Exact Extremal Rigidity and a Dimensional Phase Transition for Diamond-Saturated Families in Boolean Lattices and Hypergrids

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Abstract

In April 2026, Ivan and Jaffe settled a long-standing problem in extremal poset theory by proving that the induced saturation number of the diamond poset $\mathcal{D}_2$ in the Boolean lattice $2^{[n]}$ is $\mathrm{sat}^*(2^{[n]}, \mathcal{D}_2) = n+1$, and conjectured that every extremal diamond-saturated family of size $n+1$ is either a maximal chain, the bottom star $\{\emptyset\} \cup \binom{[n]}{1}$, or the top star $\{[n]\} \cup \binom{[n]}{n-1}$. Simultaneously, Çiçeksiz, Falgas-Ravry, Lato, and Sharifzadeh initiated induced poset saturation in the hypergrid $[t]^n$, proving a general $\Omega(\sqrt{n})$ dichotomy.

We resolve the extremal classification part of Ivan and Jaffe's conjecture for every $n \geq 1$ (leaving the second clause $|\mathcal{F}| \geq 2n-c$ open; Pillar A is proved self-containedly while Pillar B builds on Ivan and Jaffe's hypercube-layer bounds in Proposition 3.1) and uncover a sharp *dimensional rigidity phase transition* at $n = 2$ versus $n \geq 3$ across all finite mixed hypergrids $H = \prod_{i=1}^{n}[t_i]$ ($t_i \geq 2$, rank $R = \sum_{i=1}^{n}(t_i - 1)$):

1. *Boolean Extremal Rigidity (Theorem 1.1):* For every $n \geq 1$, a diamond-saturated family $\mathcal{F} \subseteq 2^{[n]}$ has $|\mathcal{F}| = n+1$ if and only if $\mathcal{F}$ is a maximal chain, $\{\emptyset\} \cup \binom{[n]}{1}$, or $\{[n]\} \cup \binom{[n]}{n-1}$ (count $n! + 2$ for $n \geq 2$, and 1 for $n = 1$). In particular, every interior diamond-saturated family ($\emptyset, [n] \notin \mathcal{F}$) satisfies $|\mathcal{F}| \geq n+2$ for all $n \geq 3$.
2. *Hypergrid Jump-Step Lower Bound and Dimensional Phase Transition (Theorem 1.2):* Every boundary-anchored diamond-saturated family $\mathcal{F} \subseteq \prod_{i=1}^{n}[t_i]$ (containing $\mathbf{0}$ or $\mathbf{t-1}$) satisfies $|\mathcal{F}| \geq R+1$. For every dimension $n \geq 3$ and all $t_1, \ldots, t_n \geq 2$, exact boundary-anchored rigidity holds: a boundary-anchored diamond-saturated family satisfies $|\mathcal{F}| = R+1$ if and only if $\mathcal{F}$ is a maximal chain or a generalized coordinate star $\mathcal{S}_{\mathbf{0}}, \mathcal{S}_{\mathbf{t-1}}$ (count $\binom{R}{t_1-1,\ldots,t_n-1} + 2$). In stark contrast, at dimension $n = 2$, rigidity breaks down whenever $\max(t_1, t_2) \geq 3$ in *both* the boundary-anchored and interior regimes, witnessed by explicit families $\mathcal{Y}_{t_1,t_2}$, $\mathcal{I}_{t_1,t_2}$, and $\mathcal{T}_t$ of size $R+1$.
3. *Sharp Singleton-Branch Stability (Theorem 1.3):* Any non-star rooted diamond-saturated family $\mathcal{F} \subseteq 2^{[n]}$ ($n \geq 3$, $\emptyset \in \mathcal{F}$, $[n] \notin \mathcal{F}$) containing $r_1 \geq 2$ singletons satisfies $|\mathcal{F}| \geq 1 + r_1(n - r_1 + 1) \geq 2n-1$, sharp at both $r_1 = 2$ and $r_1 = n-1$.
4. *Birkhoff Sub-Rank Collapse (Theorem 1.4):* In the rank-k distributive lattice $\mathcal{O}(B_k)$ of downsets of the k-element broom poset ($k \geq 3$), $\mathrm{sat}^*(\mathcal{O}(B_k), \mathcal{D}_2) = 3$ (even among top-anchored families) while $\mathrm{rk}(\mathcal{O}(B_k))+1 = k+1 \to \infty$, demonstrating that distributivity alone is insufficient to force linear rank growth.

The poset structural cores, the general mixed-hypergrid Jump-Step Theorem and threshold injection, the Pillar B and stability arithmetic closure cores, the infinite broom upper bound $\mathrm{sat}^*(\mathcal{O}(B_k), \mathcal{D}_2) \leq 3$ for all $k \geq 3$, and five concrete 2D non-rigidity witnesses are formalized in standalone Mathlib-free Lean 4.36.0 (35/35 public theorems strictly constructive, zero `Classical.choice`, zero `sorry`) alongside exhaustive pure-stdlib zero-float Python verification.

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1 Introduction

Given a finite ambient poset $(L, \leq)$ and a fixed finite poset P, a subset $\mathcal{F} \subseteq L$ is *induced P-free* if no induced subposet of $(\mathcal{F}, \leq)$ is isomorphic to P, and *induced P-saturated* in L if $\mathcal{F}$ is induced P-free while $\mathcal{F} \cup \{x\}$ contains an induced copy of P for every $x \in L \setminus \mathcal{F}$. The *induced saturation number* $\mathrm{sat}^*(L, P)$ is the minimum cardinality of an induced P-saturated family in L. Initiated for set systems by Gerbner et al. [6] and formalized in the induced Boolean setting $L = 2^{[n]}$ by Ferrara, Kay, Kramer, Martin, Reiniger, Smith, and Sullivan [4], induced poset saturation has developed rapidly through the $\log_2 n$ dichotomy of Keszegh, Lemons, Martin, Pálvölgyi, and Patkós [9], the $\Omega(\sqrt{n})$ dichotomy of Freschi, Piga, Sharifzadeh, and Treglown [5], and the polynomial upper bound of Bastide, Groenland, Ivan, and Johnston [1].

Among all target posets, the 4-element *diamond poset* $\mathcal{D}_2$ (a bottom a, a top d, and two incomparable middle elements $b \parallel c$ with $a < b, c < d$) has served as the flagship benchmark. Any maximal chain in $2^{[n]}$ is $\mathcal{D}_2$-saturated of size $n+1$, as are the bottom star $\mathcal{S}_\emptyset := \{\emptyset\} \cup \binom{[n]}{1}$ and the top star $\mathcal{S}_{[n]} := \{[n]\} \cup \binom{[n]}{n-1}$. Ferrara et al. [4] showed that any $\mathcal{D}_2$-saturated family containing $\emptyset$ or $[n]$ has size at least $n+1$, whereas lower bounds for general families stalled at $\Omega(\sqrt{n})$ until Ivan and Jaffe established first linear growth [7] and then, in April 2026, the exact saturation number $\mathrm{sat}^*(2^{[n]}, \mathcal{D}_2) = n+1$ [8]. In their concluding remarks [8, Section 4, Conjecture 21], Ivan and Jaffe explicitly posed the extremal classification problem:

> *Conjecture [8].* Let $\mathcal{F} \subseteq 2^{[n]}$ be diamond-saturated. If $|\mathcal{F}| = n+1$, then $\mathcal{F}$ is either a maximal chain, $\{\emptyset\} \cup \binom{[n]}{1}$, or $\{[n]\} \cup \binom{[n]}{n-1}$. Moreover, if $\emptyset, [n] \notin \mathcal{F}$, then $|\mathcal{F}| \geq 2n - c$ for some universal constant c.

In parallel, Çiçeksiz, Falgas-Ravry, Lato, and Sharifzadeh [3] extended induced poset saturation from the hypercube $2^{[n]} \cong [2]^n$ to the general hypergrid $[t]^n$, proving a dichotomy between $O(1)$ and $\Omega(\sqrt{n})$ growth and highlighting the structural obstructions caused by losing consecutive-level biregularity when $t \geq 3$.

1.1 Main Results

We resolve the first part of Ivan and Jaffe's conjecture completely for all $n \geq 1$, determine the exact boundary-anchored saturation number and extremal rigidity phase transition across *all* finite mixed hypergrids $H = \prod_{i=1}^n [t_i]$ (where $[t_i] := \{0, 1, \ldots, t_i - 1\}$ with coordinatewise order and rank $R := \sum_{i=1}^n (t_i - 1)$), establish sharp quadratic stability for singleton-branch rooted Boolean families, and identify the exact Birkhoff lattice boundary at which linear rank growth collapses.

For a mixed hypergrid $H = \prod_{i=1}^n [t_i]$ with minimum $\mathbf{0} = (0, \ldots, 0)$ and maximum $\mathbf{t} - \mathbf{1} = (t_1 - 1, \ldots, t_n - 1)$, define the *generalized bottom star* and *generalized top star* by

$$\mathcal{S}_\mathbf{0} := \{\mathbf{0}\} \cup \{r\mathbf{e}_i : 1 \leq i \leq n,\ 1 \leq r \leq t_i - 1\}, \qquad \mathcal{S}_{\mathbf{t}-\mathbf{1}} := \{(\mathbf{t} - \mathbf{1}) - x : x \in \mathcal{S}_\mathbf{0}\},$$

each of exact cardinality $R + 1$. A family $\mathcal{F} \subseteq H$ is *boundary-anchored* if $\mathbf{0} \in \mathcal{F}$ or $\mathbf{t} - \mathbf{1} \in \mathcal{F}$, and *interior* if $\mathbf{0}, \mathbf{t} - \mathbf{1} \notin \mathcal{F}$.

Theorem 1.1 (Complete Resolution of Ivan and Jaffe's Boolean Extremal Rigidity Conjecture)**.** *For every integer $n \geq 1$, an induced diamond-saturated family $\mathcal{F} \subseteq 2^{[n]}$ satisfies $|\mathcal{F}| = n + 1$ if and only if $\mathcal{F}$ is a maximal chain, the bottom star $\{\emptyset\} \cup \binom{[n]}{1}$, or the top star $\{[n]\} \cup \binom{[n]}{n-1}$. In particular:*

(i) *The exact number of extremal diamond-saturated families in $2^{[n]}$ is 1 for $n = 1$ and $n! + 2$ for every $n \geq 2$.*

(ii) *Every interior diamond-saturated family $\mathcal{F} \subseteq 2^{[n]}$ ($\emptyset, [n] \notin \mathcal{F}$) satisfies $|\mathcal{F}| \geq n + 2$ for all $n \geq 3$ (and no interior diamond-saturated family exists for $n \in \{1, 2\}$).*

Extending from $2^{[n]}$ to arbitrary mixed hypergrids reveals an unexpected *dimensional phase transition*: rigidity is not a Boolean artifact of $t = 2$, nor does it break merely because $t \geq 3$—rather, rigidity holds universally in all dimensions $n \geq 3$ for *all* $t_1, \ldots, t_n \geq 2$, and fails sharply at dimension $n = 2$ whenever $\max(t_1, t_2) \geq 3$.

Theorem 1.2 (Universal Dimensional Phase Transition for Hypergrid Rigidity)**.** *Let* $H = \prod_{i=1}^{n}[t_i]$ *with* $n \geq 1$, $t_i \geq 2$, *and rank* $R = \sum_{i=1}^{n}(t_i - 1)$.

(a) Universal boundary-anchored lower bound (all $n \geq 1$, all $t_i \geq 2$): *Every boundary-anchored diamond-saturated family* $\mathcal{F} \subseteq H$ *satisfies* $|\mathcal{F}| \geq R + 1 = 1 + \sum_{i=1}^{n}(t_i - 1)$ *(in particular,* $|\mathcal{F}| \geq (t-1)n + 1$ *in* $[t]^n$*).*

(b) Exact rigidity for all dimensions $n \geq 3$ and all $t_1, \ldots, t_n \geq 2$: *A boundary-anchored diamond-saturated family* $\mathcal{F} \subseteq H$ *achieves* $|\mathcal{F}| = R + 1$ *if and only if* $\mathcal{F}$ *is a maximal chain,* $\mathcal{S}_{\mathbf{0}}$*, or* $\mathcal{S}_{\mathbf{t-1}}$*. Consequently, the exact count of boundary-anchored extremal families in* H *is* $\binom{R}{t_1-1,\ldots,t_n-1} + 2$.

(c) Sharp breakdown at dimension $n = 2$: *On* $[t_1] \times [t_2]$ *with* $\max(t_1, t_2) \geq 3$ *(without loss of generality* $t_1 \geq 3$ *and* $t_2 \geq 2$ *by coordinate symmetry), rigidity fails in* both *the boundary-anchored and interior regimes at size* $R+1 = t_1+t_2-1$*, witnessed by the rooted non-chain, non-star family*

$$\mathcal{Y}_{t_1,t_2} \coloneqq \{(r,0) : 0 \leq r \leq t_1-2\} \cup \{(t_1-2, s) : 1 \leq s \leq t_2-2\} \cup \{(t_1-1, t_2-2),\ (t_1-2, t_2-1)\},$$

the interior size-$(R+1)$ *family* $(\mathbf{0}, (t_1-1, t_2-1) \notin \mathcal{I}_{t_1,t_2})$

$$\mathcal{I}_{t_1,t_2} \coloneqq \{(r,0) : 1 \leq r \leq t_1 - 1\} \cup \{(0,s) : 1 \leq s \leq t_2 - 1\} \cup \{(1,1)\},$$

and, for $t_1 = t_2 = t \geq 3$*, the anti-diagonal family* $\mathcal{T}_t \coloneqq \{(0,y) : 0 \leq y \leq t-1\} \cup \{(x, t-1-x) : 1 \leq x \leq t-1\}$.

Returning to $2^{[n]}$, we next prove a sharp stability gap above $n+1$ for rooted non-star families with at least two singleton children, complemented by an exhaustive stability census of $2^{[3]}$ and $2^{[4]}$.

Theorem 1.3 (Sharp Singleton-Branch Stability on $2^{[n]}$)**.** *Let* $n \geq 3$*, and let* $\mathcal{F} \subseteq 2^{[n]}$ *be diamond-saturated with* $\emptyset \in \mathcal{F}$*,* $[n] \notin \mathcal{F}$*, and* $\mathcal{F} \neq \{\emptyset\} \cup \binom{[n]}{1}$*. If* $\mathcal{F}$ *contains* $r_1 \geq 2$ *singletons, then* $r_1 \leq n-1$ *and*

$$|\mathcal{F}| \ \geq\ 1 + r_1(n - r_1 + 1) \ \geq\ 2n - 1.$$

Equality $|\mathcal{F}| = 2n-1$ *is achieved for every* $n \geq 3$ *at both endpoints* $r_1 = n-1$ *(the double-star* $\mathcal{R}_n \coloneqq \{\emptyset\} \cup \{\{i\}, \{i,n\} : 1 \leq i \leq n-1\}$*) and* $r_1 = 2$ *(the double-chain* $\mathcal{C}_n \coloneqq \{\emptyset, \{n\}\} \cup \{\{1,\ldots,k\} : 1 \leq k \leq n-1\} \cup \{\{n\} \cup \{2,\ldots,k\} : 2 \leq k \leq n-1\}$*).*

Finally, by Birkhoff's representation theorem [2], every finite distributive lattice is isomorphic to the downset lattice $\mathcal{O}(P)$ of its poset P of join-irreducibles (of rank $\mathrm{rk}(\mathcal{O}(P)) = |P|$), and $\mathcal{O}(P)$ is a hypergrid $\prod_{i=1}^{n}[t_i]$ if and only if P is a disjoint union of chains. We prove that distributivity alone is insufficient for linear rank growth: there exist arbitrarily large-rank distributive lattices whose diamond saturation number collapses to a constant.

Theorem 1.4 (Birkhoff Threshold: Infinite Sub-Rank Collapse in Non-Hypergrid Distributive Lattices)**.** *For every integer* $k \geq 3$*, let* B_k *denote the* k*-element broom poset consisting of a* $(k-2)$*-chain* $c_1 < \cdots < c_{k-2}$ *topped by two incomparable leaves* a, b $(c_{k-2} < a, b)$*. Then the finite distributive lattice* $L_k \coloneqq \mathcal{O}(B_k)$ *has rank* $\mathrm{rk}(L_k) = k$*, yet*

$$\mathrm{sat}^*(L_k, \mathcal{D}_2) \ =\ 3,$$

achieved by the 3*-element top-anchored family* $\{T_a, T_b, T_{ab}\}$ *where* $T_a = \{c_1, \ldots, c_{k-2}, a\}$*,* $T_b = \{c_1, \ldots, c_{k-2}, b\}$*, and* $T_{ab} = B_k$.

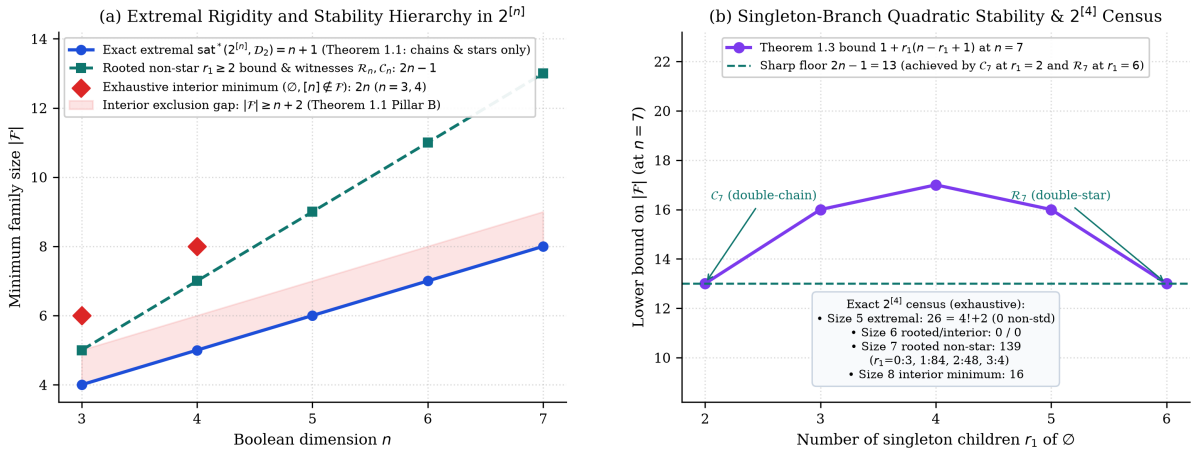

Figure 1: **Boolean extremal rigidity and stability hierarchy.** (a) Comparison of the exact extremal threshold $\mathrm{sat}^*(2^{[n]}, \mathcal{D}_2) = n+1$ (Theorem 1.1: chains and stars only), the Pillar B interior exclusion gap $|\mathcal{F}| \geq n+2$, the sharp $2n-1$ singleton-branch rooted stability floor (Theorem 1.3), and exhaustive interior minima $2n$ on $2^{[3]}, 2^{[4]}$. (b) Quadratic lower bound $1 + r_1(n - r_1 + 1)$ at $n = 7$ (sharp at $r_1 = 2$ via $\mathcal{C}_7$ and $r_1 = 6$ via $\mathcal{R}_7$) alongside the exhaustive $2^{[4]}$ census breakdown.

2 Hypergrid Jump-Step Engine and Dimensional Phase Transition: Proof of Theorem 1.2

We prove Theorem 1.2 first, as its $t_1 = \cdots = t_n = 2$ specialization immediately yields the boundary-anchored half (Pillar A) of Theorem 1.1 for all $n \geq 3$.

Throughout this section, let $H = \prod_{i=1}^{n}[t_i]$ with $t_i \geq 2$ and $R = \sum_{i=1}^{n}(t_i - 1)$, where coordinate indices range over $i \in \{1, \ldots, n\}$ and coordinate values range over $[t_i] \coloneqq \{0, \ldots, t_i - 1\}$. For $x \in H$, $i \in \{1, \ldots, n\}$, and $r \in \{0, \ldots, t_i - 1\}$, write $\mathrm{setCoord}(x, i, r)$ for the point obtained by replacing x_i with r while leaving all coordinates $j \neq i$ unchanged, and write $\mathrm{rk}(x) \coloneqq \sum_{j=1}^{n} x_j$.

We begin with two elementary poset facts (both formalized in Lean 4 as `not_incomp_below_of_bot_mem`, `not_incomp_of_bot_top_mem`, and `rooted_missing_below_must_be_middle`):

- *Out-Tree Lemma:* If $\mathcal{F} \subseteq H$ is diamond-free and $\mathbf{0} \in \mathcal{F}$, then for every $d \in \mathcal{F}$, the set $\{y \in \mathcal{F} : y < d\}$ is totally ordered (a chain); consequently, the Hasse diagram of $\mathcal{F}$ is an out-tree rooted at $\mathbf{0}$, and every $v \in \mathcal{F} \setminus \{\mathbf{0}\}$ has a unique immediate parent $p(v) \in \mathcal{F}$. If both $\mathbf{0}, \mathbf{t} - \mathbf{1} \in \mathcal{F}$, then $\mathcal{F}$ is a chain (hence a maximal chain of size $R+1$ when saturated).

- *Rooted Middle-Placement Lemma:* If $\mathbf{0} \in \mathcal{F}$ is diamond-free and $x \in H \setminus \mathcal{F}$ lies strictly below some $d_0 \in \mathcal{F}$, then in any induced diamond $\{a, b, c, d\} \subseteq \mathcal{F} \cup \{x\}$, x cannot be the bottom (else $\mathbf{0} < \{b, c\} < d$ lies in $\mathcal{F}$) nor the top (else $\mathbf{0} < \{b, c\} < d_0$ lies in $\mathcal{F}$); thus x must be a middle element.

Lemma 2.1 (Hypergrid Jump-Step Lemma)**.** *Let $\mathcal{F} \subseteq H = \prod_{i=1}^{n}[t_i]$ be diamond-saturated with $\mathbf{0} \in \mathcal{F}$. Then for every coordinate $i \in [n]$ and every threshold $r \in \{1, \ldots, t_i - 1\}$, there exists $u \in \mathcal{F}$ such that $u_i < r$ and $\mathrm{setCoord}(u, i, r) \in \mathcal{F}$.*

Proof. Consider the axis point $e_{i,r} \coloneqq r\mathbf{e}_i = \mathrm{setCoord}(\mathbf{0}, i, r) \in H$. If $e_{i,r} \in \mathcal{F}$, we may take $u = \mathbf{0}$. Otherwise $e_{i,r} \notin \mathcal{F}$. Every element of $\mathcal{F}$ below $e_{i,r}$ has zero coordinates for all $j \neq i$ and therefore forms a chain; hence $e_{i,r}$ cannot be the top of a diamond, nor can it be the bottom (since $\mathbf{0} \in \mathcal{F}$). By saturation, $e_{i,r}$ is a middle element of a diamond with some top $d_0 \in \mathcal{F}$, giving at least one crossing pair $(b, d) = (\mathbf{0}, d_0) \in \mathcal{F}^2$ satisfying $b < d$, $b_i < r$, and $r \leq d_i$.

Among all pairs $(b, d) \in \mathcal{F}^2$ with $b < d$, $b_i < r$, and $r \le d_i$, choose one maximizing $\mathrm{rk}(b)$, and set $T \coloneqq \mathrm{setCoord}(b, i, r) \in H$. Clearly $b < T \le d$. If $T \in \mathcal{F}$, taking $u = b$ completes the proof. Suppose instead that $T \notin \mathcal{F}$; then $T < d$, so by the Rooted Middle-Placement Lemma, saturation at T yields $b', d' \in \mathcal{F}$ with $T < d'$, $b' < d'$, and $T \parallel b'$. Because both $b < T < d'$ and $b' < d'$ lie in $\mathcal{F}$, the Out-Tree Lemma forces b and b' to be comparable; we cannot have $b' \le b$ (as $b < T$ would imply $b' < T$, contradicting $T \parallel b'$), so $b < b'$ and hence $\mathrm{rk}(b') > \mathrm{rk}(b)$. Because $b \le b'$ implies $b'_j \ge b_j = T_j$ for all $j \ne i$ while $T \not\le b'$, necessarily $b'_i < T_i = r$; meanwhile $T < d'$ gives $d'_i \ge T_i = r$. Thus (b', d') is a crossing pair with $\mathrm{rk}(b') > \mathrm{rk}(b)$, contradicting maximality. (Formally verified constructively in Lean 4 by `jump_step_exists`.) □

Remark 2.2. Notice why Lemma 2.1 uses $\mathrm{setCoord}(u, i, r)$ (allowing $u_i < r - 1$) rather than insisting on a unit step $u_i = r - 1$: in $[3] \times [2]$, the diamond-saturated family $\mathcal{J}_{3,2} = \{(0,0), (0,1), (1,1), (2,0)\}$ contains the jump edge $(0,0) \to (2,0)$ and has *no* unit-step edge increasing coordinate 1 from 1 to 2. In the Boolean case $t_i = 2$ ($r = 1$), $u_i < 1$ automatically forces $u_i = 0$, so every jump step is a unit bit-flip.

Lemma 2.3 (Unique-Coordinate Threshold Injection)**.** *Let $\mathcal{F} \subseteq H$ be diamond-free with $\mathbf{0} \in \mathcal{F}$. If $u, u' \in \mathcal{F}$, $i, i' \in [n]$, $u_i < r$, $u'_{i'} < r'$, and $w \coloneqq \mathrm{setCoord}(u, i, r) = \mathrm{setCoord}(u', i', r') \in \mathcal{F}$, then $i = i'$ and $r = r'$.*

Proof. Since $u < w$ and $u' < w$ in $\mathcal{F}$, the Out-Tree Lemma forces u and u' to be comparable. If $i \ne i'$, then $u_i < r = w_i = u'_i$ (so $u' \not\le u$) and $u'_{i'} < r' = w_{i'} = u_{i'}$ (so $u \not\le u'$), a contradiction. Hence $i = i'$, and then $r = w_i = w_{i'} = r'$. (Formally verified in Lean 4 by `jump_step_target_injective`.) □

Proof of Theorem 1.2. **(a) Universal lower bound.** By order duality ($x \mapsto (\mathbf{t} - \mathbf{1}) - x$), assume $\mathbf{0} \in \mathcal{F}$. By Lemma 2.1, each of the $R = \sum_{i=1}^{n}(t_i - 1)$ threshold pairs (i, r) ($1 \le i \le n$, $1 \le r \le t_i - 1$) admits a vertex $v(i, r) = \mathrm{setCoord}(u, i, r) \in \mathcal{F} \setminus \{\mathbf{0}\}$ with $u \in \mathcal{F}$ and $u_i < r$. By Lemma 2.3, the map $(i, r) \mapsto v(i, r)$ is injective into $\mathcal{F} \setminus \{\mathbf{0}\}$, whence $|\mathcal{F}| \ge 1 + R$.

(b) Exact rigidity for all $n \ge 3$ and all $t_1, \dots, t_n \ge 2$. For sufficiency, every maximal chain $C \subseteq H$ has size $R+1$, is trivially diamond-free, and for any $x \notin C$ the largest $c^- \in C$ below x and smallest $c^+ \in C$ above x differ in rank by at least 2, so any $c^* \in C$ with $c^- < c^* < c^+$ yields the induced diamond $c^- < \{x, c^*\} < c^+$. Likewise, $\mathcal{S}_{\mathbf{0}}$ has size $1 + R$, is a rooted out-tree (hence diamond-free), and any $x \notin \mathcal{S}_{\mathbf{0}}$ has at least two nonzero coordinates $i \ne j$, forming the induced diamond $\mathbf{0} < \{x_i \mathbf{e}_i, x_j \mathbf{e}_j\} < x$; duality gives the same for $\mathcal{S}_{\mathbf{t}-\mathbf{1}}$.

For necessity, let $\mathcal{F} \subseteq H$ be boundary-anchored with $|\mathcal{F}| = R+1$. By duality, assume $\mathbf{0} \in \mathcal{F}$. If $\mathbf{t} - \mathbf{1} \in \mathcal{F}$, the Out-Tree Lemma forces $\mathcal{F}$ to be a chain, hence a maximal chain of size $R + 1$. Assume now that $\mathbf{t} - \mathbf{1} \notin \mathcal{F}$.

Because $|\mathcal{F} \setminus \{\mathbf{0}\}| = R$, the injection $(i, r) \mapsto v(i, r)$ from part (a) is a *bijection*: every non-root vertex $v \in \mathcal{F} \setminus \{\mathbf{0}\}$ is $v(i, r)$ for a unique threshold (i, r), with an ancestor $u < v$ differing from v only in coordinate i. Since any intermediate element $z \in \mathcal{F}$ with $u \le z < v$ also satisfies $z_j = v_j$ for all $j \ne i$, the immediate parent $p(v)$ of v in the out-tree $\mathcal{F}$ satisfies $p(v)_j = v_j$ for all $j \ne i$ and $p(v)_i < r = v_i$. In other words, *every cover edge of $\mathcal{F}$ increases a single coordinate i to a unique threshold value r.* Consequently, for every (i, r) and every $x \in \mathcal{F}$,

$$x_i = r \quad \Longrightarrow \quad v(i, r) \le x \tag{1}$$

(as the last edge along $\mathbf{0} \to x$ modifying coordinate i must set it to r, hence must enter $v(i, r)$). Write $v_i \coloneqq v(i, t_i - 1)$ for the unique vertex setting coordinate i to its maximum $t_i - 1$.

Let $c_1, \dots, c_m$ ($m \ge 1$) be the children of $\mathbf{0}$ in $\mathcal{F}$, and let $C_q \coloneqq \{x \in \mathcal{F} : c_q \le x\}$ be the subtree rooted at c_q. Because each cover edge $(\mathbf{0}, c_q)$ modifies one coordinate, each child has the form $c_q = r_q \mathbf{e}_{i_q}$ with $1 \le r_q \le t_{i_q} - 1$; since distinct children of $\mathbf{0}$ are incomparable, the coordinates $i_1, \dots, i_m$ are pairwise distinct. Moreover, the subtrees $C_1, \dots, C_m$ are pairwise disjoint, and $x \in \mathcal{F}$ satisfies $x \ge c_q$ if and only if $x \in C_q$. We split on whether $m \ge 2$ or $m = 1$.

Case 1: $m \geq 2$. For every $j \in [n]$, let $\pi(j) \in \{1, \ldots, m\}$ be the unique branch index such that $v_j \in C_{\pi(j)}$; by (1), every $x \in \mathcal{F}$ with $x_j = t_j - 1$ belongs to $C_{\pi(j)}$.

1. *Claim 1.1:* $\pi(i_q) = q$ *for every* $q \in \{1, \ldots, m\}$. Suppose for contradiction that $\pi(i_q) = k \neq q$. Then $r_q < t_{i_q} - 1$ (else $c_q = (t_{i_q} - 1)\mathbf{e}_{i_q}$ would give $c_q \in C_q \cap C_k$). Consider $S' \coloneqq (t_{i_q} - 1)\mathbf{e}_{i_q} \notin \mathcal{F}$. Every element of $\mathcal{F}$ below S' lies on the i_q-axis (a chain), while any $D \in \mathcal{F}$ with $D > S'$ satisfies $D_{i_q} = t_{i_q} - 1$ (so $D \in C_k$) and $D > S' > c_q$ (so $D \in C_q$), impossible since $C_k \cap C_q = \emptyset$. Thus S' cannot form a diamond with $\mathcal{F}$, contradicting saturation.

2. *Claim 1.2:* $m = n$. Suppose there exists a coordinate $j \notin \{i_1, \ldots, i_m\}$. Let $k \coloneqq \pi(j)$, pick any $\ell \in \{1, \ldots, m\} \setminus \{k\}$ (since $m \geq 2$), and consider $S \coloneqq (t_j - 1)\mathbf{e}_j + r_\ell \mathbf{e}_{i_\ell} \notin \mathcal{F}$. Any $D \in \mathcal{F}$ with $D \geq S$ satisfies $D_j = t_j - 1 \implies D \in C_k$ and $D \geq c_\ell \implies D \in C_\ell$, impossible since $C_k \cap C_\ell = \emptyset$; hence nothing in $\mathcal{F}$ lies above S. Meanwhile, any nonzero $y \in \mathcal{F}$ with $y \leq S$ has support inside $\{j, i_\ell\}$; since $j \notin \{i_1, \ldots, i_m\}$, the only child of $\mathbf{0}$ below S is c_ℓ, so every such y satisfies $y \in C_\ell$, $y_{i_\ell} = r_\ell$, and $y_w = 0$ for all $w \notin \{j, i_\ell\}$. Thus $\{y \in \mathcal{F} : y \leq S\}$ is totally ordered by coordinate j (a chain), contradicting saturation at S.

3. *Claim 1.3:* $r_i = 1$ *for every* $i \in [n]$ *(using* $n \geq 3$*).* Reindex so $c_i = r_i \mathbf{e}_i$ for each $i \in [n]$. Suppose for contradiction that $r_1 \geq 2$. Since every element of C_1 has coordinate 1 at least r_1, the vertex $v(1, r_1 - 1)$ lies in some branch C_k with $k \neq 1$. Because $n \geq 3$, choose a third coordinate $\ell \in [n] \setminus \{1, k\}$, and consider $S \coloneqq (r_1 - 1)\mathbf{e}_1 + r_\ell \mathbf{e}_\ell$. Any $D \in \mathcal{F}$ with $D \geq S$ lies in C_ℓ (as $D \geq c_\ell$) and cannot have $D_1 \geq r_1$ (else $D \geq c_1 \implies D \in C_1$, contradicting $C_1 \cap C_\ell = \emptyset$); hence $D_1 = r_1 - 1$, which by (1) forces $D \geq v(1, r_1 - 1) \in C_k$, contradicting $C_k \cap C_\ell = \emptyset$. In particular, taking $D = S$ shows $S \notin \mathcal{F}$, and no element of $\mathcal{F}$ lies above S. Conversely, since $r_1 - 1 < r_1$, $c_1 \not\leq S$, so the only child of $\mathbf{0}$ below S is c_ℓ; as above, $\{y \in \mathcal{F} : y \leq S\}$ has $y_\ell = r_\ell$ and support in $\{1, \ell\}$, forming a chain. Thus $S \notin \mathcal{F}$ forms no diamond with $\mathcal{F}$, contradiction.

Hence $c_i = \mathbf{e}_i$ for all $i \in [n]$. For any $x \in \mathcal{F}$, if $x_i \geq 1$ and $x_j \geq 1$ for $i \neq j$, then $x \geq c_i$ and $x \geq c_j$ give $x \in C_i \cap C_j = \emptyset$, impossible. Therefore every $x \in \mathcal{F} \setminus \{\mathbf{0}\}$ is supported on a single coordinate, forcing $\mathcal{F} = \mathcal{S}_{\mathbf{0}}$.

Case 2: $m = 1$ *(using* $n \geq 3$*).* Reindexing coordinates so that $i_1 = 1$, $\mathbf{0}$ has a unique child $c_1 = r_1 \mathbf{e}_1$, so every $x \in \mathcal{F} \setminus \{\mathbf{0}\}$ satisfies $x_1 \geq r_1 \geq 1$. Consider $S_0 \coloneqq (0, t_2 - 1, \ldots, t_n - 1) \notin \mathcal{F}$. Only $\mathbf{0}$ lies below S_0 in $\mathcal{F}$, so saturation forces some $D \in \mathcal{F}$ with $D > S_0$; since $\mathbf{t} - \mathbf{1} \notin \mathcal{F}$, this forces $t_1 \geq 3$ and $D = (d_1, t_2 - 1, \ldots, t_n - 1)$ with $1 \leq d_1 \leq t_1 - 2$ (if $t_1 = 2$, then $D_1 \geq 1 = t_1 - 1$ would give $D = \mathbf{t} - \mathbf{1} \in \mathcal{F}$, contradiction).

Crucially, because D attains the maximum value $t_j - 1$ in every coordinate $j \geq 2$, *every* $y \in \mathcal{F}$ *with* $y_1 \leq d_1$ *satisfies* $y \leq D$, and hence by the Out-Tree Lemma lies on the single chain $\Gamma_D \coloneqq \{y \in \mathcal{F} : y \leq D\}$. In particular, all $n - 1$ vertices $v_2, \ldots, v_n$ lie on Γ_D (as $D_j = t_j - 1$ implies $v_j \leq D$ by (1)). Relabel $\{2, \ldots, n\}$ as $j_1, \ldots, j_{n-1}$ in their order along Γ_D:

$$\mathbf{0} < v_{j_1} < v_{j_2} < \cdots < v_{j_{n-1}} \leq D.$$

Because $n \geq 3$, we have $n - 1 \geq 2$, so $v_{j_1} < v_{j_{n-1}}$ strictly. Meanwhile, $v_1 = v(1, t_1 - 1)$ has $(v_1)_1 = t_1 - 1 > d_1$, so $v_1 \notin \Gamma_D$; also $v_{j_{n-1}} \not\leq v_1$ (else v_1 would have $(v_1)_j = t_j - 1$ for all $j \geq 2$ as well as $(v_1)_1 = t_1 - 1$, giving $v_1 = \mathbf{t} - \mathbf{1} \notin \mathcal{F}$). Thus $v_1 \parallel v_{j_{n-1}}$.

Let $p_{\min} \in \Gamma_D$ be the *lowest* vertex on Γ_D that has a child outside Γ_D (such a vertex exists since $v_1 \notin \Gamma_D$). Then every $x \in \mathcal{F}$ with $x_1 > d_1$ satisfies $x > p_{\min}$. Furthermore, $p_{\min} < v_{j_{n-1}}$ (if $p_{\min} \geq v_{j_{n-1}}$, then $v_1 > p_{\min} \geq v_{j_{n-1}}$ gives $v_1 = \mathbf{t} - \mathbf{1}$, contradiction). We consider two positions of $p_{\min}$ on Γ_D:

- *Subcase 2a:* $p_{\min} \geq v_{j_1}$. Then $(p_{\min})_{j_1} = t_{j_1} - 1 \geq 1$. Define $W \in H$ by $W_{j_1} \coloneqq 0$ and $W_q \coloneqq t_q - 1$ for all $q \neq j_1$. Any $z \in \mathcal{F}$ with $z \geq W$ has $z_1 = t_1 - 1 \implies z \geq v_1$ and $z_{j_{n-1}} = t_{j_{n-1}} - 1 \implies z \geq v_{j_{n-1}}$, impossible since $v_1 \parallel v_{j_{n-1}}$ in the out-tree $\mathcal{F}$; thus $W \notin \mathcal{F}$ and nothing in $\mathcal{F}$ lies above W. Conversely, any $y \in \mathcal{F}$ with $y \leq W$ has $y_{j_1} = 0 < (p_{\min})_{j_1}$, so $y \not\geq p_{\min}$, forcing $y \in \Gamma_D$ (a chain). This contradicts saturation at W.

- *Subcase 2b:* $p_{\min} < v_{j_1}$. Then $p_{\min} < v_{j_1} < v_{j_{n-1}}$. Let q_{first} be the first vertex strictly above $p_{\min}$ on Γ_D whose incoming cover edge increases some coordinate $\ell \in \{2, \ldots, n\}$ (so $(q_{\text{first}})_\ell > (p_{\min})_\ell$, while every vertex on Γ_D strictly below q_{first} has coordinate j at most $(p_{\min})_j$ for every $j \geq 2$). Note that $q_{\text{first}} \leq v_{j_1}$ (since v_{j_1} itself lies strictly above $p_{\min}$ on Γ_D and its incoming cover edge increases coordinate $j_1 \in \{2, \ldots, n\}$). Because $n \geq 3$, choose a coordinate $k \in \{2, \ldots, n\} \setminus \{\ell\}$; then $q_{\text{first}} \leq v_{j_1} \leq v_k$.

  Define $\mathcal{M} \coloneqq \{x \in \mathcal{F} : x_1 > d_1 \text{ and } x_j = (p_{\min})_j \text{ for all } j \notin \{1, k\}\}$. Notice that $\mathcal{M}$ is nonempty: by definition of $p_{\min}$, $p_{\min}$ has a child $z^* \in \mathcal{F} \setminus \Gamma_D$; since $z^* \notin \Gamma_D$ forces $z^*_1 > d_1 \geq (p_{\min})_1$ and the cover edge $(p_{\min}, z^*)$ changes only a single coordinate, $z^*_j = (p_{\min})_j$ for all $j \geq 2$, so $z^* \in \mathcal{M}$. Choose $M \in \mathcal{M}$ maximizing M_k. We have $M_k < t_k - 1$: otherwise $M \geq v_k \geq q_{\text{first}}$ by (1), giving $M_\ell \geq (q_{\text{first}})_\ell > (p_{\min})_\ell = M_\ell$, impossible.

  Now set $W \coloneqq \text{setCoord}(M, k, t_k - 1) \in H$. Note that $W_k = t_k - 1$ while $W_\ell = (p_{\min})_\ell < (q_{\text{first}})_\ell \leq (v_k)_\ell$, so $W \not\geq v_k$ and thus $W \notin \mathcal{F}$. Moreover, any $z \in \mathcal{F}$ with $z \geq W$ satisfies both $z > M$ (as $W > M$) and $z \geq v_k$ (by (1) since $z_k = t_k - 1$); by the Out-Tree Lemma, M and v_k would have to be comparable, impossible because $M_1 > d_1 \geq (v_k)_1$ while $(v_k)_k = t_k - 1 > M_k$. Hence nothing in $\mathcal{F}$ lies above W. On the other hand, let $y \in \mathcal{F}$ satisfy $y \leq W$:

  - If $y_1 \leq d_1$, then $y \in \Gamma_D$, and $y_\ell \leq W_\ell = (p_{\min})_\ell < (q_{\text{first}})_\ell$ forces $y < q_{\text{first}}$, whence $y_j \leq (p_{\min})_j \leq M_j$ for all $j \geq 2$ and $y_1 \leq d_1 < M_1$, i.e., $y \leq M$.
  - If $y_1 > d_1$, then $y > p_{\min}$, so $(p_{\min})_j \leq y_j \leq W_j = (p_{\min})_j$ for all $j \notin \{1, k\}$, placing $y \in \mathcal{M}$. By maximality of M_k, $y_k \leq M_k$; since also $y_1 \leq W_1 = M_1$, we have $y \leq M$.

  Thus every $y \in \mathcal{F}$ with $y \leq W$ satisfies $y \leq M$, so $\{y \in \mathcal{F} : y \leq W\}$ is a chain, contradicting saturation at W.

Case 2 is impossible, establishing part (b). The exact count $\binom{R}{t_1-1,\ldots,t_n-1} + 2$ follows immediately since maximal chains in H biject with coordinate-step words using each coordinate i exactly $t_i - 1$ times, and $\mathcal{S}_{\mathbf{0}}, \mathcal{S}_{\mathbf{t-1}}$ are non-chains for $n \geq 3$.

(c) Sharp breakdown at $n = 2$ ($\max(t_1, t_2) \geq 3$). Assume without loss of generality $t_1 \geq 3$ and $t_2 \geq 2$, so $R + 1 = t_1 + t_2 - 1$.

- *Rooted family $\mathcal{Y}_{t_1,t_2}$:* It has cardinality $(t_1 - 1) + (t_2 - 2) + 2 = t_1 + t_2 - 1$ and is an out-tree rooted at $(0, 0)$ with a single trunk to $q = (t_1 - 2, t_2 - 2)$ forking into $(t_1 - 1, t_2 - 2)$ and $(t_1 - 2, t_2 - 1)$, hence diamond-free, non-chain, and non-star. For any $(x, y) \notin \mathcal{Y}_{t_1,t_2}$: if $(x, y) = (t_1 - 1, t_2 - 1)$, then $q < \{(t_1 - 1, t_2 - 2), (t_1 - 2, t_2 - 1)\} < (x, y)$ is a diamond; if $x = t_1 - 1$ and $0 \leq y \leq t_2 - 3$ (when $t_2 \geq 3$), then $(t_1 - 2, y) < \{(t_1 - 2, y + 1), (x, y)\} < (t_1 - 1, t_2 - 2)$ is a diamond; and if $0 \leq x \leq t_1 - 3$ and $y \geq 1$, then $(x, 0) < \{(x + 1, 0), (x, y)\} < (t_1 - 2, t_2 - 1)$ is a diamond.

- *Interior family $\mathcal{I}_{t_1,t_2}$:* It has cardinality $(t_1 - 1) + (t_2 - 1) + 1 = t_1 + t_2 - 1$ and omits both $(0, 0)$ and $(t_1 - 1, t_2 - 1)$ (since $t_1 \geq 3 \implies (1, 1) \neq (t_1 - 1, t_2 - 1)$). The two chains $\{(r, 0) : r \geq 1\}$ and $\{(0, s) : s \geq 1\}$ are mutually incomparable, and $(1, 1)$ lies only above $(1, 0)$ and $(0, 1)$, which have no common lower bound in $\mathcal{I}_{t_1,t_2}$; hence $\mathcal{I}_{t_1,t_2}$ is diamond-free. For saturation: $(0, 0) < \{(1, 0), (0, 1)\} < (1, 1)$ is a diamond, and for any missing (x, y) with $x \geq 1, y \geq 1$, either $x \geq 2$ (giving $(1, 0) < \{(1, 1), (x, 0)\} < (x, y)$) or $y \geq 2$ (giving $(0, 1) < \{(1, 1), (0, y)\} < (x, y)$).

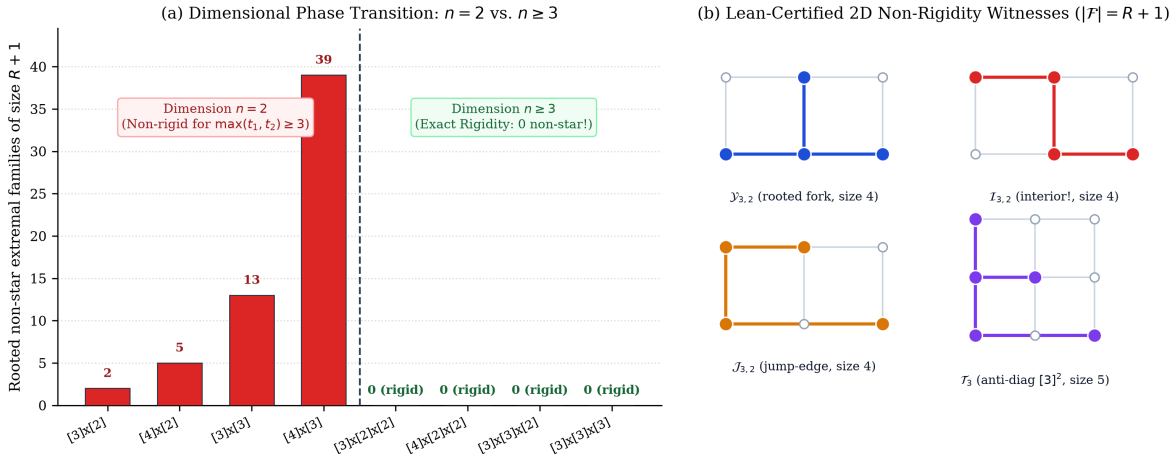

Figure 2: **Dimensional phase transition for hypergrid extremal rigidity.** (a) Exhaustive rooted jump-step out-tree census from `verify_diamond_rigidity.py`: the count of non-star rooted extremal families of size $R + 1$ grows rapidly in dimension $n = 2$ ($2, 5, 13, 39$ across $[3] \times [2]$, $[4] \times [2]$, $[3]^2$, $[4] \times [3]$) and drops abruptly to **exact zero** across all $n \geq 3$ hypergrids ($[3] \times [2]^2$, $[4] \times [2]^2$, $[3]^2 \times [2]$, $[3]^3$), confirming Theorem 1.2(b)–(c). (b) Grid geometry of four Lean 4-certified 2D non-rigidity witnesses ($\mathcal{Y}_{3,2}$, $\mathcal{I}_{3,2}$, $\mathcal{J}_{3,2}$, $\mathcal{T}_3$).

- *Anti-diagonal family $\mathcal{T}_t$ ($t_1 = t_2 = t \geq 3$):* It has cardinality $t + (t-1) = 2t - 1$, is an out-tree rooted at $(0, 0)$ (where each $(x, t-1-x)$ lies above $(0, 0), \ldots, (0, t-1-x)$ and is incomparable to all other $(x', t-1-x')$), hence diamond-free. For any missing (x, y) ($x \geq 1$): if $x + y < t - 1$, then $(0, y) < \{(x, y), (0, y+1)\} < (x, t-1-x)$ is a diamond; if $x + y > t - 1$, then $(0, 0) < \{(0, y), (x, t-1-x)\} < (x, y)$ is a diamond (note $y > t-1-x$ and $x \geq 1$ guarantee $(0, y) \parallel (x, t-1-x)$). □

3 Boolean Extremal Rigidity: Proof of Theorem 1.1

For $n = 1$, $2^{[1]} = \{\emptyset, \{1\}\}$ is the unique diamond-saturated family (simultaneously a maximal chain and both stars). For $n = 2$, no interior family exists (any $\mathcal{F} \subseteq \{\{1\}, \{2\}\}$ allows adding $\emptyset$ without creating a diamond), while any boundary-anchored family of size 3 is a maximal chain (2 chains), $\{\emptyset, \{1\}, \{2\}\}$, or $\{\{1\}, \{2\}, \{1, 2\}\}$, giving count $4 = 2! + 2$.

For $n \geq 3$, the boundary-anchored case ($\emptyset \in \mathcal{F}$ or $[n] \in \mathcal{F}$) is already completely proved by Theorem 1.2(b) with $t_1 = \cdots = t_n = 2$. It remains to prove **Pillar B**: every interior diamond-saturated family $\mathcal{F} \subseteq 2^{[n]}$ ($\emptyset, [n] \notin \mathcal{F}$, $n \geq 3$) satisfies $|\mathcal{F}| \geq n + 2$.

We build on the layer decomposition of Ivan and Jaffe [8, Sections 2–3], proving the separation and boundary-witness properties directly below so that Pillar B relies externally only on their three general hypercube-layer bounds (Proposition 3.1), and strengthening their Section 4 counting arguments at all three critical bottlenecks. For an interior diamond-saturated family $\mathcal{F} \subseteq 2^{[n]}$ ($\emptyset, [n] \notin \mathcal{F}$):

- $\mathcal{A}$ and $\mathcal{X}$ denote the families of minimal and maximal elements of $\mathcal{F}$, respectively, with $m_{\mathcal{A}} := \max_{A \in \mathcal{A}} |A|$, $W := \{i \in [n] : i \notin \bigcup_{A \in \mathcal{A}} A\}$, and $\overline{W} := \bigcap_{X \in \mathcal{X}} X$.

- $\mathcal{B}$ is the set of inclusion-maximal sets $B \in 2^{[n]} \setminus \mathcal{F}$ that contain no element of $\mathcal{F}$ (i.e., $A \not\subseteq B$ for all $A \in \mathcal{A}$, so by saturation every $B \in \mathcal{B}$ must be the bottom element of an induced diamond with three elements in $\mathcal{F}$); $G(\mathcal{B}) \subseteq \mathcal{F}$ is the set of middle elements of diamonds whose bottom lies in $\mathcal{B}$. Dually, $\mathcal{Y}$ is the set of inclusion-minimal sets $Y \in 2^{[n]} \setminus \mathcal{F}$ contained in no element of $\mathcal{F}$ (i.e., $Y \not\subseteq X$ for all $X \in \mathcal{X}$), and $H(\mathcal{Y}) \subseteq \mathcal{F}$ is the set of

middle elements of diamonds whose top lies in $\mathcal{Y}$. (Note that $\mathcal{B}$ contains the subfamily $\mathcal{B}_{\mathrm{IJ}} \subseteq \mathcal{B}$ of [8], which additionally excludes subsets of minimal elements: any strictly larger $\mathcal{F}$-avoiding set also avoids being contained in a minimal element, so maximality within $\mathcal{B}_{\mathrm{IJ}}$ implies maximality within $\mathcal{B}$, and dually $\mathcal{Y}_{\mathrm{IJ}} \subseteq \mathcal{Y}$. Thus $G(\mathcal{B}_{\mathrm{IJ}}) \subseteq G(\mathcal{B})$ and $H(\mathcal{Y}_{\mathrm{IJ}}) \subseteq H(\mathcal{Y})$, so all lower bounds in Proposition 3.1 hold a fortiori while preserving all immediate properties and Lemma 3.5.)

- *Immediate self-contained properties:* (1) $\mathcal{A} \cap H(\mathcal{Y}) = \emptyset$ and $\mathcal{X} \cap G(\mathcal{B}) = \emptyset$ (every $M \in G(\mathcal{B})$ is a middle vertex of a diamond $B < \{M, M'\} < D$ with $D \in \mathcal{F}$, so $M < D \implies M \notin \mathcal{X}$; dually for $H(\mathcal{Y})$; cf. [8, Proposition 19]); (2) for every $B \in \mathcal{B}$ and every $i \notin B$, maximality of B forces $B \cup \{i\}$ to contain some $A_i \in \mathcal{A}$ with $i \in A_i$ (cf. [8, Lemma 8]), and dually for every $Y \in \mathcal{Y}$ and $j \in Y$ there exists $X_j \in \mathcal{X}$ with $Y \setminus \{j\} \subseteq X_j$ and $j \notin X_j$; (3) whenever $W \neq \emptyset$, there automatically exist incomparable $A \in \mathcal{A}$ and $X_0 \in \mathcal{X}$: indeed, if every $A \in \mathcal{A}$ were contained in every $X \in \mathcal{X}$, then $U := \bigcup_{A \in \mathcal{A}} A \subseteq X$ for all $X \in \mathcal{X}$, so no $F \in \mathcal{F}$ could satisfy $F \subseteq W = [n] \setminus U$ (as every $F \in \mathcal{F}$ contains some nonempty $A \in \mathcal{A} \subseteq U$) and no $F \in \mathcal{F}$ could satisfy $W \subseteq F$ (as $F \subseteq X \in \mathcal{X}$ would force $X \supseteq U \cup W = [n] \notin \mathcal{F}$), leaving $W \notin \mathcal{F}$ completely isolated in $\mathcal{F} \cup \{W\}$, contradicting saturation.

The sole external input used in Pillar B is the following trio of general hypercube-layer bounds from Ivan and Jaffe [8], which hold for every interior diamond-saturated family $\mathcal{F} \subseteq 2^{[n]}$ with no upper bound on $|\mathcal{F}|$:

Proposition 3.1 (Hypercube-Layer Bounds, Ivan and Jaffe [8])**.** *Let $\mathcal{F} \subseteq 2^{[n]}$ be an interior diamond-saturated family. Then:*

(a) $|\mathcal{A} \cup G(\mathcal{B})| \geq n + 1 - m_{\mathcal{A}} - |W|$ *and dually* $|\mathcal{X} \cup H(\mathcal{Y})| \geq n + 1 - \max_{X \in \mathcal{X}}(n - |X|) - |\overline{W}|$ [8, Corollary 15].

(b) *If $W \neq \emptyset$ and $\mathcal{F}_1 := \{M \in G(\mathcal{B}) : W \subseteq M\}$, then $|\mathcal{F}_1 \cup \mathcal{A}| \geq n + 1 - |W|$* [8, Corollary 17], *and for every $A \in \mathcal{A}$ and every $i \in W$ there exist $X_i, X_i \cup \{i\} \in \mathcal{F}$ with $i \notin X_i$ and $A \subseteq X_i$* (from the upward-step proof of [8, Lemma 6] as applied in [8, Section 4], where $i \notin X_i$ holds by construction).

We now establish the five self-contained lemmas needed to push $|\mathcal{F}| \geq n + 1$ to $|\mathcal{F}| \geq n + 2$, starting with a short direct proof that $\mathcal{A}$ and $\mathcal{X}$ are disjoint (strengthening [8, Proposition 18]).

Lemma 3.2 (Minimal–Maximal Separation)**.** *If $\mathcal{F} \subseteq 2^{[n]}$ is an interior diamond-saturated family containing an element $E \in \mathcal{A} \cap \mathcal{X}$, then $|\mathcal{F}| \geq n + 1 + \lceil n/2 \rceil$. In particular, whenever $|\mathcal{F}| \leq n + 1$, we have $\mathcal{A} \cap \mathcal{X} = \emptyset$.*

Proof. Suppose $E \in \mathcal{A} \cap \mathcal{X}$; since $\emptyset, [n] \notin \mathcal{F}$, we have $1 \leq |E| \leq n - 1$. For each $i \in E$, the set $E \setminus \{i\} \notin \mathcal{F}$ has no element of $\mathcal{F}$ below it (as $E \in \mathcal{A}$), so saturation gives a diamond $E \setminus \{i\} < \{P_i, C_i\} < D_i$ in $\mathcal{F}$. At least one middle vertex, say P_i, omits i (else $E \subseteq P_i, C_i$ gives the diamond $E < \{P_i, C_i\} < D_i$ in $\mathcal{F}$), so $P_i \cap E = E \setminus \{i\}$ and $|P_i| \geq |E|$. Because $E \in \mathcal{X}$ is maximal and $D_i \neq E$, we cannot have $E \subseteq D_i$, so $D_i \cap E = E \setminus \{i\}$ and $|D_i| \geq |P_i| + 1 \geq |E| + 1$. Set $S_1 := \{P_i : i \in E\}$ ($|S_1| = |E|$) and $T_1 := \{D_i : i \in E\}$ ($|T_1| = |E|$, since $D_i \cap E = E \setminus \{i\}$ uniquely identifies i). Note $T_1 \cap S_1 = \emptyset$ (if $D_i = P_{i'}$ then $E \setminus \{i\} = E \setminus \{i'\}$, forcing $i = i'$ and contradicting $P_i \subsetneq D_i$). Dually, for each $j \notin E$, $E \cup \{j\}$ yields a diamond $L_j < \{Q_j, R_j\} < E \cup \{j\}$ with $L_j, Q_j, R_j \in \mathcal{F}$, $Q_j \cup E = L_j \cup E = E \cup \{j\}$, $|Q_j| \leq |E|$, and $|L_j| \leq |E| - 1$, giving $S_2 := \{Q_j : j \notin E\}$ ($|S_2| = n - |E|$) and $T_2 := \{L_j : j \notin E\}$ ($|T_2| = n - |E|$) with $T_2 \cap S_2 = \emptyset$. We verify that $\{E\}, T_1, T_2, S_1 \cup S_2$ are pairwise disjoint: every element of T_1 has size $\geq |E| + 1$ (hence is disjoint from $\{E\} \cup S_2 \cup T_2$) and $T_1 \cap S_1 = \emptyset$ by the index-uniqueness check above; dually, every element of T_2 has size $\leq |E| - 1$ (hence is disjoint from $\{E\} \cup S_1 \cup T_1$) and $T_2 \cap S_2 = \emptyset$; and $E \notin S_1 \cup S_2$ since $P_i \cap E = E \setminus \{i\} \neq E$ and $Q_j \cup E = E \cup \{j\} \neq E$. Consequently,

$$|\mathcal{F}| \;\geq\; 1 + |T_1| + |T_2| + |S_1 \cup S_2| \;\geq\; 1 + |E| + (n - |E|) + \max(|E|, n - |E|) \;\geq\; n + 1 + \lceil n/2 \rceil. \;\square$$

Lemma 3.3 (Non-Unique Extrema)**.** *If $\mathcal{F} \subseteq 2^{[n]}$ is diamond-saturated with $[n] \notin \mathcal{F}$ (resp. $\emptyset \notin \mathcal{F}$), then $|\mathcal{X}| \geq 2$ (resp. $|\mathcal{A}| \geq 2$).*

Proof. First, $\mathcal{F}$ cannot be empty: if $\mathcal{F} = \emptyset$ then $[n] \notin \mathcal{F}$ could be added without forming a diamond, contradicting saturation; hence $|\mathcal{X}| \geq 1$. Suppose for contradiction that $\mathcal{X} = \{X\}$ is a unique maximal element of $\mathcal{F}$. Since $X \neq [n]$, pick $j \in [n] \setminus X$ and set $s \coloneqq X \cup \{j\} \notin \mathcal{F}$. Every element $y \in \mathcal{F}$ satisfies $y \subseteq X \subsetneq s$, so nothing in $\mathcal{F}$ lies above s. By saturation, s must be the top of a diamond $a < \{b, c\} < s$ with $a, b, c \in \mathcal{F}$. Since $b, c \subseteq X$ and $b \parallel c$, neither b nor c equals X, so $a < \{b, c\} < X$ is already an induced diamond in $\mathcal{F}$, a contradiction. The statement $|\mathcal{A}| \geq 2$ follows by complementation. (Formally verified in Lean 4 by `no_unique_maximal_below_missing`.) □

Lemma 3.4 (Strict Cardinality Separation)**.** *Let $\mathcal{F} \subseteq 2^{[n]}$ be an interior diamond-saturated family. For every minimal element $A \in \mathcal{A}$, $|\{F \in \mathcal{F} : |F| \geq |A|\}| \geq |A| + 1$. Dually, for every maximal element $X \in \mathcal{X}$, $|\{F \in \mathcal{F} : |F| \leq |X|\}| \geq n - |X| + 1$.*

Proof. Fix $A \in \mathcal{A}$. For each $i \in A$, the set $A \setminus \{i\}$ does not belong to $\mathcal{F}$ and has no element of $\mathcal{F}$ below it, so $A \setminus \{i\}$ must be the bottom of a diamond $A \setminus \{i\} < \{B_i, C_i\} < D_i$ with $B_i, C_i, D_i \in \mathcal{F}$. At least one of B_i, C_i does not contain i (otherwise $A \subseteq B_i, C_i$ would give the diamond $A < \{B_i, C_i\} < D_i$ in $\mathcal{F}$); call this set $P_i \in \mathcal{F}$. Then $P_i \cap A = A \setminus \{i\}$ and $P_i \neq A \setminus \{i\}$, so $|P_i| \geq |A|$ and $P_i \neq A$. Since $P_i \cap A = A \setminus \{i\}$ uniquely identifies i, the $|A|$ sets $\{P_i : i \in A\}$ are pairwise distinct and distinct from A itself, yielding $|A| + 1$ elements of $\mathcal{F}$ of size at least $|A|$. The dual bound follows by complementation. □

Lemma 3.5 (Extended Disjointness of $G(\mathcal{B})$ and $H(\mathcal{Y})$)**.** *If $\mathcal{F} \subseteq 2^{[n]}$ is interior diamond-saturated with $|\mathcal{F}| \leq n + 1$, then $G(\mathcal{B}) \cap H(\mathcal{Y}) = \emptyset$.*

Proof. Suppose for contradiction that $P \in G(\mathcal{B}) \cap H(\mathcal{Y})$. By definition of $G(\mathcal{B})$ and $H(\mathcal{Y})$, there exist $B \in \mathcal{B}$, $Y \in \mathcal{Y}$, and $M, D, R, Q \in \mathcal{F}$ forming two diamonds $B < \{P, M\} < D$ and $Q < \{P, R\} < Y$. Since $\mathcal{F}$ is diamond-free, $P \cap M$ contains no element of $\mathcal{F}$ (else $A \subseteq P \cap M$ for some $A \in \mathcal{A}$ would yield the diamond $A < \{P, M\} < D$ in $\mathcal{F}$); as $B \subseteq P \cap M$, maximality of $B \in \mathcal{B}$ forces $B = P \cap M \in \mathcal{B}$. Dually, $P \cup R$ is contained in no element of $\mathcal{F}$, so minimality of $Y \in \mathcal{Y}$ forces $Y = P \cup R \in \mathcal{Y}$ (cf. [8, Proposition 20]). By immediate property (2) above (cf. [8, Lemma 8]), for every $i \notin P \cap M$ there exists $A_i \in \mathcal{A} \subseteq \mathcal{F}$ with $A_i \subseteq (P \cap M) \cup \{i\}$ and $i \in A_i$; choose one such A_i per $i \notin P \cap M$ and let $S_1 \coloneqq \{A_i : i \notin P \cap M\}$ ($|S_1| = n - |P \cap M|$). Dually, for every $j \in P \cup R$ choose one $U_j \in \mathcal{X} \subseteq \mathcal{F}$ with $(P \cup R) \setminus \{j\} \subseteq U_j$ and $j \notin U_j$, and let $S_2 \coloneqq \{U_j : j \in P \cup R\}$ ($|S_2| = |P \cup R|$).

Because $|P \cap M| \leq |P| - 1$ and $|P \cup R| \geq |P| + 1$, we have

$$|S_1| + |S_2| = n - |P \cap M| + |P \cup R| \;\geq\; n + 2.$$

If $|S_1 \cap S_2| = 0$, then $|\mathcal{F}| \geq |S_1 \cup S_2| \geq n + 2$, a contradiction. Otherwise let $Z \in S_1 \cap S_2$. Since every set in S_1 has size at most $|P \cap M| + 1$ and every set in S_2 has size at least $|P \cup R| - 1$,

$$|P| \leq |P \cup R| - 1 \leq |Z| \leq |P \cap M| + 1 \leq |P|,$$

forcing $|Z| = |P|$, $P \cap M = P \setminus \{b\}$ for some $b \in P$, and $P \cup R = P \cup \{a\}$ for some $a \notin P$. Moreover, $(P \cup \{a\}) \setminus \{j\} \subseteq Z \subseteq (P \setminus \{b\}) \cup \{i\}$ forces either $(i, j) = (b, a)$ with $Z = P$, or $(i, j) = (a, b)$ with $Z = (P \setminus \{b\}) \cup \{a\}$. Thus $S_1 \cap S_2 \subseteq \{P, (P \setminus \{b\}) \cup \{a\}\}$ and $|S_1 \cap S_2| \in \{1, 2\}$.

Now observe two dual exclusions:

1. *If $P \in S_2$, then $D \notin S_1 \cup S_2$:* Because $|D| > |P| = |P \cap M| + 1$, $D \notin S_1$. If $D \in S_2$ corresponded to $\ell \in P \cup \{a\}$, then $(P \cup \{a\}) \setminus \{\ell\} \subseteq D$ and $\ell \notin D$; since $P \subsetneq D$, $\ell \notin P$, forcing $\ell = a$, which contradicts $P \in S_2$ already being the unique element of S_2 chosen for index a (as $P \neq D$).

2. *If $P \in S_1$, then $Q \notin S_1 \cup S_2$:* Because $|Q| < |P| = |P \cup R| - 1$, $Q \notin S_2$. If $Q \in S_1$ corresponded to $\ell \notin P \setminus \{b\}$, then $Q \subseteq (P \setminus \{b\}) \cup \{\ell\}$ and $\ell \in Q$; since $Q \subsetneq P$, $\ell \in P$, forcing $\ell = b$, which contradicts $P \in S_1$ already being the unique element of S_1 chosen for index b (as $P \neq Q$).

Consequently:

- If $|S_1 \cap S_2| = 1$, then $|S_1 \cup S_2| \geq n + 1$, and at least one of P (if $P \notin S_1 \cup S_2$), D (if $P \in S_2$), or Q (if $P \in S_1$) lies in $\mathcal{F} \setminus (S_1 \cup S_2)$, giving $|\mathcal{F}| \geq n + 2$.
- If $|S_1 \cap S_2| = 2$, then $|S_1 \cup S_2| \geq n$ and $P \in S_1 \cap S_2$, so *both* D and Q lie in $\mathcal{F} \setminus (S_1 \cup S_2)$; since $|Q| < |P| < |D|$, $Q \neq D$, giving $|\mathcal{F}| \geq |S_1 \cup S_2 \cup \{Q, D\}| \geq n + 2$.

(The arithmetic closure is verified in Lean 4 by `pillar_b_gh_counting_core`.) In every case $|\mathcal{F}| \geq n + 2$, a contradiction. □

Proof of Theorem 1.1 (Pillar B: $|\mathcal{F}| \geq n + 2$ for interior $\mathcal{F}$, $n \geq 3$). Suppose for contradiction that $\mathcal{F} \subseteq 2^{[n]}$ ($n \geq 3$) is interior diamond-saturated with $|\mathcal{F}| \leq n+1$. By Lemma 3.2, $\mathcal{A} \cap \mathcal{X} = \emptyset$; together with $\mathcal{A} \cap H(\mathcal{Y}) = \mathcal{X} \cap G(\mathcal{B}) = \emptyset$ and Lemma 3.5, the sets $\mathcal{A} \cup G(\mathcal{B})$ and $\mathcal{X} \cup H(\mathcal{Y})$ are completely disjoint.

Case 1: $W = \overline{W} = \emptyset$. If $\max_{A \in \mathcal{A}} |A| \geq \min_{X \in \mathcal{X}} |X| + 1$, pick $A \in \mathcal{A}$ and $X \in \mathcal{X}$ achieving these extremes. By Lemma 3.4, the disjoint sets $\{F \in \mathcal{F} : |F| \geq |A|\}$ and $\{F \in \mathcal{F} : |F| \leq |X|\}$ yield

$$|\mathcal{F}| \;\geq\; (|A| + 1) + (n - |X| + 1) \;=\; n + 2 + (|A| - |X|) \;\geq\; n + 3,$$

contradiction. Otherwise $\min_{X \in \mathcal{X}} |X| \geq \max_{A \in \mathcal{A}} |A|$, and Proposition 3.1(a) immediately yields

$$\begin{aligned} |\mathcal{F}| \;&\geq\; |\mathcal{A} \cup G(\mathcal{B})| + |\mathcal{X} \cup H(\mathcal{Y})| \\ &\geq\; \big(n + 1 - \max_{A \in \mathcal{A}} |A|\big) + \big(n + 1 - \max_{X \in \mathcal{X}} (n - |X|)\big) \;=\; n + 2 + \min_{X \in \mathcal{X}} |X| - \max_{A \in \mathcal{A}} |A| \;\geq\; n + 2, \end{aligned}$$

contradiction (verified in Lean 4 by `pillar_b_case1_bound`).

Case 2: $W \neq \emptyset$ or $\overline{W} \neq \emptyset$. By complementation, assume $W \neq \emptyset$. Set $\mathcal{F}_1 := \{M \in G(\mathcal{B}) : W \subseteq M\}$. Since $W \neq \emptyset$ is contained in every $M \in \mathcal{F}_1$ whereas every $A \in \mathcal{A}$ is disjoint from W, we have $\mathcal{F}_1 \cap \mathcal{A} = \emptyset$, and Proposition 3.1(b) gives $|\mathcal{F}_1| + |\mathcal{A}| \geq n + 1 - |W|$. By immediate property (3) above, fix an incomparable pair $A \in \mathcal{A}$ and $X_0 \in \mathcal{X}$, and by Proposition 3.1(b) choose for each $i \in W$ elements $X_i, X_i \cup \{i\} \in \mathcal{F}$ with $i \notin X_i$ and $A \subseteq X_i$. Let $\mathcal{F}_2 := \{X_i, X_i \cup \{i\} : i \in W\} \subseteq \mathcal{F}$, viewed as a graph with edges $\{X_i, X_i \cup \{i\}\}$ ($i \in W$). As in [8, Section 4], this graph is a forest with $|W|$ edges and some number $t \geq 1$ of connected components, so $|\mathcal{F}_2| = |W| + t$. Moreover, $|\mathcal{F}_2 \cap \mathcal{A}| \leq 1$ (only A itself can lie in $\mathcal{F}_2 \cap \mathcal{A}$), and each connected component of $\mathcal{F}_2$ contains at most one vertex of $\mathcal{F}_1$ (since toggling elements of W one by one cannot move from one superset of W to a distinct superset of W).

Let $c_{\text{hit}} \leq t$ be the number of connected components of $\mathcal{F}_2$ that intersect $\mathcal{F}_1$. Then

$$|\mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{A}| \;\geq\; |\mathcal{F}_1| + |\mathcal{A}| + (|W| + t) - c_{\text{hit}} - |\mathcal{F}_2 \cap \mathcal{A}| \;\geq\; n + (t - c_{\text{hit}}).$$

- If $c_{\text{hit}} < t$, then $|\mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{A}| \geq n + 1$. Since $X_0 \in \mathcal{X}$ is incomparable to A (so $X_0 \notin \mathcal{F}_2$), $X_0 \notin \mathcal{A}$, and $X_0 \notin \mathcal{F}_1 \subseteq G(\mathcal{B})$, adding X_0 gives $|\mathcal{F}| \geq n + 2$.
- If $c_{\text{hit}} = t$, then *every* connected component C of $\mathcal{F}_2$ contains a vertex $M_C \in \mathcal{F}_1 \subseteq G(\mathcal{B})$ with $W \subseteq M_C$. Because every edge in C toggles only an element of W, every vertex $Y \in C$ satisfies $Y \setminus W = M_C \setminus W$, and therefore

  $$Y = (Y \setminus W) \cup (Y \cap W) = (M_C \setminus W) \cup (Y \cap W) \;\subseteq\; (M_C \setminus W) \cup W = M_C.$$

  Since $M_C \in G(\mathcal{B})$ is a middle vertex of a diamond formed upon inserting some bottom $B_0 \in \mathcal{B}$ into $\mathcal{F}$, M_C is strictly contained in the top element $X' \in \mathcal{F}$ of that diamond,

and thus *no* vertex $Y \in \mathcal{F}_2$ can be maximal in $\mathcal{F}$. That is, $\mathcal{X} \cap \mathcal{F}_2 = \emptyset$. Combined with $\mathcal{X} \cap \mathcal{F}_1 = \emptyset$ and $\mathcal{X} \cap \mathcal{A} = \emptyset$, the entire maximal layer $\mathcal{X}$ is disjoint from $\mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{A}$. By Lemma 3.3, $|\mathcal{X}| \geq 2$, yielding

$$|\mathcal{F}| \;\geq\; |\mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{A}| + |\mathcal{X}| \;\geq\; n + 2.$$

(The arithmetic closure is verified in Lean 4 by `pillar_b_case2_bound`.) This contradiction proves $|\mathcal{F}| \geq n + 2$ for every interior diamond-saturated family on $n \geq 3$, completing the proof of Theorem 1.1. □

4 Sharp Singleton-Branch Stability and Birkhoff Collapse

Proof of Theorem 1.3. Let $\mathcal{F} \subseteq 2^{[n]}$ ($n \geq 3$) be diamond-saturated with $\emptyset \in \mathcal{F}$, $[n] \notin \mathcal{F}$, and $\mathcal{F} \neq \{\emptyset\} \cup \binom{[n]}{1}$. Let $I_1 := \{i \in [n] : \{i\} \in \mathcal{F}\}$ ($|I_1| = r_1 \geq 2$) be the singleton children of $\emptyset$ in the out-tree $\mathcal{F}$. Note $r_1 \leq n - 1$: if $r_1 = n$, then $\mathcal{F}$ contains the bottom star $\{\emptyset\} \cup \binom{[n]}{1}$, which is already diamond-saturated, forcing $\mathcal{F} = \{\emptyset\} \cup \binom{[n]}{1}$.

For each $a \in I_1$, let $C_a := \{F \in \mathcal{F} : \{a\} \subseteq F\}$ be the subtree rooted at $\{a\}$; the subtrees $\{C_a : a \in I_1\}$ are pairwise vertex-disjoint. Fix any $a \in I_1$ and any coordinate $x \in [n] \setminus I_1$. We claim that C_a contains a single-step cover edge $(B, B \cup \{x\})$ with $x \notin B$.

Indeed, consider $S := \{a, x\} \in 2^{[n]}$. If $S \in \mathcal{F}$, then $(\{a\}, \{a, x\})$ is the desired cover edge in C_a. Suppose $S \notin \mathcal{F}$. What lies below S in $\mathcal{F}$? Because $x \notin I_1$, $\{x\} \notin \mathcal{F}$, so the only elements of $\mathcal{F}$ strictly below $\{a, x\}$ are $\emptyset$ and $\{a\}$ (a chain). Hence S cannot be the top of a diamond, nor the bottom (since $\emptyset \in \mathcal{F}$), so saturation forces S to be a middle element of a diamond with some top $D_0 \in \mathcal{F}$; since $\{a\} \subsetneq S \subsetneq D_0$, we have $D_0 \in C_a$. Now, among all pairs $(B, D) \in C_a^2$ with $B \subsetneq D$, $x \notin B$, and $x \in D$ (nonempty since $(\{a\}, D_0)$ is one), choose one maximizing $|B|$, and set $T := B \cup \{x\}$. Clearly $B \subsetneq T \subseteq D$. If $T \in \mathcal{F}$, then $(B, B \cup \{x\})$ lies in C_a. Otherwise $T \notin \mathcal{F}$ satisfies $T \subsetneq D$, so by the Rooted Middle-Placement Lemma, saturation at T yields $B', D' \in \mathcal{F}$ with $T \subsetneq D'$, $B' \subsetneq D'$, and $T \parallel B'$. Since $\{a\} \subseteq B \subsetneq T \subsetneq D'$, we have $D' \in C_a$; by the Out-Tree Lemma, B and B' are comparable, forcing $\{a\} \subseteq B \subsetneq B' \subsetneq D'$ (so $B' \in C_a$) with $x \notin B'$ (else $T = B \cup \{x\} \subseteq B'$) and $x \in D'$, contradicting maximality of $|B|$.

By Lemma 2.3 (applied to $2^{[n]}$), the $n - r_1$ single-step targets $B \cup \{x\}$ ($x \in [n] \setminus I_1$) are pairwise distinct vertices of $C_a \setminus \{\{a\}\}$, so $|C_a| \geq 1 + (n - r_1)$ for every $a \in I_1$. Summing over the r_1 disjoint subtrees C_a plus $\emptyset$ yields

$$|\mathcal{F}| \;\geq\; 1 + \sum_{a \in I_1} |C_a| \;\geq\; 1 + r_1(n - r_1 + 1) \;\geq\; 2n - 1,$$

where the last inequality is the identity $r_1(n - r_1 + 1) - (2n - 2) = (r_1 - 2)(n - 1 - r_1) \geq 0$ for $2 \leq r_1 \leq n - 1$ (verified in Lean 4 by `singleton_branch_quadratic_bound`).

For sharpness across all $n \geq 3$:

- *Double-star $\mathcal{R}_n$ ($r_1 = n - 1$, size $2n - 1$):* Every $\{i, n\}$ ($1 \leq i \leq n - 1$) has only $\emptyset < \{i\}$ below it, so $\mathcal{R}_n$ is an out-tree (diamond-free). For $S \notin \mathcal{R}_n$: if $S = \{n\}$, then $\emptyset < \{\{1\}, \{n\}\} < \{1, n\}$; if $|S| \geq 2$ and $n \notin S$, pick $i \neq j \in S$ giving $\emptyset < \{\{i\}, \{j\}\} < S$; if $|S| \geq 3$ and $n \in S$, pick $i \neq j \in S \setminus \{n\}$ giving $\emptyset < \{\{i, n\}, \{j, n\}\} < S$.

- *Double-chain $\mathcal{C}_n$ ($r_1 = 2$, size $2n - 1$):* Write $U_k := \{1, \ldots, k\}$ and $V_k := \{n\} \cup \{2, \ldots, k\}$ ($1 \leq k \leq n - 1$, with $V_1 = \{n\}$). Every U_k contains 1 and lacks n, while every V_k contains n and lacks 1, so $\mathcal{C}_n$ consists of $\emptyset$ below two mutually incomparable chains (diamond-free). For $S \notin \mathcal{C}_n$: if $1, n \in S$, then $\emptyset < \{\{1\}, \{n\}\} < S$; if $1 \in S, n \notin S$, let $k = \max S \geq 2$ (so $S \subsetneq U_k$ since $S \neq U_k$) and pick the smallest $j \in \{2, \ldots, k\} \setminus S$, giving the diamond $U_{j-1} < \{U_j, S\} < U_k$ (as $j \notin S$ and $k \in S \setminus U_j$ force $S \parallel U_j$); if $n \in S, 1 \notin S$, let

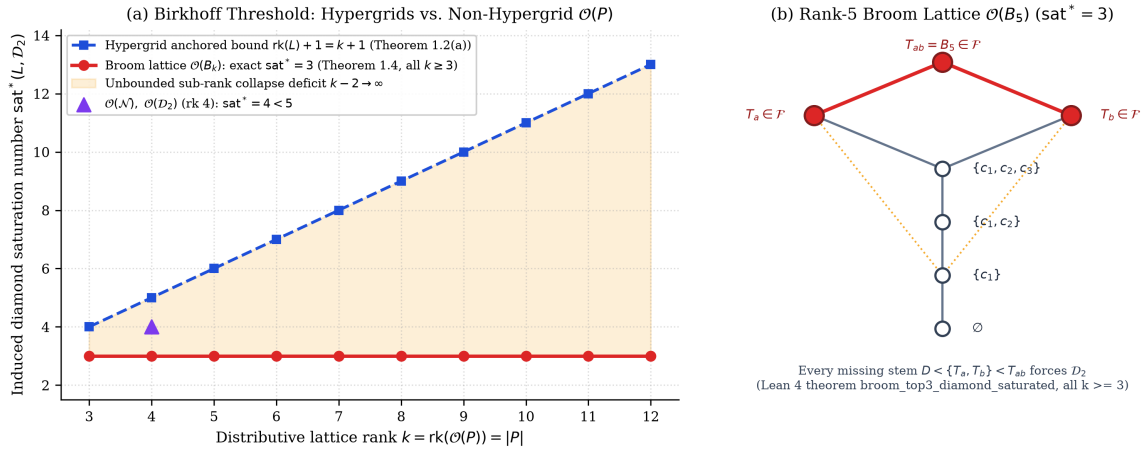

Figure 3: **Birkhoff sub-rank collapse in non-hypergrid distributive lattices.** (a) Comparison between the hypergrid anchored lower bound $\mathrm{rk}(L) + 1 = k + 1$ (Theorem 1.2(a)) and the exact constant saturation number $\mathrm{sat}^*(\mathcal{O}(B_k), \mathcal{D}_2) = 3$ for the rank-k broom distributive lattices ($k = 3, \ldots, 12$, Theorem 1.4), together with small non-hypergrid distributive lattices $\mathcal{O}(\mathcal{N}), \mathcal{O}(\mathcal{D}_2)$ ($\mathrm{sat}^* = 4 < 5$). (b) Hasse diagram of $\mathcal{O}(B_5)$ illustrating how every stem downset forms an induced diamond beneath the 3-element top-anchored family $\{T_a, T_b, T_{ab}\}$.

$k = \max(S \setminus \{n\}) \geq 2$ and pick the smallest $j \in \{2, \ldots, k\} \setminus S$, symmetrically giving $V_{j-1} < \{V_j, S\} < V_k$; if $1, n \notin S$, then $S \neq \emptyset$ satisfies $S \subsetneq U_{n-1}$ and $S \parallel \{1\}$, giving $\emptyset < \{S, \{1\}\} < U_{n-1}$. □

Proof of Theorem 1.4. Let $k \geq 3$ and $L_k = \mathcal{O}(B_k)$. Every downset of the broom poset B_k is either one of the $k-1$ chain downsets $D_j := \{c_1, \ldots, c_j\}$ ($0 \leq j \leq k-2$, with $D_0 = \emptyset$) or one of the three top downsets T_a, T_b, T_{ab}; hence $|L_k| = k + 2$ and $\mathrm{rk}(L_k) = |B_k| = k$. Because L_k contains an induced diamond (e.g., $\emptyset < \{T_a, T_b\} < T_{ab}$) and adding a single element to any family of size at most 2 yields at most $3 < 4$ elements, we immediately have $\mathrm{sat}^*(L_k, \mathcal{D}_2) \geq 3$. Conversely, $\mathcal{F} := \{T_a, T_b, T_{ab}\}$ has height 2 (so it cannot contain a 3-chain $x < y < z$, hence is diamond-free), contains the top element T_{ab}, and for every missing element $D_j \in L_k \setminus \mathcal{F}$ ($0 \leq j \leq k-2$) we have $D_j < \{T_a, T_b\} < T_{ab}$ with $T_a \parallel T_b$, forming an induced diamond. Thus $\mathrm{sat}^*(L_k, \mathcal{D}_2) = 3$. (Formally verified for all $k \geq 3$ in Lean 4 by `broom_top3_diamond_saturated`.) □

5 Formal Verification in Lean 4.36.0 and Exhaustive Computational Census

5.1 Lean 4.36.0 Constructive Formalization

Core structural, hypergrid, counting, and infinite-collapse theorems are formalized in the standalone, Mathlib-free Lean 4.36.0 module `DiamondRigidityCore.lean` (828 lines, 35,995 bytes, SHA-256 `c3114bc9e5cf473e8b5d83d672faae1bbd66c435a47e2a09c4e11a474f299e62`), which compiles standalone with zero warnings under `lean -DwarningAsError=true` and zero external dependencies. Exhaustive `#print axioms` auditing confirms that all 35/35 public theorems are strictly constructive (16/35 zero-axiom, 19/35 depending only on `[propext, Quot.sound]`), with **zero** `Classical.choice` and **zero** `sorry`. Formalized results include:

- `not_incomp_below_of_bot_mem`, `not_incomp_of_bot_top_mem`, `no_unique_maximal_below_missing` (the core unique-maximum obstruction underlying Lemma 3.3), and `rooted_missing_below_must_be_middle`;

Table 1: **Exhaustive hypergrid extremal census from `verify_diamond_rigidity.py`.** The fourth column reports the exact boundary-anchored saturation number $\mathrm{sat}^*_{\mathrm{anch}}(H, \mathcal{D}_2) = R + 1$ (Theorem 1.2(a)); for every row with a non-dashed fifth column, full exhaustive search over all subsets of size $\leq R + 1$ confirms unconditional $\mathrm{sat}^*(H, \mathcal{D}_2) = R + 1$ as well, whereas for $[4] \times [3]$ we have $\mathrm{sat}^* \leq 6$ and for the three larger 3D grids unconditional sat^* remains open (Section 6, Problem 1). The sixth column reports the count of rooted non-chain, non-star extremal families of size $R + 1$ omitting $\mathbf{t} - \mathbf{1}$: obtained from the full exhaustive census for $[2]^2$ and $[2]^3$ (`anchored_nonstandard = 0`) and from exhaustive enumeration of all single-coordinate jump-step rooted out-trees (Lemmas 2.1–2.3) for the remaining eight grids.

Hypergrid H	Dim n	Rank R	$\mathrm{sat}^*_{\mathrm{anch}}$	Full Extremal (Ch/St/Non-std/Int)	Rooted Non-Star	Rigid?
$[2]^2$	2	2	3	4 (2 / 2 / 0 / 0)	0	Yes
$[3] \times [2]$	2	3	4	10 (3 / 2 / 4 / 1)	2	**No**
$[4] \times [2]$	2	4	5	20 (4 / 2 / 10 / 4)	5	**No**
$[3]^2$	2	4	5	43 (6 / 2 / 26 / 9)	13	**No**
$[5] \times [2]$	2	5	6	35 (5 / 2 / 18 / 10)	—	**No**
$[4] \times [3]$	2	5	6	—	39	**No**
$[2]^3$	3	3	4	8 (6 / 2 / 0 / 0)	0	**Yes**
$[3] \times [2]^2$	3	4	5	14 (12 / 2 / 0 / 0)	0	**Yes**
$[4] \times [2]^2$	3	5	6	—	0	**Yes**
$[3]^2 \times [2]$	3	5	6	—	0	**Yes**
$[3]^3$	3	6	7	—	0	**Yes**

Table 2: **Exhaustive Boolean census ($2^{[1]}$–$2^{[4]}$) and Birkhoff distributive lattice collapse.** All numbers are reproduced deterministically by `verify_diamond_rigidity.py` to `data/verification_summary.json`.

Boolean $2^{[n]}$	sat^*	Ext ($n! + 2$)	Rooted 2nd	Int min	Lattice $\mathcal{O}(P)$	$\lvert L\rvert$	rk +1	$\mathrm{sat}^*_{\mathrm{all}}$	$\mathrm{sat}^*_{\mathrm{anch}}$
$2^{[1]}$	2	1 (0 non-std)	—	—	$\mathcal{O}(\mathcal{V}_2)$	5	4	3	3
$2^{[2]}$	3	4 (0 non-std)	—	—	$\mathcal{O}(\Lambda_2)$	5	4	3	3
$2^{[3]}$	4	8 (0 non-std)	5 (6 fam.)	6 (1 fam.)	$\mathcal{O}(\mathcal{N})$	8	5	4	4
$2^{[4]}$	5	26 (0 non-std)	7 (139 fam.)	8 (16 fam.)	$\mathcal{O}(\mathcal{D}_2)$	6	5	4	4
					$\mathcal{O}(B_k)$ $(k = 3..12)$	$k + 2$	$k + 1$	**3**	**3**

- `jump_step_from_crossing`, `jump_step_exists` (full general-dimension, general-mixed-hypergrid proof of Lemma 2.1), and `jump_step_target_injective` (full proof of Lemma 2.3, establishing the injection behind Theorem 1.2(a));
- `pillar_b_gh_counting_core`, `pillar_b_case1_bound`, `pillar_b_case2_bound`, and `singleton_branch_quadratic_bound` (exact arithmetic closure cores of Lemmas 3.5, Theorem 1.1 Pillar B, and Theorem 1.3);
- `broom_top3_diamond_saturated` (full proof of the upper bound $\mathrm{sat}^*(\mathcal{O}(B_k), \mathcal{D}_2) \leq 3$ in Theorem 1.4 for all $k \geq 3$);
- Five zero-axiom `rfl`-certified 2D non-rigidity witnesses: `y32_rooted_nonstar_saturated`, `i32_interior_extremal_saturated`, `j32_jump_edge_extremal_saturated`, `t3_antidiag_extremal_saturated`, and `i33_interior_extremal_saturated`.

The higher-level case analyses of Theorem 1.2(b), the set-theoretic construction of $S_1, S_2, \mathcal{F}_1, \mathcal{F}_2$ in Theorem 1.1 Pillar B (building on [8]), and the general-n verification of $\mathcal{R}_n, \mathcal{C}_n$ in Theorem 1.3 are proved in Sections 2–4 and cross-checked computationally in Python.

5.2 Pure-Stdlib Zero-Float Computational Census

The deterministic pure-standard-library Python verifier `verify_diamond_rigidity.py` writes all results in Tables 1–2 and Figures 1–3 to `data/verification_summary.json`, certifying: (i) exhaustive Boolean censuses for $n \in \{1,2,3,4\}$ (exact extremal counts $1,4,8,26$, zero non-standard extremal families, exact second-minimum rooted non-star sizes 5 and 7, and exact minimum interior sizes 6 and 8); (ii) full exhaustive hypergrid censuses on 7 grids ($[2]^2, [3]\times[2], [4]\times[2], [3]^2, [5]\times[2], [2]^3, [3]\times[2]^2$) plus exhaustive rooted jump-step out-tree censuses on 8 grids up to $[3]^3$; (iii) direct verification of $\mathcal{Y}_{t_1,t_2}$, $\mathcal{I}_{t_1,t_2}$, and $\mathcal{T}_t$ across all 20 pairs $3 \le t_1 \le 6$, $2 \le t_2 \le 6$; (iv) direct verification of $\mathcal{R}_n$ and $\mathcal{C}_n$ for $3 \le n \le 7$; (v) exhaustive Birkhoff collapse on 5 small non-hypergrid lattices and exact $\mathrm{sat}^*(\mathcal{O}(B_k), \mathcal{D}_2) = 3$ for $3 \le k \le 12$; and (vi) direct verification of Lemmas 3.3–3.4 on all 16 minimum interior families of $2^{[4]}$ and 40 interior families of $2^{[5]}$.

6 Discussion and Open Directions

Our results settle extremal uniqueness for the diamond in $2^{[n]}$ and identify both the dimensional threshold ($n = 2$ versus $n \ge 3$) and the lattice-theoretic threshold (hypergrids versus general Birkhoff lattices) governing diamond saturation. Three natural problems remain open:

1. **Unanchored hypergrid saturation for $n \ge 3$:** Theorem 1.2(a)–(b) proves $\mathrm{sat}^*_{\mathrm{anch}}(\prod_{i=1}^n [t_i], \mathcal{D}_2) = R+1$ with exact rigidity for $n \ge 3$, and our exhaustive census on $[3]\times[2]^2$ confirms $\mathrm{sat}^*([3]\times[2]^2, \mathcal{D}_2) = 5$ with zero interior extremal families. Determining whether $\mathrm{sat}^*(\prod_{i=1}^n [t_i], \mathcal{D}_2) = R+1$ unconditionally (including interior families) for all $n \ge 3$ and $t_i \ge 2$ remains open; note that extending Ivan and Jaffe's Boolean interior argument to $t_i \ge 3$ must navigate the loss of consecutive-level biregularity noted in [3].

2. **Second-minimum rooted stability for $r_1 \in \{0,1\}$:** Theorem 1.3 proves $|\mathcal{F}| \ge 2n-1$ whenever a non-star rooted diamond-saturated family $\mathcal{F} \subseteq 2^{[n]}$ has $r_1 \ge 2$ singletons, and our exhaustive census of $2^{[4]}$ shows $|\mathcal{F}| \ge 7 = 2n-1$ across all $r_1 \in \{0,1,2,3\}$ (with $r_1 = 0$ contributing 3 families and $r_1 = 1$ contributing 84). Resolving whether $|\mathcal{F}| \ge 2n-1$ holds unconditionally for $r_1 \in \{0,1\}$ and all $n \ge 5$, or whether $r_1 = 1$ with large non-singleton children admits families of size cn with $c < 2$, is an inviting stability question.

3. **Linear interior lower bound in $2^{[n]}$:** Theorem 1.1(ii) proves $|\mathcal{F}| \ge n+2$ whenever $\emptyset, [n] \notin \mathcal{F}$ ($n \ge 3$), while the second half of Ivan and Jaffe's conjecture [8] asks for $|\mathcal{F}| \ge 2n - c$ (matching our exhaustive minima $6 = 2(3)$ in $2^{[3]}$ and $8 = 2(4)$ in $2^{[4]}$).

AI Assistance and Reproducibility Disclosure

This research, including proof discovery, Lean 4.36.0 formalization, zero-float Python verification, and manuscript preparation, was developed using Google Antigravity autonomous research tooling under the direction of Lluis Eriksson. All mathematical claims are backed by complete human-readable proofs, standalone Lean 4.36.0 kernel verification, and deterministic pure-standard-library Python scripts archived with the AIRR record.

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