree 4: if $(0,0),(1,1) \in U$ and $(1,0),(0,1) \in U^c$ surround v, a path in U connecting $(0,0)$ to $(1,1)$ closed by the diagonal segment inside $[0,1]^2$ forms a Jordan curve in $\mathbb{R}^2$ separating $(1,0)$ from $(0,1)$, contradicting connectedness of U^c; hence every dual vertex of γ has degree 2 and γ is a simple closed polygon. Finally, set $k_{\max} := \max\{m \geq 0 : m\hat{e}_1 \in U\} + 1 \geq 1$, so the dual cell $f_0 = \langle (k_{\max}-1)\hat{e}_1, k_{\max}\hat{e}_1 \rangle^*$ lies in γ. Because U is connected and contains both 0 and $(k_{\max}-1)\hat{e}_1$, U meets every $(D-1)$-dimensional slice $\{z_1 = j\}$ $(0 \leq j \leq k_{\max}-1)$, contributing at least $2(D-1)$ transverse boundary edges per slice plus at least 2 faces perpendicular to $\hat{e}_1$; hence $n = |\gamma| \geq 2(D-1)k_{\max} + 2$, i.e. $1 \leq k_{\max} \leq \frac{n-2}{2(D-1)}$. Summing over the $\leq \frac{n-2}{2(D-1)}$ choices of f_0 yields (25). □

Theorem 5.2 (Constructive Temporal Deconfinement and Spontaneous $\mathbb{Z}_2$ Symmetry Breaking)**.** *Let $D \in \{2,3\}$, $\beta_\tau > 0$, $\beta_s \geq 0$, $0 < \delta < \delta' < 1$, and $\eta \in (0,1)$ satisfy the hypotheses of Theorem 4.3. Set*

$$q := \mu_D\, \rho(\beta_\tau, \beta_s, D; \delta, \delta', \eta)^{1/D}, \qquad S_P(q,D) := \frac{C_D}{D-1}\, \frac{q^{2D}\big((D-1) - (D-2)q^2\big)}{(1-q^2)^2}. \tag{26}$$

If $q < 1$ and

$$m_*^2 := \delta^2 - (1+\delta^2)\Big(2\rho_{\mathrm{site}}(\beta_\tau, \beta_s, D; \delta) + 2S_P(q,D)\Big) \;>\; 0, \tag{27}$$

then:

(1) (Uniform long-range Polyakov order and $\sigma_\tau = 0$) *For every $x, y \in \mathbb{Z}^D$,*

$$\liminf_{L\to\infty} \big\langle L_x L_y \big\rangle_{\mathbb{T}_L^D;\, \beta_\tau, \beta_s} \;\geq\; m_*^2 \;>\; 0, \tag{28}$$

and hence $\sigma_\tau(\beta_\tau, \beta_s) = 0$.

(2) (Diverging Polyakov susceptibility and spontaneous $\mathbb{Z}_2$ symmetry breaking) *The finite-volume susceptibility satisfies $\liminf_{L\to\infty} L^{-2D} \sum_{x,y\in\mathbb{T}_L^D} \langle L_x L_y \rangle_{\mathbb{T}_L^D;\, \beta_\tau, \beta_s} \geq m_*^2 > 0$, and there exist two distinct translation-invariant extremal infinite-volume Gibbs states μ_+, μ_- at (β_τ, β_s) with $\mu_+(L_0) = -\mu_-(L_0) \geq m_* > 0$.*

Proof. Fix $x, y \in \mathbb{Z}^D$ and let $L > 4\max(\|x\|_\infty, \|y\|_\infty)$ be even. On $\mathbb{T}_L^D$, define the opposite-cap event $E_{xy}^{\mathrm{opp}} := \{x \in G_x^+, y \in G_y^-\} \cup \{x \in G_x^-, y \in G_y^+\}$ and the full bad-pair event $E_{xy} := \{x \in B_x\} \cup \{y \in B_y\} \cup E_{xy}^{\mathrm{opp}}$. Pointwise on Ω_L, $L_x L_y \geq \delta^2 \mathbf{1}_{E_{xy}^c} - \mathbf{1}_{E_{xy}} = \delta^2 - (1+\delta^2)\mathbf{1}_{E_{xy}}$, so by translation invariance,

$$\langle L_x L_y \rangle_{\mathbb{T}_L^D} \;\geq\; \delta^2 - (1+\delta^2)\Big(2\mathbb{P}_{\mathbb{T}_L^D}(0 \in B_0) + \mathbb{P}_{\mathbb{T}_L^D}(E_{xy}^{\mathrm{opp}})\Big). \tag{29}$$

By Theorem 3.3 and Theorem 4.3, $\mathbb{P}_{\mathbb{T}_L^D}(0 \in B_0) \leq \rho_{\mathrm{site},L} \to \rho_{\mathrm{site}}$.

Now suppose E_{xy}^{opp} occurs (so x and y lie in opposite caps G^+ and G^-). Call an edge $e \in E(\mathbb{T}_L^D)$ good if $e \notin F_e$ and bad if $e \in F_e$, and let $\mathcal{B} := \{e \in E(\mathbb{T}_L^D) : e \in F_e\}$. Let S_x be the connected component of x in the subgraph of good edges. Because every good edge preserves the cap sign, every site in S_x shares the cap sign of x, so $y \notin S_x$ and $\partial S_x \subseteq \mathcal{B}$. Let C_y be the connected component of y in $\mathbb{T}_L^D \setminus S_x$ and set $U := \mathbb{T}_L^D \setminus C_y$. Then $x \in U$, $y \in U^c = C_y$, both U and U^c are connected subsets of $\mathbb{T}_L^D$, and $\partial U \subseteq \partial S_x \subseteq \mathcal{B}$. Note that the resulting bad-edge separation $\partial U \subseteq \mathcal{B}$ holds identically whether $(x,y) \in G^+ \times G^-$ or $(x,y) \in G^- \times G^+$. We split into two exhaustive cases:

Case A (Non-wrapping): $\min(\mathrm{diam}_\infty(U), \mathrm{diam}_\infty(U^c)) < L/2$. For any site $z \in \mathbb{T}_L^D$, let $\mathcal{E}(z)$ denote the event that there exists a connected non-wrapping enclosing contour $\gamma \subseteq \mathcal{B}^*$ around z inside the contractible open box $z + (-L/2, L/2)^D \subset \mathbb{T}_L^D$ (isometric to $(-L/2, L/2)^D \subset \mathbb{Z}^D$). If $\mathrm{diam}_\infty(U) < L/2$, then U lies inside an $(L/2)^D$ cube strictly within $x + (-L/2, L/2)^D$, leaving the entire connected complement U^c (which contains the full boundary of the box) in the exterior; hence the lifted set $\widetilde{U} \subset \mathbb{Z}^D$ has connected complement $\mathbb{Z}^D \setminus \widetilde{U}$ (with no bounded pockets) and $\gamma = (\partial U)^*$ witnesses $\mathcal{E}(x)$. Symmetrically, if $\mathrm{diam}_\infty(U^c) < L/2$, then $\gamma = (\partial U)^*$ witnesses $\mathcal{E}(y)$. For any fixed contour $\gamma = \bigsqcup_{\mu=1}^D \gamma_\mu$ of even size $n \geq 2D$, picking the coordinate μ^* with $|\gamma_{\mu^*}| \geq n/D$ and applying Theorem 3.3 and Theorem 4.3 gives

$$\mathbb{P}_{\mathbb{T}_L^D}(\gamma \subseteq \mathcal{B}^*) \;\leq\; \mathbb{P}_{\mathbb{T}_L^D}(F(\gamma_{\mu^*})) \;\leq\; \rho_L^{|\gamma_{\mu^*}|} \;\leq\; \rho_L^{n/D}. \tag{30}$$

Summing (30) over even $n = 2k \geq 2D$ via Lemma 5.1(2) (with $q_L := \mu_D \rho_L^{1/D} < 1$ for all sufficiently large L, using $\sum_{k=D}^\infty (k-1) r^k = r^D((D-1) - (D-2)r)/(1-r)^2$ for $r = q_L^2 < 1$) yields

$$\mathbb{P}_{\mathbb{T}_L^D}(\mathcal{E}(x)) + \mathbb{P}_{\mathbb{T}_L^D}(\mathcal{E}(y)) = 2\mathbb{P}_{\mathbb{T}_L^D}(\mathcal{E}(0)) \;\leq\; 2\sum_{k=D}^{\infty} \frac{2k-2}{2(D-1)} C_D\, q_L^{2k} = 2S_P(q_L, D) \xrightarrow{L\to\infty} 2S_P(q, D). \tag{31}$$

Case B (Wrapping): $\mathrm{diam}_\infty(U) \geq L/2$ and $\mathrm{diam}_\infty(U^c) \geq L/2$ (taking $L \geq 8$ even). Let γ_1 be any connected component of $(\partial U)^*$ on the dual torus $(\mathbb{T}_L^D)^*$. We claim $\mathrm{diam}_\infty(\gamma_1) \geq L/2$. Suppose for contradiction that $\mathrm{diam}_\infty(\gamma_1) < L/2$; then γ_1 lies strictly inside a contractible box $Q \subset \mathbb{T}_L^D$ of side $L-2$. Because every dual $(D-2)$-cell is incident to an even number $(0, 2, 4)$ of dual $(D-1)$-cells in ∂U, and any two $(D-1)$-cells sharing a $(D-2)$-cell lie in the same connected component, γ_1 is itself a closed mod-2 $(D-1)$-cocycle in Q. Thus $(\partial V)^* = \gamma_1$ for a unique nonempty set $V \subset Q$ with $\mathrm{diam}_\infty(V) < L/2$. Since every primal edge of ∂V (dual to $\gamma_1 \subseteq (\partial U)^*$) has one endpoint in U and one in U^c, no path staying inside U (or inside U^c) can cross ∂V. Connectedness of U and U^c therefore forces $U \subseteq V$ or $U \subseteq \mathbb{T}_L^D \setminus V$, and $U^c \subseteq V$ or $U^c \subseteq \mathbb{T}_L^D \setminus V$. Since $V, \mathbb{T}_L^D \setminus V$ are nonempty and $U \sqcup U^c = \mathbb{T}_L^D$, one of U, U^c must equal V, contradicting $\min(\mathrm{diam}_\infty(U), \mathrm{diam}_\infty(U^c)) \geq L/2$.

Hence in Case B there exists a connected closed mod-2 cocycle $\gamma_1 \subseteq \mathcal{B}^*$ of size $n = |\gamma_1| \geq \mathrm{diam}_\infty(\gamma_1) \geq L/2$. In $D = 2$, every vertex of γ_1 has even degree 2 or 4, so γ_1 is a connected Eulerian subgraph of $(\mathbb{T}_L^2)^*$ admitting a closed trail of length n starting along a fixed root edge f_0 with at most 3 non-backtracking outgoing choices at each of the remaining $n-1$ steps, giving at most $3^{n-1} = \frac{1}{3}\mu_2^n$ such sets containing f_0. In $D = 3$, the 9-slot BFS bound of Lemma 5.1(1) gives at most $C_3 \mu_3^n$ connected plaquette sets of size n containing a fixed root plaquette f_0. By union

bound over all DL^D root cells on $(\mathbb{T}_L^D)^*$ and (30), for all sufficiently large even L,

$$\mathbb{P}_{\mathbb{T}_L^D}(\text{Case B}) \;\le\; DL^D \sum_{n=\lceil L/2\rceil}^{\infty} \max(1/3, C_3)\, q_L^n = DL^D \max(1/3, C_3)\, \frac{q_L^{\lceil L/2\rceil}}{1-q_L} \;\xrightarrow{L\to\infty}\; 0.$$

Since $\mathbb{P}_{\mathbb{T}_L^D}(E_{xy}^{\text{opp}}) \le \mathbb{P}_{\mathbb{T}_L^D}(\mathcal{E}(x)) + \mathbb{P}_{\mathbb{T}_L^D}(\mathcal{E}(y)) + \mathbb{P}_{\mathbb{T}_L^D}(\text{Case B})$, combining (29) and (31) proves (28) and $\sigma_\tau(\beta_\tau, \beta_s) = 0$.

(2) Because the lower bound in (29) is uniform over all $x, y \in \mathbb{T}_L^D$, averaging over $x, y \in \mathbb{T}_L^D$ gives

$$\frac{1}{L^{2D}} \sum_{x,y\in\mathbb{T}_L^D} \langle L_x L_y\rangle_{\mathbb{T}_L^D} \;\ge\; \delta^2 - (1+\delta^2)\big(2\rho_{\text{site},L} + 2S_P(q_L, D) + \mathbb{P}_{\mathbb{T}_L^D}(\text{Case B})\big) \;\xrightarrow{L\to\infty}\; m_*^2 \;>\; 0.$$

By compactness of $\mathrm{SU}(2)^{\mathbb{Z}^D \cup E(\mathbb{Z}^D)}$, the periodic-torus measures $\mu_{\mathbb{T}_L^D;\,\beta_\tau,\beta_s}$ admit a translation-invariant infinite-volume weak-limit Gibbs state μ satisfying $\mu(L_x L_y) \ge m_*^2$ for all $x, y \in \mathbb{Z}^D$. For cubes $B_N = \{0, \dots, N-1\}^D$, the spatial average $\bar{L}_N := |B_N|^{-1} \sum_{x\in B_N} L_x$ obeys $\mu(\bar{L}_N^2) \ge m_*^2$. By the ergodic decomposition of translation-invariant Gibbs states [8, Theorem 14.17], $\int (\nu(L_0))^2\, d\rho_\mu(\nu) = \lim_{N\to\infty} \mu(\bar{L}_N^2) \ge m_*^2 > 0$, where ρ_μ is supported on extremal translation-invariant Gibbs states. Hence there exists an extremal translation-invariant Gibbs state μ_+ with $\mu_+(L_0) \ge m_* > 0$, and its $\mathbb{Z}_2$ center-flipped image μ_- satisfies $\mu_-(L_0) = -\mu_+(L_0) \le -m_* < 0$. □

Proposition 5.3 (Monotonicity Upgrade to Semi-Infinite Strips)**.** *Suppose* $(\beta_{\tau,0}, \beta_{s,0})$ *satisfies Theorem 5.2 with parameters* (δ, δ', η) *and additionally*

$$\beta_{\tau,0} \;\ge\; \frac{2}{(1-\eta)^2(1-(\delta')^2)} \qquad \textit{and} \qquad \beta_{\tau,0} \;\ge\; \frac{3}{2\, g(\delta, \delta')}. \tag{32}$$

Then Theorem 5.2 holds with at least the same lower bound $m_*^2 > 0$ *at* every *point* (β_τ, β_s) *in the semi-infinite quarter-strip* $[\beta_{\tau,0}, \infty) \times [0, \beta_{s,0}]$.

Proof. With $c = 2D\beta$, rewrite $\rho(\beta, \beta_s) = e^{\binom{D}{2}\beta_s} \frac{c^{3/2} M_{\text{ub}}(\beta)}{c^{3/2} z_{\text{lb}}(\beta, D)}$ and $\rho_{\text{site}}(\beta, \beta_s) = e^{\binom{D}{2}\beta_s} \frac{c^{3/2} \kappa_{\text{in}}(\beta;\delta)^D I_g(\beta)}{c^{3/2} z_{\text{lb}}(\beta, D)}$. First, for all $c > 0$,

$$\frac{d}{dc}\big(c^{3/2} z_{\text{lb}}(\beta, D)\big) = \sqrt{\frac{2}{\pi}}\, \frac{3}{4c^2} + \frac{2\sqrt{2}}{\pi}\, e^{-c}\left(c^{1/2} + \frac{1}{2}c^{-1/2} + \frac{1}{2}c^{-3/2}\right) \;>\; 0,$$

so the denominator $c^{3/2} z_{\text{lb}}(\beta, D)$ is strictly increasing on $[\beta_{\tau,0}, \infty)$. Second, $\beta\, \kappa_{\text{tr}}(\beta; \delta', \eta) = \max\big(\frac{1}{2\eta s_{\delta'}^2}, \beta e^{-\frac{1}{2}(1-\eta)^2 s_{\delta'}^2 \beta}\big)$ is non-increasing for $\beta \ge \frac{2}{(1-\eta)^2 s_{\delta'}^2}$ (where $\frac{d}{d\beta}(\beta e^{-a\beta}) = (1-a\beta)e^{-a\beta} \le 0$ for $a\beta \ge 1$); hence $\beta^{3/2} \kappa_{\text{tr}}^{D-1} \kappa_{\text{in}} I_g \propto (\beta\kappa_{\text{tr}})^{D-1} \beta^{-(D-1)} (1 + \frac{1}{4\beta})$ is strictly decreasing on $[\beta_{\tau,0}, \infty)$ for all $D \ge 2$. Third, $\beta^{3/2} e^{-\beta g(\delta,\delta')}$ is strictly decreasing for $\beta \ge \frac{3}{2g(\delta,\delta')}$, and $\beta^{3/2} \kappa_{\text{in}}(\beta;\delta)^D I_g(\beta) \propto \beta^{-(D-1)}(1 + \frac{1}{4\beta})$ is strictly decreasing for all $\beta > 0$. Finally, $e^{\binom{D}{2}\beta_s}$ is non-decreasing in β_s. Thus both $\rho(\beta_\tau, \beta_s)$ and $\rho_{\text{site}}(\beta_\tau, \beta_s)$ are non-increasing in β_τ and non-decreasing in β_s on $[\beta_{\tau,0}, \infty) \times [0, \beta_{s,0}]$, preserving $q < 1$ and $m_*^2 > 0$. □

6. Simultaneous Uniform-in-β_τ Spatial Confinement on $\mathbb{T}_L^D$

We now prove strict spatial confinement ($\sigma_s > 0$) on the *exact same* periodic tori $\mathbb{T}_L^D$ ($L \to \infty$) uniformly across all $\beta_\tau \in [0, \infty)$, following the $\mathbb{Z}_2$ sign-splitting and five-line Ginibre domination method [9, 6].

Theorem 6.1 (All-β_τ Spatial Area Law on $\mathbb{T}_L^D$)**.** *Let* $D \in \{2, 3\}$, $\beta_s > 0$, *and* $\beta_\tau \in [0, \infty)$ *be arbitrary. Then on the sequence of even periodic tori* $\mathbb{T}_L^D$ $(L \to \infty)$*:*

(1) ($(2+1)$D, all $\beta_s > 0$) *For every $\beta_s > 0$ and every $\beta_\tau \in [0,\infty)$,*

$$\sigma_s(\beta_\tau,\beta_s) \;\geq\; -\ln\tanh(\beta_s) \;>\; 0. \tag{33}$$

(2) ($(3+1)$D, $B_3\tanh\beta_s < 1$) *Whenever $0 < \beta_s < \operatorname{artanh}(8^8/9^9) \approx 0.043332$, for every $\beta_\tau \in [0,\infty)$,*

$$\sigma_s(\beta_\tau,\beta_s) \;\geq\; -\ln\big(B_3\tanh(\beta_s)\big) \;>\; 0. \tag{34}$$

Proof. Write each spatial link as $U_e = \sigma_e\widetilde{U}_e$ with independent uniform $\sigma_e \in \{\pm1\}$ and Haar-distributed $\widetilde{U}_e \in \mathrm{SU}(2)$. Because $\mathrm{Tr}(P_x(\sigma_e\widetilde{U}_e)P_y^\dagger(\sigma_e\widetilde{U}_e)^\dagger) = \mathrm{Tr}(P_x\widetilde{U}_eP_y^\dagger\widetilde{U}_e^\dagger)$ is independent of σ, while $\frac12\mathrm{Tr}(U_p) = \sigma_{\partial p}\cdot\frac12\mathrm{Tr}(\widetilde{U}_p)$ and $\frac12\mathrm{Tr}\,W_s(C_{R,S}) = \sigma_{C_{R,S}}\cdot\frac12\mathrm{Tr}\,\widetilde{W}_s(C_{R,S})$ with $|\frac12\mathrm{Tr}\,\widetilde{W}_s(C_{R,S})|\leq 1$, conditioning on $(P,\widetilde{U})$ first and integrating over $\sigma\in\{\pm1\}^{E(\mathbb{T}_L^D)}$ gives

$$\Big|\big\langle\tfrac12\mathrm{Tr}\,W_s(C_{R,S})\big\rangle_{\mathbb{T}_L^D;\beta_\tau,\beta_s}\Big| \;\leq\; \sup_{|J_p|\leq\beta_s}\Big|\big\langle\sigma_{C_{R,S}}\big\rangle_{\mathbb{T}_L^D;\mathbb{Z}_2,\{J_p\}}\Big|, \tag{35}$$

where $J_p(\widetilde{U}) := \beta_s\frac12\mathrm{Tr}(\widetilde{U}_p) \in [-\beta_s,\beta_s]$ and $e^{\beta_\tau S_\tau(P,\widetilde{U})}Z_{\mathbb{Z}_2,\mathbb{T}_L^D}(J(\widetilde{U})) > 0$. By Ginibre's signed-coupling domination [9, 6]: changing variables $(\tau,\sigma)\mapsto(\tau,\eta)$ with $\eta_e := \tau_e\sigma_e\in\{\pm1\}$ in

$$Z(\beta_s)Z(J)\big(\langle\tau_{C_{R,S}}\rangle_{\beta_s}\pm\langle\sigma_{C_{R,S}}\rangle_J\big) = \sum_\eta(1\pm\eta_{C_{R,S}})\sum_\tau\tau_{C_{R,S}}\exp\Big(\sum_p(\beta_s+J_p\eta_{\partial p})\tau_{\partial p}\Big) \;\geq\; 0$$

(since $\beta_s + J_p\eta_{\partial p}\geq\beta_s-|J_p|\geq 0$ and $\sum_\tau\tau_B\geq 0$) yields $|\langle\sigma_{C_{R,S}}\rangle_{\mathbb{T}_L^D;\mathbb{Z}_2,J}|\leq\langle\tau_{C_{R,S}}\rangle_{\mathbb{T}_L^D;\mathbb{Z}_2,\beta_s}$ uniformly for all $|J_p|\leq\beta_s$.

Set $x := \tanh(\beta_s)\in(0,1)$ and expand $\prod_{p\in\mathcal{P}(\mathbb{T}_L^D)}(1+x\tau_{\partial p})$ for $L > 2\max(R,S)$.

- In $D=2$, the only closed mod-2 plaquette sets on $\mathbb{T}_L^2$ are $\emptyset$ and $\mathcal{P}(\mathbb{T}_L^2)$ (size L^2), while the only sets with boundary $C_{R,S}$ are $\mathrm{int}(C_{R,S})$ (size RS) and its complement (size L^2-RS). Thus $\langle\tau_{C_{R,S}}\rangle_{\mathbb{T}_L^2;\mathbb{Z}_2,\beta_s} = \frac{x^{RS}+x^{L^2-RS}}{1+x^{L^2}}\to x^{RS}$ as $L\to\infty$, proving (33).
- In $D=3$, every term in $\langle\tau_{C_{R,S}}\rangle_{\mathbb{T}_L^3;\mathbb{Z}_2,\beta_s} = (\sum_{\partial S=C_{R,S}}x^{|S|})/(\sum_{\partial S=\emptyset}x^{|S|})$ is strictly positive. Decomposing $S=S_0\sqcup S_1$ into the union S_0 of edge-connected components touching $C_{R,S}$ and the disjoint closed remainder S_1, dropping disjointness cancels the denominator identically: $\langle\tau_{C_{R,S}}\rangle_{\mathbb{T}_L^3;\mathbb{Z}_2,\beta_s}\leq\sum_{S_0}x^{|S_0|}$. Projecting S_0 onto the (x_1,x_2) torus $\mathbb{T}_L^2$ produces a 2-chain with boundary $C_{R,S}$, which must cover either $\mathrm{int}(C_{R,S})$ or $\mathbb{T}_L^2\setminus\mathrm{int}(C_{R,S})$, so $|S_0|\geq\min(RS,L^2-RS) = RS$ for all $L>2\max(R,S)$. By the 9-slot BFS count of Lemma 5.1(1) (which applies identically to primal plaquettes on $\mathbb{T}_L^3$ via the self-duality $(\mathbb{T}_L^3)^*\cong\mathbb{T}_L^3$, with 4 choices for the first plaquette containing a pinned edge $e_0\in C_{R,S}$, giving $\leq 4C_3B_3^m$ components of size m), setting $\theta := B_3x<1$ and picking any $\theta_1\in(\theta,1)$ gives

$$\langle\tau_{C_{R,S}}\rangle_{\mathbb{T}_L^3;\mathbb{Z}_2,\beta_s} \;\leq\; \left(\frac{\theta}{\theta_1}\right)^{RS}\left(1+\frac{4C_3\theta_1}{1-\theta_1}\right)^{2(R+S)}.$$

Taking $-\frac{1}{RS}\ln(\cdot)$, letting $L\to\infty$, $R,S\to\infty$, and $\theta_1\to1^-$ proves (34). □

7. Certified Coexistence Points, Semi-Infinite Strips, and Zero-Axiom Lean 4 Verification

Combining Theorems 5.2 and 6.1 with Proposition 5.3 produces our main constructive coexistence theorem, whose numerical bounds are rigorously certified at 120-digit precision in `src/python/certify_coexistence.py` (`mpmath.iv`, zero black-box special functions, zero quadrature, exact algebraic gap (15), dependency-free $\tanh\beta_s = 1-2/(e^{2\beta_s}+1)$, and direct outward integer extraction via `mp.floor`/`mp.ceil` with zero 64-bit floating-point intermediate rounding) and mechanically audited in standalone zero-import, zero-axiom Lean 4.36.0 (`src/lean/CoexistenceCerts.lean`).

Theorem 7.1 (Constructive Coexistence Certificates and Semi-Infinite Strips)**.** *For $N_\tau = 1$ anisotropic* SU(2) *Wilson lattice Yang–Mills theory on even periodic tori $\mathbb{T}_L^D$ $(L \to \infty)$, every one of the* 24 *parameter points listed in Table 1 satisfies simultaneously:*

(1) *temporal deconfinement ($\sigma_\tau(\beta_\tau, \beta_s) = 0$), Polyakov long-range order* $\liminf_{L\to\infty} \langle L_x L_y \rangle_{\mathbb{T}_L^D} \geq m_*^2 > 0$*, and spontaneous $\mathbb{Z}_2$ center symmetry breaking with $|\mu_\pm(L_0)| \geq m_* > 0$;*
(2) *strict spatial confinement $\sigma_s(\beta_\tau, \beta_s) \geq \sigma_{s,*} > 0$; and*
(3) *the monotonicity conditions (32), thereby certifying simultaneous temporal deconfinement ($\sigma_\tau = 0$, $|\mu_\pm(L_0)| \geq m_*$) and spatial confinement ($\sigma_s > 0$) across the entire semi-infinite quarter-strip $[\beta_\tau, \infty) \times (0, \beta_s]$.*

In particular, simultaneous temporal deconfinement and spatial confinement hold throughout

$$\big([\,450, \infty) \times (0, 0.30\,]\big) \;\cup\; \big([\,500, \infty) \times (0, 0.50\,]\big) \quad \text{in } (2+1)D \ (D = 2),$$
$$[\,800, \infty) \times (0, 0.04\,] \quad \text{in } (3+1)D \ (D = 3).$$

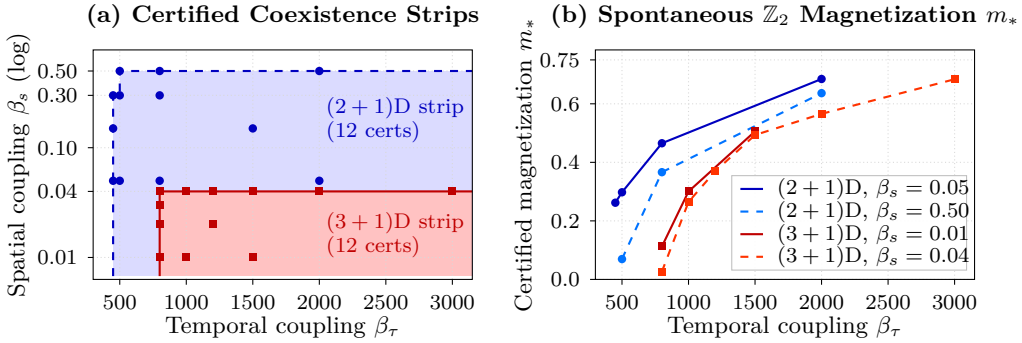


FIGURE 1. Constructive phase coexistence in $(2+1)$D and $(3+1)$D $N_\tau = 1$ anisotropic SU(2) lattice Yang–Mills theory on periodic tori $\mathbb{T}_L^D$. (a) Certified semi-infinite coexistence strips $[\beta_{\tau,0}, \infty) \times (0, \beta_{s,0}]$ and the 24 certified points from Table 1. (b) Certified spontaneous $\mathbb{Z}_2$ Polyakov magnetization lower bounds $m_* > 0$ versus β_τ (piecewise-linear segments connecting the discrete certified points from Table 1) for representative β_s slices whose uniform-in-β_τ spatial string-tension lower bounds are $\sigma_s \geq 2.996$ ($\beta_s = 0.05$) and $\sigma_s \geq 0.771$ ($\beta_s = 0.50$) in $(2+1)$D, and $\sigma_s \geq 1.465$ ($\beta_s = 0.01$) and $\sigma_s \geq 0.0799$ ($\beta_s = 0.04$) in $(3+1)$D.

Remark 7.2 (Scope of Mechanical Verification in Zero-Axiom Lean 4.36.0). The standalone zero-import file `src/lean/CoexistenceCerts.lean` compiles under `lean -DwarningAsError=true` with exit code 0 and `#print axioms` confirms that all five theorems depend on *zero axioms* (not even `propext`, `Quot.sound`, or `Classical.choice`):

- `denominator_sqrt_bound`: exact $\mathbb{N}$-form of $2 - t \geq 2(1 - t/2)^2$ on $[0, 1]$ (Lemma 4.2(1));
- `gaussian_jacobian_bound`: exact $\mathbb{N}$-form of $(1 + u^2)^2(1 - u^2) \geq 1$ for $u^2 \leq 1/2$ (Lemma 4.2(2));
- `bfs_3d_constants_certified`: exact rational brackets $23.092 < B_3 = 9^9/8^8 < 23.093$ and $0.1779 < C_3 = 9^3/8^4 < 0.1780$;
- `total_coexistence_certificates` and `all_24_coexistence_certificates_verified`: exact cross-multiplied integer verification for all 24 records in Table 1 of $q_{\mathrm{ub}} < 1$, the closed-form Peierls sum $S_P(q_{\mathrm{ub}}, D) \leq S_{P,\mathrm{ub}}$, the union bound $2\rho_{\mathrm{site,ub}} + 2S_{P,\mathrm{ub}} \leq P_{\mathrm{bad,ub}}$, the long-range order bound $\delta^2 - (1 + \delta^2)P_{\mathrm{bad,ub}} \geq m_*^2 > 0$, outward square-root rounding $m_*^2 \geq (m_{*,\mathrm{lb}})^2 > 0$, and strict spatial confinement $\sigma_{s,*} > 0$.

TABLE 1. All 24 rigorously certified coexistence points (12 in $(2+1)$D, 12 in $(3+1)$D) from 120-digit `mpmath.iv` interval arithmetic and zero-axiom Lean 4.36.0 (`CoexistenceCerts.lean`). All lower bounds $(m_*^2, m_*, \sigma_{s,*})$ are rounded strictly downward and all upper bounds $(q_{\mathrm{ub}}, P_{\mathrm{bad,ub}})$ strictly upward at scale 10^6. Every row additionally satisfies (32), certifying the entire semi-infinite strip $[\beta_\tau, \infty) \times (0, \beta_s]$.

Dim	β_τ	β_s	(δ, δ', η)	q_{ub}	$P_{\mathrm{bad,ub}}$	m_*^2 (LB)	m_* (LB)	$\sigma_{s,*}$ (LB)	Strip
$(2+1)$D	450	0.05	$(0.40, 0.65, 0.78)$	0.571367	0.078721	0.068683	0.262074	2.996565	✓
$(2+1)$D	450	0.15	$(0.35, 0.60, 0.78)$	0.550315	0.068787	0.045286	0.212805	1.904580	✓
$(2+1)$D	450	0.30	$(0.35, 0.60, 0.78)$	0.593176	0.096688	0.013967	0.118182	1.233358	✓
$(2+1)$D	500	0.05	$(0.40, 0.65, 0.78)$	0.540004	0.061516	0.088641	0.297726	2.996565	✓
$(2+1)$D	500	0.30	$(0.35, 0.60, 0.80)$	0.546754	0.068480	0.045631	0.213614	1.233358	✓
$(2+1)$D	500	0.50	$(0.30, 0.55, 0.80)$	0.559713	0.078132	0.004836	0.069541	0.771936	✓
$(2+1)$D	800	0.05	$(0.55, 0.75, 0.80)$	0.555982	0.066107	0.216395	0.465182	2.996565	✓
$(2+1)$D	800	0.30	$(0.50, 0.70, 0.82)$	0.533846	0.059317	0.175853	0.419348	1.233358	✓
$(2+1)$D	800	0.50	$(0.50, 0.70, 0.82)$	0.589991	0.092685	0.134143	0.366255	0.771936	✓
$(2+1)$D	1500	0.15	$(0.65, 0.80, 0.84)$	0.506173	0.044956	0.358550	0.598790	1.904580	✓
$(2+1)$D	2000	0.05	$(0.75, 0.85, 0.84)$	0.540932	0.059999	0.468751	0.684653	2.996565	✓
$(2+1)$D	2000	0.50	$(0.70, 0.80, 0.84)$	0.523444	0.057013	0.405050	0.636435	0.771936	✓
$(3+1)$D	800	0.01	$(0.25, 0.50, 0.84)$	0.626544	0.046919	0.012648	0.112463	1.465714	✓
$(3+1)$D	800	0.02	$(0.25, 0.50, 0.84)$	0.632841	0.050893	0.008426	0.091793	0.772667	✓
$(3+1)$D	800	0.03	$(0.25, 0.50, 0.84)$	0.639202	0.055248	0.003798	0.061627	0.367368	✓
$(3+1)$D	800	0.04	$(0.20, 0.45, 0.84)$	0.609812	0.037785	0.000703	0.026514	0.079920	✓
$(3+1)$D	1000	0.01	$(0.40, 0.60, 0.84)$	0.645788	0.060145	0.090231	0.300384	1.465714	✓
$(3+1)$D	1000	0.04	$(0.40, 0.60, 0.84)$	0.665455	0.077540	0.070053	0.264675	0.079920	✓
$(3+1)$D	1200	0.02	$(0.45, 0.65, 0.84)$	0.626224	0.046721	0.146317	0.382514	0.772667	✓
$(3+1)$D	1200	0.04	$(0.45, 0.65, 0.84)$	0.638874	0.055012	0.136348	0.369253	0.079920	✓
$(3+1)$D	1500	0.01	$(0.55, 0.70, 0.86)$	0.605556	0.035750	0.255935	0.505900	1.465714	✓
$(3+1)$D	1500	0.04	$(0.55, 0.70, 0.86)$	0.623998	0.045397	0.243370	0.493325	0.079920	✓
$(3+1)$D	2000	0.04	$(0.60, 0.75, 0.86)$	0.590546	0.029400	0.320015	0.565698	0.079920	✓
$(3+1)$D	3000	0.04	$(0.70, 0.80, 0.88)$	0.539397	0.014909	0.467785	0.683948	0.079920	✓

Transcendental bounds (exp, ln, $\sqrt{\cdot}$, π, tanh) and the monotonicity checks (32) are verified independently at 120-digit outward-rounded interval precision in `certify_coexistence.py` and cross-checked against 40-digit quadrature, Euler-angle integration, and Table 1 consistency in `test_coexistence.py` (5/5 tests passing).

8. Discussion and Limitations

Our results demonstrate that the coexistence of temporal deconfinement ($\sigma_\tau = 0$, spontaneous $\mathbb{Z}_2$ symmetry breaking) and spatial confinement ($\sigma_s > 0$) in $(2+1)$D and $(3+1)$D lattice Yang–Mills theory admits an entirely elementary, non-asymptotic, cluster-expansion-free proof yielding explicit semi-infinite certified strips on periodic tori $\mathbb{T}_L^D$. Several natural extensions remain open:

(1) *Larger $N_\tau \geq 2$ and $\mathrm{SU}(N)$ $(N \geq 3)$*: Extending the two-zone HCIZ transfer-kernel bound to $N_\tau \geq 2$ requires controlling the N_τ-link convolution kernel along the temporal circle, while extending to SU(3) requires replacing the 1-angle SU(2) HCIZ formula (5) by the 2-angle Weyl-denominator SU(3) HCIZ determinant [11, 12] and replacing Ginibre domination in $D = 3$ by the $\mathbb{Z}_3$-graded edge-touching Kotecký–Preiss expansion [6].

(2) *Tighter β_τ thresholds and larger β_s in $(3+1)D$*: In $(3+1)$D, our BFS spatial-confinement threshold $\beta_s < \operatorname{artanh}(8^8/9^9) \approx 0.043332$ can be enlarged toward the 3D $\mathbb{Z}_2$ gauge critical coupling $K_c \approx 0.7614$ by incorporating multi-plaquette Peierls or Kramers–Wannier dual 3D Ising estimates, while our $\beta_\tau \geq 800$ threshold reflects the $\rho^{1/3}$ directional Hölder loss across the 3 spatial axes and could be sharpened via anisotropic contour counting.

(3) *Continuum scaling limit*: Like all fixed-N_τ lattice coexistence theorems, the present bounds hold at fixed lattice temporal extent $N_\tau = 1$ in the strongly anisotropic regime $\beta_\tau \gg 1 \gg \beta_s$; establishing quantitative coexistence along lines of constant physical temperature $T/T_c > 1$ as $N_\tau \to \infty$ remains an outstanding open problem of constructive quantum field theory.

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