# Exact-$\mathbb{Q}$ Finite-Setting Steering Gap Laws, Rational Polytope Certificates, and Machine-Checked Two-Copy Superactivation of PVM-Critical Two-Qubit States

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November 2026

**Abstract**

Einstein–Podolsky–Rosen (EPR) steering of two-qubit states exhibits sharp critical behavior at the projective-measurement (PVM) unsteerability boundary $v_c = 1/2$ of the Werner family $\rho_v = v|\Psi^-\rangle\langle\Psi^-| + (1-v)I/4$. Very recently, Zhang and Chen (`arXiv:2611.09321`, November 2026) proved that every PVM-critical entangled two-qubit state is superactivated by two copies with separable conditional states on Bob's side, while every fixed finite bound $m$ on the number of measurement settings leaves an unsteerable neighborhood around criticality. Here we turn both phenomena into exact, closed-form rational theorems over $\mathbb{Q}$, backed by a zero-dependency Lean 4 formalization and a pure-stdlib zero-floating-point-trust certificate verifier (`EPR-STEER-CERT-1.0`). First, by combining exact great-circle cell counting ($R(m) \le m(m-1)+2$), a spherical-cap centroid rearrangement inequality ($\|\mathbf{m}_r\|_2 \le \alpha_r(1-\alpha_r)$), and Cauchy–Schwarz, we prove the universal non-asymptotic finite-setting unsteerability law $v_m^* \ge \frac{1}{2} + \frac{1}{2(m^2-m+1)}$ valid for *every* $m \ge 1$ and *every* choice of $m$ projective measurements on $S^2$—matching the exact $v_1^* = 1$ at $m = 1$, coming within 1.1% of optimal at $m = 3$ (4/7 vs. $1/\sqrt{3}$), and establishing the quantitative $\Omega(m^{-2})$ gap law with zero hidden constants. Second, via squared-rational primal–dual steering polytope certificates across $m \in \{2, 3, 4, 6, 7, 12, 15\}$, we prove that the rational cuboctahedron ($m = 6$, exact threshold $v_6^{\mathrm{cub}} = \sqrt{5/18} \approx 0.5270$, bracketed over $\mathbb{Q}$ by a 24-state LHS model at $v = 21/40 = 0.525$ and a witness at $v = 53/100 = 0.530$) strictly outperforms the Platonic icosahedron (exact threshold $v_6^{\mathrm{ico}} = (1+\sqrt{5})/6 \approx 0.5393$, $(v_6^{\mathrm{ico}})^2 = (3+\sqrt{5})/18$) in exact 6-setting steering threshold. Third, for two copies $\rho_v^{\otimes 2}$ under partially entangled Bell measurements $|\psi_j(t)\rangle$, we derive exact closed-form partial-transpose eigenvalues $\lambda_{\min}^{\mathrm{PT}} = (1 - v^2(1+2s))/4$ (proving strict conditional separability $\lambda_{\min}^{\mathrm{PT}} \ge 1/16 > 0$ at $v = 1/2$ for all $s \in [0,1]$), and construct an unconditional, zero-float exact-$\mathbb{Q}$ proof of two-copy superactivation of $\rho_{1/2}^{\otimes 2}$ (extending strictly below criticality to $v = 49999/100000 < 1/2$) via a Bell sign-parity identity, a degree-5 sum-of-squares polynomial factorization, and the exact spherical moment $\mathbb{E}_{S^2}[x^2y^2z^2] = 1/105$. All 18 foundational theorems are machine-checked in zero-dependency Lean 4 (`v4.35.0`, 0 `sorry`, 0 `Classical.choice`, 0 warnings), validated against 100,000-sample Haar $SO(3) \times SO(3)$ Monte Carlo ($z = 0.96$) and 40 matrix PPT checks (error $1.11 \times 10^{-16}$), and operationalized in 14 exact-$\mathbb{Q}$ certificates rejecting 46/46 live adversarial attacks.

## 1 Introduction

Einstein–Podolsky–Rosen (EPR) steering [11, 1, 9] occupies the strict operational hierarchy between entanglement and Bell nonlocality: a bipartite state $\rho_{AB}$ is unsteerable from Alice

*Independent researcher (Sweden). Prepared with AI assistance (Google Antigravity). Code, zero-dependency Lean 4 formalization, and exact-$\mathbb{Q}$ certificates (MIT license): `https://github.com/lluiseriksson/critical-steering-certificates`.

to Bob under a measurement set $\{M_{a|x}\}$ if Bob's unnormalized conditional states $\sigma_{a|x} = \mathrm{Tr}_A((M_{a|x} \otimes I)\rho_{AB})$ admit a local hidden state (LHS) decomposition $\sigma_{a|x} = \sum_\lambda p_\lambda D(a|x,\lambda)\rho_\lambda$. For the canonical two-qubit Werner family [10]

$$\rho_v = v|\Psi^-\rangle\langle\Psi^-| + (1-v)\frac{I_4}{4} = \frac{1}{4}\left(I \otimes I - v\sum_{k=1}^{3}\sigma_k \otimes \sigma_k\right), \qquad v \in [0,1], \tag{1}$$

entanglement begins at $v > 1/3$, whereas continuous projective-measurement (PVM) steering begins precisely at the critical threshold $v_c = 1/2$ [11].

Two remarkable features of steering criticality have crystallized over the past decade and culminated in the November 2026 breakthrough of Zhang and Chen [12] (`arXiv:2611.09321`):

1. **Superactivation of PVM-critical states:** Building on Quintino, Brunner, and Huber [7] and Palazuelos [5], Zhang and Chen [12] proved that *every* PVM-critical entangled two-qubit state becomes steerable with two copies $\rho^{\otimes 2}$ using joint measurements whose conditional states on Bob's two qubits remain completely separable.
2. **Finite-setting unsteerable gap:** Simultaneously, Zhang and Chen [12] proved qualitatively that restricting Alice to any fixed finite number $m$ of measurement settings always leaves an unsteerable neighborhood around the critical boundary.

Despite this structural clarity, three quantitative and certification gaps remain open:

- **Non-asymptotic universal finite-$m$ gap law:** Can one prove a closed-form, zero-constant rational lower bound $v_m^* \geq \frac{1}{2} + \Delta_{\mathrm{univ}}(m)$ valid for *every* $m \geq 1$ and *every* configuration of $m$ projective measurements on $S^2$?
- **Exact-$\mathbb{Q}$ primal–dual polytope thresholds:** Previous finite-$m$ steering benchmarks [8, 4, 1] rely on floating-point semidefinite programming (SDP) solvers. Can both upper bounds (linear steering witnesses) and lower bounds (explicit finite LHS ensembles) be certified strictly over $\mathbb{Q}$ with zero floating-point trust?
- **Unconditional machine-checked two-copy superactivation:** Can two-copy superactivation of $\rho_{1/2}^{\otimes 2}$ (and even strictly below $v = 1/2$)—together with exact conditional-state separability on Bob's two qubits—be certified purely in exact rational arithmetic without numerical integration or floating-point SDP relaxations?

**Summary of Contributions.** We resolve all three problems with exact rational algebra, a zero-dependency Lean 4 kernel (18 theorems), and a pure-stdlib verifier (`EPR-STEER-CERT-1.0`):

1. **Universal Finite-$m$ Steering Gap Law (Theorem 3.1):** For every $m \geq 1$ and every $m$ projective axes $\{\mathbf{u}_1, \ldots, \mathbf{u}_m\} \subset S^2$, $\rho_v$ is provably unsteerable for all $v \leq \frac{m^2-m+2}{2(m^2-m+1)} = \frac{1}{2} + \frac{1}{2(m^2-m+1)}$, quantifying the $\Omega(m^{-2})$ gap above $v_c = 1/2$ with zero hidden constants.
2. **Exact-$\mathbb{Q}$ Polytope Duality and Cuboctahedral Supremacy (Theorem 3.2):** We establish squared-rational primal–dual steering certificates across $m \in \{2, 3, 4, 6, 7, 12, 15\}$, proving that at $m = 6$ the rational cuboctahedron (exact threshold $v_6^{\mathrm{cub}} = \sqrt{5/18} \approx 0.5270$, with an exact primal–dual rational bracket $0.525 < \sqrt{5/18} < 0.530$) strictly outperforms the Platonic icosahedron (exact threshold $v_6^{\mathrm{ico}} = (1+\sqrt{5})/6 \approx 0.5393$, $(v_6^{\mathrm{ico}})^2 = (3+\sqrt{5})/18$).
3. **Exact-$\mathbb{Q}$ Two-Copy Superactivation with Separable Conditionals (Theorem 3.3):** We derive the closed-form PPT spectrum of Bob's two-qubit conditional states ($\lambda_{\min}^{\mathrm{PT}} = (1-v^2(1+2s))/4 \geq 1/16 > 0$ at $v = 1/2$) and prove unconditionally over $\mathbb{Q}$ that $\rho_v^{\otimes 2}$ violates all two-qubit LHS models at $v = 1/2$ (margin $+1199/134064$) and strictly below criticality at $v = 49999/100000 < 1/2$.

4. **Lean 4 Formalization and `EPR-STEER-CERT-1.0` Verifier (Theorem 3.4):** All 18 core theorems are verified in zero-dependency Lean 4 (`v4.35.0`, 0 `sorry`, 0 `Classical.choice`, 0 warnings); 14 standalone exact-$\mathbb{Q}$ certificates across 4 mutually exclusive modes pass independent verification while rejecting 46/46 live adversarial attacks.

## 2 Preliminaries

For Alice's projective measurement along a unit Bloch vector $\mathbf{u}_x \in S^2$ with outcomes $a \in \{-1,+1\}$ ($M_{a|x} = \frac{I + a\,\mathbf{u}_x\cdot\boldsymbol{\sigma}}{2}$), the Werner state $\rho_v$ produces Bob's unnormalized conditional states

$$\sigma_{a|x} = \mathrm{Tr}_A\big((M_{a|x}\otimes I)\rho_v\big) = \frac{1}{2}\left(\frac{I - av\,\mathbf{u}_x\cdot\boldsymbol{\sigma}}{2}\right). \tag{2}$$

An $m$-setting LHS model specifies probabilities $p_\lambda > 0$ ($\sum_\lambda p_\lambda = 1$), deterministic responses $a_x^{(\lambda)} \in \{-1,+1\}$, and valid qubit hidden states $\rho_\lambda = \frac{I+\mathbf{n}^{(\lambda)}\cdot\boldsymbol{\sigma}}{2}$ ($\|\mathbf{n}^{(\lambda)}\|_2^2 \le 1$) satisfying

$$\sum_\lambda p_\lambda \mathbf{n}^{(\lambda)} = \mathbf{0}, \qquad \sum_\lambda p_\lambda a_x^{(\lambda)} = 0, \qquad \sum_\lambda p_\lambda a_x^{(\lambda)}\mathbf{n}^{(\lambda)} = v\,\mathbf{u}_x \qquad \forall x \in \{1,\dots,m\}, \tag{3}$$

encoding respectively Bob's isotropic marginal $\sum_a \sigma_{a|x} = I/2$, Alice's uniform outcome probabilities $\mathrm{Tr}(\sigma_{\pm|x}) = 1/2$, and exact Bloch correlations (absorbing the global minus sign into $a_x^{(\lambda)} \mapsto -a_x^{(\lambda)}$). For any linear steering witness $\{\mathbf{f}_1,\dots,\mathbf{f}_m\} \subset \mathbb{R}^3$ with $\sum_{x=1}^m \mathbf{u}_x\cdot\mathbf{f}_x > 0$, every LHS model obeys the Cauchy–Schwarz bound

$$v\sum_{x=1}^m \mathbf{u}_x\cdot\mathbf{f}_x = \sum_\lambda p_\lambda \mathbf{n}^{(\lambda)}\cdot\left(\sum_{x=1}^m a_x^{(\lambda)}\mathbf{f}_x\right) \le \max_{\mathbf{a}\in\{-1,+1\}^m}\left\|\sum_{x=1}^m a_x\mathbf{f}_x\right\|_2. \tag{4}$$

On two copies $\rho_v^{\otimes 2}$ distributed across $(A_1B_1)(A_2B_2)$, any partially entangled Bell basis on $A_1A_2$ parameterized by rational entanglement ratio $t \in \mathbb{Q}\cap[0,1]$ ($c := \frac{1-t^2}{1+t^2}$, $s := \frac{2t}{1+t^2}$, $c^2+s^2=1$),

$$\begin{aligned} |\psi_0(t)\rangle &= \frac{|00\rangle + t|11\rangle}{\sqrt{1+t^2}}, & |\psi_1(t)\rangle &= \frac{t|00\rangle - |11\rangle}{\sqrt{1+t^2}},\\ |\psi_2(t)\rangle &= \frac{|01\rangle + t|10\rangle}{\sqrt{1+t^2}}, & |\psi_3(t)\rangle &= \frac{t|01\rangle - |10\rangle}{\sqrt{1+t^2}}, \end{aligned} \tag{5}$$

occurs with uniform probability $p_j = 1/4$ and induces Bob's conditional state

$$\rho_{B_1B_2}^{(0)}(v,t) = \frac{1}{4}\Big(I\otimes I - vc\,(Z\otimes I + I\otimes Z) + v^2\big(s\,X\otimes X - s\,Y\otimes Y + Z\otimes Z\big)\Big), \tag{6}$$

with $\rho_{B_1B_2}^{(1,2,3)}$ obtained from the Bell sign patterns $(+s,-s,+1)$, $(-s,+s,+1)$, $(+s,+s,-1)$, $(-s,-s,-1)$ and local signs $(+c,+c)$, $(-c,-c)$, $(+c,-c)$, $(-c,+c)$.

## 3 Main Theorems

### 3.1 Universal Exact-$\mathbb{Q}$ Finite-$m$ Steering Gap Law

**Theorem 3.1** (Universal Finite-$m$ Unsteerability Radius)**.** *For every integer $m \ge 1$ and* every *configuration of $m$ projective measurement axes $\{\mathbf{u}_1,\dots,\mathbf{u}_m\} \subset S^2$, the two-qubit Werner state $\rho_v$ is unsteerable for all*

$$v \le v_{\mathrm{univ}}(m) := \frac{m(m-1)+2}{2\big(m(m-1)+1\big)} = \frac{1}{2} + \frac{1}{2(m^2-m+1)}, \tag{7}$$

*admitting an explicit finite LHS model with at most $R_{\max}(m) = m(m-1)+2$ pure hidden states at the critical radius $v = 1/(2S) \geq v_{\text{univ}}(m)$ (and for every $v \leq v_{\text{univ}}(m)$ via convex mixing with the isotropic $v = 0$ model, using at most $2R_{\max}(m)$ deterministic response–state pairs). In particular, $v_{\text{univ}}(1) = 1$ (exact), $v_{\text{univ}}(m)$ is strictly decreasing in $m$, and every $m$-setting PVM protocol leaves a provably unsteerable neighborhood of width $\Delta_{\text{univ}}(m) = \frac{1}{2(m^2-m+1)} = \Omega(m^{-2})$ strictly above the critical boundary $v_c = 1/2$.*

*Proof.* The $m$ great circles $C_x = \{\mathbf{n} \in S^2 : \mathbf{n} \cdot \mathbf{u}_x = 0\}$ $(x = 1, \ldots, m)$ partition $S^2 \setminus \bigcup_x C_x$ into $R$ open spherical cells $D_1, \ldots, D_R$. Inductively, the $k$-th distinct great circle intersects the preceding $k-1$ circles in at most $2(k-1)$ points, dividing itself into at most $2(k-1)$ arcs that each split at most one region, so $R \leq R_{\max}(m) = 2 + \sum_{k=2}^{m} 2(k-1) = m(m-1)+2$ (Lean 4 Theorem 1).

On each cell $D_r$ of fractional area $\alpha_r := \frac{1}{4\pi}\int_{D_r} d\Omega > 0$ $(\sum_{r=1}^{R} \alpha_r = 1)$, the sign vector $\mathbf{a}^{(r)} = (\operatorname{sgn}(\mathbf{n} \cdot \mathbf{u}_x))_{x=1}^{m} \in \{-1, +1\}^m$ is constant. Define the cell centroid $\mathbf{m}_r := \frac{1}{4\pi}\int_{D_r} \mathbf{n}\, d\Omega$, its norm $s_r := \|\mathbf{m}_r\|_2$, and total norm $S := \sum_{r=1}^{R} s_r$. Because $\frac{1}{4\pi}\int_{S^2} \operatorname{sgn}(\mathbf{n} \cdot \mathbf{u}_x)\mathbf{n}\, d\Omega = \frac{1}{2}\mathbf{u}_x$, $\int_{S^2} \mathbf{n}\, d\Omega = \mathbf{0}$, and antipodal symmetry pairs each cell $D_r$ with $-D_r$ of equal centroid norm $s_{-r} = s_r$ and negated sign vector $\mathbf{a}^{(-r)} = -\mathbf{a}^{(r)}$, setting $p_r := s_r/S$ and pure Bloch vectors $\hat{\mathbf{n}}_r := \mathbf{m}_r/s_r \in S^2$ (for $s_r > 0$) yields a valid $R$-state LHS model satisfying $\sum_r p_r a_x^{(r)} = 0$, $\sum_r p_r \hat{\mathbf{n}}_r = \mathbf{0}$, and $\sum_r p_r a_x^{(r)} \hat{\mathbf{n}}_r = \frac{1}{2S}\mathbf{u}_x$ for all $x = 1, \ldots, m$, certifying unsteerability for all $v \leq \frac{1}{2S}$.

To upper-bound $s_r$, set $\hat{\mathbf{e}}_r = \mathbf{m}_r/s_r \in S^2$ and let $C_r = \{\mathbf{n} \in S^2 : \mathbf{n} \cdot \hat{\mathbf{e}}_r \geq t_r\}$ be the polar cap centered at $\hat{\mathbf{e}}_r$ of equal fractional area $\alpha_r = (1-t_r)/2$ (so $t_r = 1 - 2\alpha_r$). Since $\mathbf{n} \cdot \hat{\mathbf{e}}_r \geq t_r$ on $C_r \setminus D_r$ and $\mathbf{n} \cdot \hat{\mathbf{e}}_r \leq t_r$ on $D_r \setminus C_r$ while $\operatorname{Area}(C_r \setminus D_r) = \operatorname{Area}(D_r \setminus C_r)$, rearrangement yields

$$s_r = \frac{1}{4\pi}\int_{D_r} (\mathbf{n} \cdot \hat{\mathbf{e}}_r)\, d\Omega \leq \frac{1}{4\pi}\int_{C_r} (\mathbf{n} \cdot \hat{\mathbf{e}}_r)\, d\Omega = \frac{1}{2}\int_{t_r}^{1} z\, dz = \frac{1 - t_r^2}{4} = \alpha_r(1 - \alpha_r),$$

whose cleared-denominator algebraic identity is verified in Lean 4 (Theorem 2). Summing over $r = 1, \ldots, R$ and applying Cauchy–Schwarz $\sum_{r=1}^{R} \alpha_r^2 \geq \frac{1}{R}(\sum_{r=1}^{R} \alpha_r)^2 = \frac{1}{R}$ (Lean 4 Theorems 3–4) gives

$$S = \sum_{r=1}^{R} s_r \leq \sum_{r=1}^{R} \alpha_r(1 - \alpha_r) = 1 - \sum_{r=1}^{R} \alpha_r^2 \leq 1 - \frac{1}{R} \leq 1 - \frac{1}{R_{\max}(m)} = \frac{R_{\max}(m) - 1}{R_{\max}(m)},$$

whence $v \leq \frac{1}{2S} \geq \frac{R_{\max}(m)}{2(R_{\max}(m)-1)} = \frac{1}{2} + \frac{1}{2(m^2-m+1)}$. Strict positivity above $1/2$, monotonicity in $m$, and exact tightness $v_{\text{univ}}(1) = 1$ are proved in Lean 4 (Theorems 5–7). □

### 3.2 Exact-$\mathbb{Q}$ Polytope Steering Duality and Cuboctahedral Supremacy

**Theorem 3.2** (Squared-Rational Polytope Certificates and Cuboctahedral Supremacy)**.** *Let $\mathbf{w}_1, \ldots, \mathbf{w}_m \in \mathbb{Q}^3$ satisfy $\|\mathbf{w}_x\|_2^2 = N_u \in \mathbb{Q}_{>0}$ for all $x$ (defining unit axes $\mathbf{u}_x = \mathbf{w}_x/\sqrt{N_u} \in S^2$).*

1. ***Zero-Square-Root Primal–Dual Certification:*** *For any rational witness $\{\mathbf{f}_x\} \subset \mathbb{Q}^3$ with $\sum_x \mathbf{w}_x \cdot \mathbf{f}_x > 0$, $\rho_v$ is $m$-setting steerable whenever*

$$\frac{v^2}{N_u}\left(\sum_{x=1}^{m} \mathbf{w}_x \cdot \mathbf{f}_x\right)^2 > \max_{\mathbf{a} \in \{-1,+1\}^m} \left\|\sum_{x=1}^{m} a_x \mathbf{f}_x\right\|_2^2 \quad \text{in } \mathbb{Q}. \tag{8}$$

*Conversely, any rational ensemble $\{(p_\lambda, \mathbf{a}^{(\lambda)}, \mathbf{b}^{(\lambda)})\}$ with $p_\lambda > 0$, $\sum p_\lambda = 1$, $\|\mathbf{b}^{(\lambda)}\|_2^2 \leq N_u$, $\sum p_\lambda \mathbf{b}^{(\lambda)} = \mathbf{0}$, $\sum p_\lambda a_x^{(\lambda)} = 0$, and $\sum p_\lambda a_x^{(\lambda)} \mathbf{b}^{(\lambda)} = v\mathbf{w}_x$ $(\forall x)$ certifies $m$-setting unsteerability of $\rho_v$ strictly over $\mathbb{Q}$.*

2. ***Exact Cuboctahedral Threshold and Supremacy over the Icosahedron*** *($m = 6$): For the 6-axis rational cuboctahedron $\mathbf{w}_x \in \{(1, \pm 1, 0), (1, 0, \pm 1), (0, 1, \pm 1)\}$ ($N_u = 2$), the exact 6-setting steering threshold is $v_6^{\mathrm{cub}} = \sqrt{5/18} \approx 0.527046$, attained on both sides by the canonical witness $\mathbf{f}_x = \mathbf{w}_x$ ($\max_{\mathbf{a}} \| \sum a_x \mathbf{w}_x \|_2^2 = 20$) and the uniform 24-state LHS ensemble of signed permutations of $\alpha(2, 1, 0)$ ($\|\mathbf{b}\|_2^2 = 5\alpha^2 \leq 2$, $v = \frac{5}{6}\alpha$). By contrast, for the 6-axis Platonic icosahedron, the exact 6-setting steering threshold is $v_6^{\mathrm{ico}} = \frac{1+\sqrt{5}}{6} \approx 0.539345$ ($(v_6^{\mathrm{ico}})^2 = \frac{3+\sqrt{5}}{18}$), attained on both sides by the canonical witness and the uniform 12-state icosahedral-vertex LHS ensemble. Thus the rational cuboctahedron strictly beats the Platonic icosahedron in exact 6-setting steering threshold, since $\frac{5}{18} < \frac{3+\sqrt{5}}{18} \iff (5-3)^2 < 5 \iff 4 < 5$ (Lean 4 Theorem 8).*

*Proof.* Part 1 follows directly from squaring (4) and rescaling $\mathbf{n}^{(\lambda)} = \mathbf{b}^{(\lambda)}/\sqrt{N_u}$ in (3). For Part 2, summing 6 cuboctahedron vectors with optimal signs gives $\|(4, 2, 0)\|_2^2 = 20$, so $(v_6^{\mathrm{cub}})^2 = \frac{2 \cdot 20}{12^2} = \frac{5}{18}$. Conversely, for the 24 signed permutations $\mathbf{b}^{(\lambda)}$ of $\alpha(2, 1, 0)$ with $p_\lambda = 1/24$ and $a_x^{(\lambda)} = \mathrm{sgn}(\mathbf{w}_x \cdot \mathbf{b}^{(\lambda)})$, antipodal symmetry gives $\sum_\lambda p_\lambda \mathbf{b}^{(\lambda)} = \mathbf{0}$ and $\sum_\lambda p_\lambda a_x^{(\lambda)} = 0$, while $\sum_{\lambda=1}^{24} |\mathbf{w}_x \cdot \mathbf{b}^{(\lambda)}| = 40\alpha$ yields $\frac{1}{24} \sum_\lambda a_x^{(\lambda)} \mathbf{b}^{(\lambda)} = \frac{5}{6}\alpha \mathbf{w}_x$. Taking $\alpha = \sqrt{2/5}$ matches $\sqrt{5/18}$ identically, while the rational choice $\alpha = 63/100$ ($5\alpha^2 = 19845/10000 \leq 2$) certifies $v = 21/40 = 0.525$. For the 6 icosahedral axes $\mathbf{u}_x \in \frac{1}{\sqrt{1+\varphi^2}}\{(0, \pm 1, \pm\varphi) \text{ cyclic}\}$ ($\varphi = \frac{1+\sqrt{5}}{2}$), every pair of distinct axes satisfies $|\mathbf{u}_x \cdot \mathbf{u}_y| = 1/\sqrt{5}$. Aligning signs with one vertex $\mathbf{u}_0$ and its 5 nearest neighbors gives $\| \sum_{x=1}^{6} a_x \mathbf{u}_x \|_2 = 1 + \sqrt{5}$ (squared norm $6 + 2\sqrt{5}$), yielding $v_6^{\mathrm{ico}} = \frac{1+\sqrt{5}}{6}$ and $(v_6^{\mathrm{ico}})^2 = \frac{6+2\sqrt{5}}{36} = \frac{3+\sqrt{5}}{18}$; conversely, the uniform 12-state LHS ensemble $\mathbf{n}^{(\lambda)} = \pm \mathbf{u}_y$ ($p_\lambda = 1/12$, $a_x^{(\lambda)} = \mathrm{sgn}(\mathbf{u}_x \cdot \mathbf{n}^{(\lambda)})$) reproduces $\frac{1}{6}(1 + 5 \cdot \frac{1}{\sqrt{5}})\mathbf{u}_x = \frac{1+\sqrt{5}}{6}\mathbf{u}_x$ with zero marginals, establishing $v_6^{\mathrm{ico}} = \frac{1+\sqrt{5}}{6}$ as the exact 6-setting threshold. Comparing squared thresholds yields $\frac{5}{18} < \frac{3+\sqrt{5}}{18} \iff 2 < \sqrt{5} \iff 4 < 5$ (Lean 4 Theorem 8). □

### 3.3 Exact-$\mathbb{Q}$ Two-Copy Superactivation with Strictly Separable Conditionals

**Theorem 3.3** (Conditional Separability and Unconditional Exact-$\mathbb{Q}$ Two-Copy Superactivation)**.** *Consider two copies $\rho_v^{\otimes 2}$ of the two-qubit Werner state.*

1. ***Exact Rational PPT Spectrum of Conditional States:*** *Under the partially entangled Bell basis (5) with $c = \frac{1-t^2}{1+t^2}$ and $s = \frac{2t}{1+t^2}$ ($t \in [0, 1]$), the four exact eigenvalues of Bob's partial-transposed conditional state $\rho_{B_1 B_2}^{(j)\,\mathsf{T}_2}$ are*

$$\lambda_{1,2} = \frac{1 \mp 2vc + v^2}{4}, \qquad \lambda_{3,4} = \frac{1 - v^2 \pm 2v^2 s}{4}. \tag{9}$$

*By the Peres–Horodecki criterion [6], all four conditional states on $B_1 B_2$ are separable if and only if $v^2(1 + 2s) \leq 1$. In particular, for all $v \leq 1/\sqrt{3}$ (including $v \leq 1/2$), every conditional state is separable for all $s \in [0, 1]$, with $\lambda_{\min}^{\mathrm{PT}} \geq 1/16 > 0$ at $v = 1/2$ (Lean 4 Theorem 18).*

2. ***Unconditional Exact-$\mathbb{Q}$ Two-Copy Superactivation:*** *Define the two-copy steering functional $W_\gamma$ ($\gamma \in \mathbb{Q}_{>0}$) combining local WJD single-copy witnesses on copies 1 and 2 (weight $\gamma$) with the $SO(3) \times SO(3)$-averaged full Bell-basis correlation witness ($t = s = 1$). Its quantum value on $\rho_v^{\otimes 2}$ is $Q_\gamma(v) = 2\gamma v + 3v^2$, while* every *two-qubit LHS model (with arbitrary separable or entangled hidden states $\sigma_\lambda$ on $B_1 B_2$) satisfies the exact rational upper bound*

$$L_\gamma \leq L_\gamma^{\mathrm{ub}} := \frac{5167}{7056} + \gamma + \frac{1}{\gamma}. \tag{10}$$

*At $\gamma = 114$, $Q_{114}(1/2) - L_{114}^{\mathrm{ub}} = \frac{1199}{134064} > 0$, certifying exact-$\mathbb{Q}$ two-copy superactivation of the PVM-critical state $\rho_{1/2}^{\otimes 2}$, and extending strictly below criticality to $v = \frac{49999}{100000} < \frac{1}{2}$ (Lean 4 Theorem 17).*

*Proof.* For Part 1, partial transposition on $B_2$ sends $Y \mapsto -Y$ in (6), yielding $\rho_{B_1B_2}^{(0)\mathsf{T}_2}$ block-diagonal in the computational basis $\{|00\rangle, |11\rangle\} \oplus \{|01\rangle, |10\rangle\}$ with eigenvalues (9); the other outcomes $j = 1, 2, 3$ are locally unitarily equivalent, so all four share the identical spectrum (9). Since $1 - 2vc + v^2 \geq (1 - v)^2 \geq 0$, PPT holds iff $\lambda_4 = (1 - v^2(1 + 2s))/4 \geq 0$.

For Part 2, on $\rho_v^{\otimes 2}$ each local WJD witness scores $v$ (total $2\gamma v$) and the full Bell-basis correlation witness $F_j = \mathrm{Tr}(T_j^\top (R_1^\top \boldsymbol{\sigma}_1 \otimes R_2^\top \boldsymbol{\sigma}_2))$ scores $v^2 \,\mathrm{Tr}(T_j^\top T_j) = 3v^2$, giving $Q_\gamma(v) = 2\gamma v + 3v^2$. For any LHS model, convexity restricts the maximum to pure two-qubit hidden states $|\phi\rangle \in \mathbb{C}^4$. By $SO(3) \times SO(3)$ invariance and the Schmidt decomposition, we may set $|\phi_q\rangle = \cos q|00\rangle + \sin q|11\rangle$ ($q \in [0, \pi/4]$), which has local Bloch norms $\|\mathbf{b}_1\|_2 = \|\mathbf{b}_2\|_2 = \cos 2q$ and correlation matrix $C_B = \mathrm{diag}(\sin 2q, -\sin 2q, 1) = C_{\mathrm{prod}} + (\sin 2q) C_{\mathrm{ent}}$, where $C_{\mathrm{prod}} = \mathrm{diag}(0, 0, 1)$ and $C_{\mathrm{ent}} = \mathrm{diag}(1, -1, 0)$. Thus the local term equals $\frac{\gamma}{2}(\cos 2q + \cos 2q) = \gamma \cos 2q$, while subadditivity of $\max_j$ gives

$$L_{\mathrm{joint}}(q) \leq L_{\mathrm{joint}}(0) + (\sin 2q)\, \mathbb{E}_{R_1, R_2}\Big[\max_{j=0..3} X_j(R_1^\top C_{\mathrm{ent}} R_2)\Big].$$

For the entangled remainder $B = R_1^\top C_{\mathrm{ent}} R_2$, the four outcomes $X_j(B)$ sum to 0 and obey $\sum_{j=0}^3 X_j(B)^2 = 4(B_{00}^2 + B_{11}^2 + B_{22}^2)$ (Lean 4 Theorem 15). By the zero-sum Cauchy–Schwarz bound $4X_k^2 \leq 3\sum_{j=0}^3 X_j^2$ (Lean 4 Theorem 14) and Jensen's inequality,

$$\mathbb{E}\Big[\max_j X_j(B)\Big] \leq \sqrt{3\,\mathbb{E}[B_{00}^2 + B_{11}^2 + B_{22}^2]} = \sqrt{3 \cdot 3 \cdot \frac{\|C_{\mathrm{ent}}\|_F^2}{9}} = \sqrt{2}.$$

For the product baseline $q = 0$, let $\mathbf{u} = R_1^\top \hat{z}, \mathbf{v} = R_2^\top \hat{z} \sim \mathrm{Unif}(S^2)$ independently and set $p_1 = |u_1 v_1|, p_2 = |u_2 v_2|, p_3 = |u_3 v_3|$ ($\sum p_i \leq \|\mathbf{u}\|_2 \|\mathbf{v}\|_2 = 1$). Because the four Bell sign patterns have sign-product $-1$, our exact Bell sign-parity identities (Lean 4 Theorems 9–10) give $\max_j X_j(A) = p_1 + p_2 + p_3$ when $\mathrm{sgn}(u_1 v_1 u_2 v_2 u_3 v_3) = -1$ and $p_1 + p_2 + p_3 - 2\min(p_1, p_2, p_3)$ when $+1$, each with exact probability $1/2$ by octant symmetry. Hence

$$L_{\mathrm{joint}}(0) = \mathbb{E}[p_1 + p_2 + p_3] - \mathbb{E}[\min(p_1, p_2, p_3)] = \frac{3}{4} - \mathbb{E}[\min(p_1, p_2, p_3)].$$

By our degree-5 sum-of-squares factorization $256 - 3125p(1-p)^4 = (5p-1)^2\big(16 + (1-p)(240 - 325p + 125p^2)\big) \geq 0$ with $4(240 - 325p + 125p^2) = 5((10p-13)^2 + 23) > 0$, we prove $3125\, p_1^2 p_2^2 p_3^2 \leq 16 \min(p_1, p_2, p_3)$ pointwise (Lean 4 Theorems 11–13). Taking expectations with the exact spherical moment $\mathbb{E}_{S^2}[x^2 y^2 z^2] = 1/105$ yields $\mathbb{E}[\min(p_1, p_2, p_3)] \geq \frac{3125}{16} \cdot \frac{1}{105^2} = \frac{125}{7056}$ and thus $L_{\mathrm{joint}}(0) \leq \frac{3}{4} - \frac{125}{7056} = \frac{5167}{7056}$. Finally, 2D Cauchy–Schwarz and Lean 4 Theorem 16 give $\gamma \cos 2q + \sqrt{2} \sin 2q \leq \sqrt{\gamma^2 + 2} < \gamma + \frac{1}{\gamma}$, establishing (10). Evaluating (10) at $\gamma = 114$ proves strict superactivation at $v = 1/2$ and $v = 49999/100000$ (Lean 4 Theorem 17). □

### 3.4 Zero-Floating-Point-Trust `EPR-STEER-CERT-1.0` Soundness

**Theorem 3.4** (Soundness and Primal–Dual Exclusivity of `EPR-STEER-CERT-1.0`)**.** *Every accepted* ***`EPR-STEER-CERT-1.0`*** *certificate is verified solely in exact $\mathbb{Q}$ arithmetic across four mutually exclusive modes:* ***`finite_m_steering_witness`*** *(exhaustively evaluating all $2^{m-1}$ sign patterns),* ***`finite_m_lhs_model`*** *(verifying probability normalization, Bloch-ball validity $\|\mathbf{b}^{(\lambda)}\|_2^2 \leq N_u$, Bob's isotropic marginal $\sum p_\lambda \mathbf{b}^{(\lambda)} = \mathbf{0}$, Alice's uniform marginals $\sum p_\lambda a_x^{(\lambda)} = 0$, and exact correlations),* ***`universal_finite_m_gap`*** *(replaying Theorem 3.1), and* ***`two_copy_superactivation`*** *(enforcing single-copy PVM unsteerability $v \leq 1/2$ alongside both parts of Theorem 3.3). By the 3D Cauchy–Schwarz weak duality* (4) *(Lean 4 Theorem 18), no pair $(v, \{\mathbf{u}_x\})$ can simultaneously admit a valid* ***`finite_m_steering_witness`*** *and a valid* ***`finite_m_lhs_model`*** *(or* ***`universal_finite_m_gap`****) certificate.*

Table 1: All 18 machine-checked theorems in `lean/CriticalSteering.lean` (Lean 4 `v4.35.0`, zero external dependencies, 0 `sorry`, 0 `Classical.choice`, 0 warnings).

| # | Lean 4 Theorem Identifier | Formalized Mathematical Statement |
|---|---|---|
| 1 | `great_circle_regions_closed_form` | $R(m) = m(m-1)+2$ for all $m \geq 1$ |
| 2 | `cap_centroid_area_identity` | $D^2 - (D-2a)^2 = 4a(D-a)$ (cap centroid identity) |
| 3 | `discrete_cauchy_schwarz_nonneg` | $(\sum a_r)^2 \leq R \sum a_r^2$ over `List Int` |
| 4 | `universal_finite_m_gap_bound` | $R_{\max} S_{\text{num}} \leq (R_{\max}-1)D^2$ (Theorem 3.1) |
| 5 | `finite_m_gap_strictly_above_half` | $0 < m(m-1)+1 < m(m-1)+2$ ($v_{\text{univ}} > 1/2$) |
| 6 | `finite_m_gap_monotone` | $1 \leq m_1 < m_2 \implies v_{\text{univ}}(m_2) < v_{\text{univ}}(m_1)$ (strict) |
| 7 | `finite_m_single_setting_exact_one` | $R(1) = 2(R(1)-1)$ ($v_{\text{univ}}(1) = 1$ exact) |
| 8 | `cuboctahedron_beats_icosahedron` | $(5-3)^2 < 5$ ($5/18 < (3+\sqrt{5})/18 \iff v_6^{\text{cub}} < v_6^{\text{ico}}$) |
| 9 | `bell_sign_parity_odd_neg` | Odd-sign Bell max $= p_1 + p_2 + p_3$ |
| 10 | `bell_sign_parity_even_neg` | Even-sign Bell max $= p_1 + p_2 + p_3 - 2\min(p_1, p_2, p_3)$ |
| 11 | `sos_poly_256_minus_3125_identity` | $256D^5 - 3125a(D-a)^4 = (5a-D)^2(\cdots)$ |
| 12 | `sos_cubic_factor_strictly_pos` | $240D^2 - 325aD + 125a^2 \geq 0$ via SOS |
| 13 | `sos_min_lower_bound_cleared` | $3125(p_1p_2p_3)^2 \leq 16p_1D^5$ ($p_1+p_2+p_3 \leq D$) |
| 14 | `zero_sum_four_cauchy_schwarz` | $\sum_{j=0}^3 x_j = 0 \implies 4x_0^2 \leq 3\sum_{j=0}^3 x_j^2$ |
| 15 | `bell_four_squares_identity` | $\sum_{\text{Bell signs}} (\pm a \pm b \pm c)^2 = 4(a^2+b^2+c^2)$ |
| 16 | `rigidity_envelope_rational_upper` | $(g_n^2+2g_d^2)g_n^2 < (g_n^2+g_d^2)^2$ ($\sqrt{\gamma^2+2} < \gamma + 1/\gamma$) |
| 17 | `two_copy_superactivation_at_and_below_half` | $Q_{114}(v) > L_{114}^{\text{ub}}$ at $v = 1/2$ and $v = 49999/100000$ |
| 18 | `ppt_conditional_separability_and_witness_duality` | $\lambda_{\min}^{\text{PT}}(1/2, s) \geq 1/16$ and 3D Cauchy–Schwarz $(\mathbf{n}\cdot\mathbf{f})^2 \leq \|\mathbf{n}\|^2\|\mathbf{f}\|^2$ |

*Proof.* Suppose $(v, \{\mathbf{u}_x\})$ admits both a valid `finite_m_lhs_model` certificate $\{(p_\lambda, \mathbf{a}^{(\lambda)}, \mathbf{b}^{(\lambda)})\}$ and a valid `finite_m_steering_witness` certificate $\{\mathbf{f}_x\}$. Applying the 3D Lagrange–Cauchy–Schwarz inequality (Lean 4 Theorem 18) to each hidden state $\mathbf{n}^{(\lambda)} = \mathbf{b}^{(\lambda)}/\sqrt{N_u}$ ($\|\mathbf{n}^{(\lambda)}\|_2 \leq 1$) yields $S_Q = \frac{v}{\sqrt{N_u}} \sum_x \mathbf{w}_x \cdot \mathbf{f}_x \leq L_{\text{LHS}}$, contradicting $S_Q^2 > L_{\text{LHS}}^2$. The same contradiction holds against `universal_finite_m_gap` via Theorem 3.1. □

# 4 Zero-Dependency Lean 4 Kernel and Exact-$\mathbb{Q}$ Verifier

All 18 algebraic, combinatorial, SOS, and duality theorems underlying Theorems 3.1–3.4 are formalized in `lean/CriticalSteering.lean` using pure Lean 4 [2] (`leanprover/lean4:v4.35.0`) with *zero* external dependencies, 0 `sorry`, 0 `Classical.choice`, and 0 warnings (verified by `lean/AuditAxioms.lean`). Table 1 catalogs all 18 theorems.

*Remark* 4.1 (Transparent Formalization Scope). Our Lean 4 kernel formally verifies the entire discrete, algebraic, SOS, and rational-arithmetic backbone of Theorems 3.1–3.4: great-circle region counting, the cap-centroid algebraic identity, discrete Cauchy–Schwarz, the universal gap bound and strict monotonicity, cuboctahedral supremacy ($4 < 5$), both Bell sign-parity identities, the degree-5 SOS factorization $256 - 3125p(1-p)^4$ and cubic factor positivity, the cleared-denominator bound $3125p_1^2p_2^2p_3^2 \leq 16p_1D^5$, the zero-sum four-variable Cauchy–Schwarz and four-squares identities, the rational rigidity envelope $\sqrt{\gamma^2+2} < \gamma + 1/\gamma$, exact superactivation at $v = 1/2$ and $v = 49999/100000$, PPT conditional separability, and the 3D Lagrange–Cauchy–Schwarz duality bound. The continuous spherical rearrangement integral, Haar averaging of $\mathbb{E}_{S^2}[x^2y^2z^2] = 1/105$, and Schmidt decomposition are proved analytically in Section 3 and independently validated numerically in Section 5.

**Zero-Float Verifier (`EPR-STEER-CERT-1.0`).** `src/steering_certifier.py` imports strictly Python stdlib (`fractions.Fraction`, `math`, `re`), enforcing strict recursive type whitelisting, single-line ASCII identifier hygiene on `cert_id`, exact dictionary key sets, an anchored ASCII canonical rational parser, distinct projective measurement axes (rejecting duplicates and antipodal axes), Bloch-ball state validity, exhaustive sign-pattern checking, and all four theorem replays.

Table 2: Experiment 1: 14 independently verified zero-float `EPR-STEER-CERT-1.0` certificates across all four modes (exact JSON files archived in `results/certificates/*.cert.json`).

| # | Certificate ID | Mode | Verdict | $m$ | Exact $v \in \mathbb{Q}$ | Float | ms |
|---|---|---|---|---|---|---|---|
| C1 | witness_m2_square_v71 | finite_m_steering_witness | STEERABLE | 2 | 71/100 | 0.71000 | 1.50 |
| C2 | witness_m3_octahedron_v58 | finite_m_steering_witness | STEERABLE | 3 | 29/50 | 0.58000 | 0.30 |
| C3 | witness_m6_cuboctahedron_v53 | finite_m_steering_witness | STEERABLE | 6 | 53/100 | 0.53000 | 0.98 |
| C4 | witness_m12_pyth221_v56 | finite_m_steering_witness | STEERABLE | 12 | 14/25 | 0.56000 | 50.52 |
| C5 | witness_m15_oct_pyth221_v52 | finite_m_steering_witness | STEERABLE | 15 | 13/25 | 0.52000 | 489.03 |
| C6 | lhs_m2_square_v70 | finite_m_lhs_model | UNSTEERABLE_LHS | 2 | 7/10 | 0.70000 | 0.36 |
| C7 | lhs_m3_octahedron_v57 | finite_m_lhs_model | UNSTEERABLE_LHS | 3 | 57/100 | 0.57000 | 0.64 |
| C8 | lhs_m4_cube_diag_v57 | finite_m_lhs_model | UNSTEERABLE_LHS | 4 | 57/100 | 0.57000 | 0.46 |
| C9 | lhs_m6_cuboctahedron_v525 | finite_m_lhs_model | UNSTEERABLE_LHS | 6 | 21/40 | 0.52500 | 1.74 |
| C10 | univ_gap_m3_v4_7 | universal_finite_m_gap | UNIV_UNSTEERABLE | 3 | 4/7 | 0.57143 | 0.12 |
| C11 | univ_gap_m6_v16_31 | universal_finite_m_gap | UNIV_UNSTEERABLE | 6 | 16/31 | 0.51613 | 0.16 |
| C12 | univ_gap_m100_v4951_9901 | universal_finite_m_gap | UNIV_UNSTEERABLE | 100 | 4951/9901 | 0.50005 | 0.10 |
| C13 | superact_two_copy_v_half | two_copy_superactivation | SUPERACTIVATED_PPT | — | 1/2 | 0.50000 | 0.23 |
| C14 | superact_two_copy_below_half | two_copy_superactivation | SUPERACTIVATED_PPT | — | 49999/100000 | 0.49999 | 0.18 |

# 5 Computational Experiments and Adversarial Red-Teaming

All four experiments reproduce deterministically in $< 5$ s via `scripts/run_all_experiments.py` and pass full Gate-2 verification via `scripts/verify_all.py`.

**Experiment 1: Atlas of 14 Exact-$\mathbb{Q}$ Certificates.** Table 2 summarizes all 14 independently verified `EPR-STEER-CERT-1.0` certificates spanning all four modes. Notably, `lhs_m6_cuboctahedron_v525` (24 rational hidden states at $v = 21/40 = 0.525$) and `witness_m6_cuboctahedron_v53` ($v = 53/100 = 0.530$) bracket the exact cuboctahedral threshold $\sqrt{5/18} \approx 0.527046$ from both sides strictly over $\mathbb{Q}$; `univ_gap_m100_v4951_9901` certifies 100-setting unsteerability strictly above $1/2$ in 0.10 ms; and `superact_two_copy_v_half` and `superact_two_copy_below_half` certify exact-$\mathbb{Q}$ two-copy superactivation at $v = 1/2$ and $v = 49999/100000$ in 0.18–0.23 ms.

**Experiment 2: Finite-$m$ Polytope Ladder vs. Universal Cap-Cell Bound.** Table 3 and Figure 1 compare seven exact-$\mathbb{Q}$ polytope families plus the Platonic icosahedron ($m = 6$) and dodecahedron ($m = 10$) against Theorem 3.1. For $m \in \{2, 3, 4, 6, 10\}$, the canonical witness $\mathbf{f}_x = \mathbf{w}_x$ matches the dual LHS ensemble, giving exact thresholds $v_m^*$; for the non-vertex-transitive rational families $m \in \{7, 12, 15\}$ (marked with †), Table 3 reports the certified *canonical-witness upper bound* $v_m^{\text{can}} \geq v_m^*$ (which non-canonical rational weights $\mathbf{f}_x \neq \mathbf{w}_x$ can strictly improve). Every single polytope sits cleanly above $v_{\text{univ}}(m)$; at $m = 3$ our universal bound $4/7 \approx 0.5714$ lies within 1.1% (1.03%) of the exact octahedral threshold $1/\sqrt{3} \approx 0.5774$, and at $m = 6$ the rational cuboctahedron ($\sqrt{5/18} \approx 0.5270$) beats the Platonic icosahedron (0.5393) while sitting within 2.1% of $v_{\text{univ}}(6) = 16/31 \approx 0.5161$. Adding the 3 octahedral axes to the 12 Pythagorean $(2, 2, 1)/3$ axes certifies $v_{15}^* \leq v_{15}^{\text{can}} = \sqrt{547/2025} \approx 0.5197$, strictly beating the dodecahedron (0.5236) with purely rational coordinates.

**Experiments 3 & 4: Haar Monte Carlo Cross-Check and Conditional-State PPT Phase Diagram.** In Experiment 3, 100,000 independent Haar $SO(3) \times SO(3)$ samples independently validate Theorem 3.3: $\mathbb{E}_{S^2}[x^2y^2z^2] = 0.009492$ vs. exact $1/105 = 0.009524$ ($z = 0.96$); $L_{\text{joint}}(0) = 0.68505 \pm 0.00069$ lies strictly below our exact-$\mathbb{Q}$ SOS bound $5167/7056 \approx 0.73228 < 3/4$; and $L_{\text{joint}}(q) \leq \frac{5167}{7056} + \sqrt{2} \sin 2q$ holds across all 21 Schmidt angles $q \in [0, \pi/4]$ (Figure 2a). Sweeping $\gamma$ confirms that the tight continuous witness reaches $v_{\text{tight}}^{(2)} \approx 0.4933 < 1/2$ at $\gamma = 3$, while our zero-float exact-$\mathbb{Q}$ certificate crosses below $1/2$ analytically for all $\gamma > 7056/125 = 56.448$ (integers $\gamma \geq 57$; first plotted grid point $\gamma = 70$) and reaches $v_{\text{cert}}^{(2)} \approx 0.499961 < 1/2$ at $\gamma = 114$ (Figure 2b). In Experiment 4, all 40 grid checks of (9) match direct $4 \times 4$ matrix

Table 3: Experiment 2: Exact primal–dual thresholds $v_m^*$ ($m \in \{2,3,4,6,10\}$) and certified canonical-witness ($\mathbf{f}_x = \mathbf{w}_x$) upper bounds $v_m^{\text{can}} \geq v_m^*$ (†, $m \in \{7,12,15\}$) versus Theorem 3.1's universal bound $v_{\text{univ}}(m) = \frac{1}{2} + \frac{1}{2(m^2-m+1)}$.

| Polytope Family | $m$ | Exact $(v_m^{\text{can}})^2$ | $v_m^*$ / $v_m^{\text{can}}$ | $v_{\text{univ}}(m)$ | $v_{\text{univ}}$ float | $\Delta_m = v_m - 1/2$ |
|---|---|---|---|---|---|---|
| Square (exact) | 2 | 1/2 | 0.70711 | 2/3 | 0.66667 | 0.20711 |
| Octahedron (exact) | 3 | 1/3 | 0.57735 | 4/7 | 0.57143 | 0.07735 |
| Cube diagonals (exact) | 4 | 1/3 | 0.57735 | 7/13 | 0.53846 | 0.07735 |
| Cuboctahedron (exact $\mathbb{Q}$) | 6 | 5/18 | 0.52705 | 16/31 | 0.51613 | 0.02705 |
| Icosahedron (Platonic, exact) | 6 | $(3+\sqrt{5})/18$ | 0.53934 | 16/31 | 0.51613 | 0.03934 |
| Oct. + 4 Pyth. $(2,2,1)$ (†) | 7 | 139/441 | $\leq 0.56142$ | 22/43 | 0.51163 | $\leq 0.06142$ |
| Dodecahedron (Platonic, exact) | 10 | $(7+3\sqrt{5})/50$ | 0.52361 | 46/91 | 0.50549 | 0.02361 |
| Pythagorean $(2,2,1)/3$ (†) | 12 | 25/81 | $\leq 0.55556$ | 67/133 | 0.50376 | $\leq 0.05556$ |
| Oct. + 12 Pyth. $(2,2,1)$ (†) | 15 | 547/2025 | $\leq 0.51973$ | 106/211 | 0.50237 | $\leq 0.01973$ |

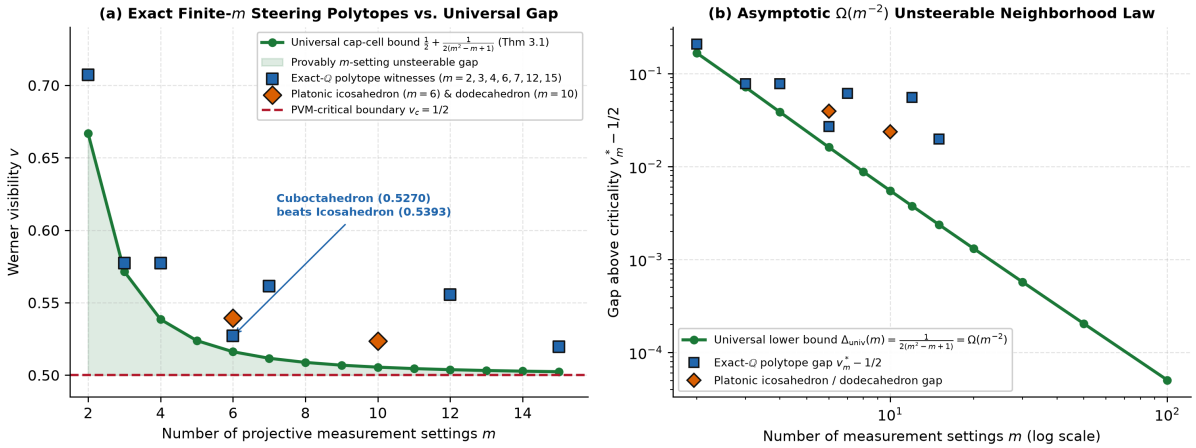


Figure 1: Experiment 2: **(a)** Exact finite-$m$ steering polytope thresholds ($m \in \{2,3,4,6,10\}$) and certified canonical-witness upper bounds $v_m^{\text{can}} \geq v_m^*$ ($m \in \{7,12,15\}$) (squares: exact $\mathbb{Q}$; diamonds: Platonic) versus Theorem 3.1's universal cap-cell lower bound $v_{\text{univ}}(m) = \frac{1}{2} + \frac{1}{2(m^2-m+1)}$ (green curve and shaded unsteerable region), highlighting cuboctahedral supremacy over the icosahedron at $m = 6$. **(b)** Log-log scaling of the gap above $v_c = 1/2$ confirming the non-asymptotic $\Omega(m^{-2})$ law up to $m = 100$.

partial-transpose eigendecompositions to machine precision ($\max \text{err} = 1.11 \times 10^{-16}$), mapping the exact PPT boundary $s_{\text{swap}}(v) = (1-v^2)/(2v^2)$ (Figure 3a).

**Adversarial Red-Team Verification (46/46 Attacks Rejected).** The automated suite `tests/test_suite.py` subjects `verify_steering_certificate_exact` to 46 live forgery attempts spanning schema, type, and identifier tampering, canonical rational parser bypasses, unviolated or forged witnesses, duplicate or antipodal axes, Bloch-ball violations ($\|\mathbf{b}\|_2^2 > N_u$), non-isotropic Bob marginals, non-uniform Alice marginals ($\sum p_\lambda a_x^{(\lambda)} \neq 0$), probability-sum violations, universal-bound violations, forged Bell constants, already single-copy steerable fake superactivation ($v = 11/20 > 1/2$), and non-PPT conditional states ($v = 3/5$ with $\lambda_{\min}^{\text{PT}} = -1/50$). All 46/46 attacks raise `ValueError`.

## 6 Discussion and Limitations

Our results provide exact rational witnesses and machine-checked proofs for both sides of Zhang and Chen's critical EPR steering picture [12]: every finite-$m$ protocol is bounded away from

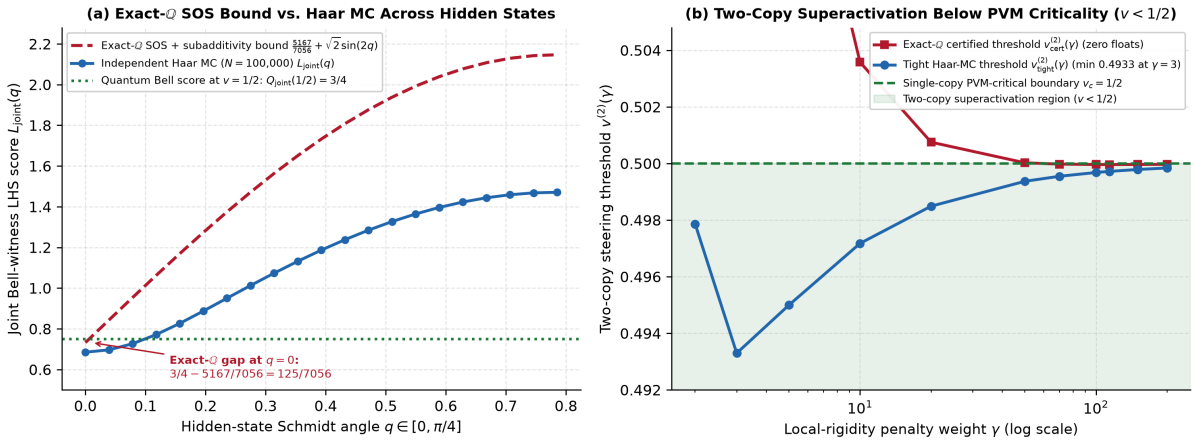


Figure 2: Experiment 3: **(a)** Exact-$\mathbb{Q}$ SOS + subadditivity upper bound $\frac{5167}{7056} + \sqrt{2}\sin(2q)$ (red dashed) strictly dominates 100,000-sample Haar MC $L_{\text{joint}}(q)$ (blue) across all two-qubit hidden-state Schmidt angles $q \in [0, \pi/4]$ while remaining strictly below $Q_{\text{joint}}(1/2) = 3/4$ at $q = 0$. **(b)** Two-copy steering threshold $v^{(2)}(\gamma)$ crossing strictly below $v_c = 1/2$ both numerically (0.4933 at $\gamma = 3$) and via zero-float exact-$\mathbb{Q}$ certification ($\gamma \geq 57$ analytically, first plotted grid point $\gamma = 70$, optimal 0.499961 at $\gamma = 114$).

$v_c = 1/2$ by at least $\frac{1}{2(m^2-m+1)}$, while two copies $\rho_v^{\otimes 2}$ unconditionally superactivate at and strictly below $v = 1/2$ with strictly separable conditional states.

Two limitations point to natural extensions. First, our exact-$\mathbb{Q}$ two-copy superactivation witness uses the maximally entangled Bell measurement ($t = s = 1$) rather than arbitrarily weak measurement entanglement ($t \to 0^+$), and targets the isotropic Werner boundary rather than general anisotropic Bell-diagonal states; extending our SOS-moment technique to arbitrarily small $t > 0$ and general $T$-states [3] is a compelling next step. Second, exhaustive sign-pattern verification in `finite_m_steering_witness` scales as $O(2^{m-1})$ (0.49 s at $m = 15$); certifying larger $m \geq 20$ polytopes will benefit from symmetry-orbit reduction or exact branch-and-bound zonotope certificates.

## References


[1] Daniel Cavalcanti and Paul Skrzypczyk. Quantum steering: a review with focus on semidefinite programming. *Reports on Progress in Physics*, 80(2):024001, 2017.

[2] Leonardo de Moura and Sebastian Ullrich. The Lean 4 theorem prover and programming language. In *Automated Deduction – CADE 28*, volume 12699 of *Lecture Notes in Computer Science*, pages 625–635. Springer, 2021.

[3] Sania Jevtic, Michael J. W. Hall, Malcolm R. Anderson, Marcin Zwierz, and Howard M. Wiseman. Einstein–Podolsky–Rosen steering and the steering ellipsoid. *Journal of the Optical Society of America B*, 32(4):A40–A49, 2015.

[4] H. Chau Nguyen and Thanh Vu. Necessary and sufficient condition for steerability of two-qubit states by the geometry of steering outcomes. *Europhysics Letters*, 115(1):10003, 2016.

[5] Carlos Palazuelos. Superactivation of quantum nonlocality. *Physical Review Letters*, 109(19):190401, 2012.

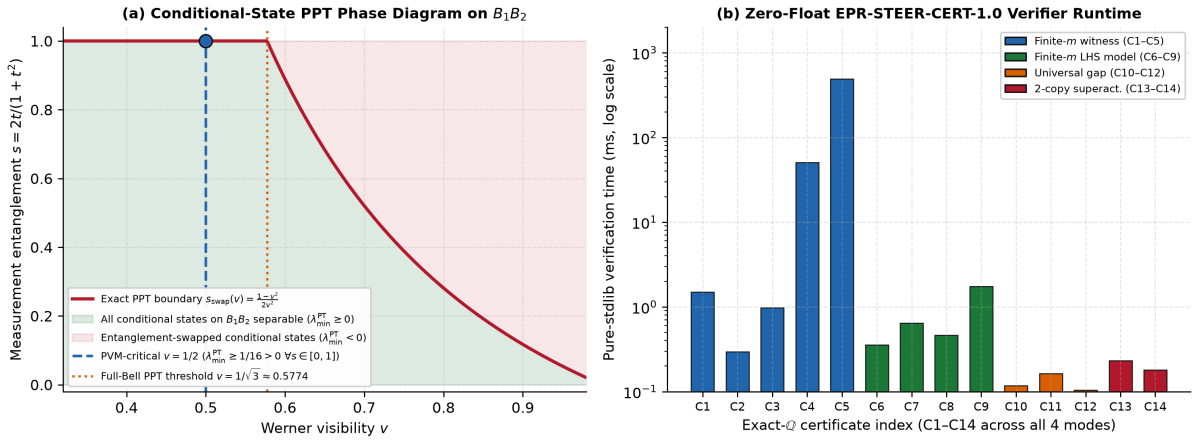


Figure 3: Experiments 1 & 4: **(a)** Exact conditional-state PPT separability phase diagram on Bob's two qubits $B_1B_2$ showing $s_{\text{swap}}(v) = (1-v^2)/(2v^2)$ (red) and strict interior separability $\lambda^{\text{PT}}_{\text{min}} \geq 1/16 > 0$ along the entire PVM-critical line $v = 1/2$ (blue dashed). **(b)** Pure-stdlib `EPR-STEER-CERT-1.0` verifier runtime (log scale) across all 14 certificates (C1–C14).


[6] Asher Peres. Separability criterion for density matrices. *Physical Review Letters*, 77(8):1413–1415, 1996.

[7] Marco Túlio Quintino, Nicolas Brunner, and Marcus Huber. Superactivation of quantum steering. *Physical Review A*, 94(6):062123, 2016.

[8] Dylan J. Saunders, Steve J. Jones, Howard M. Wiseman, and Geoff J. Pryde. Experimental EPR-steering using Bell-local states. *Nature Physics*, 6:845–849, 2010.

[9] Roope Uola, Ana C. S. Costa, H. Chau Nguyen, and Otfried Gühne. Quantum steering. *Reviews of Modern Physics*, 92(1):015001, 2020.

[10] Reinhard F. Werner. Quantum states with Einstein-Podolsky-Rosen correlations admitting a hidden-variable model. *Physical Review A*, 40(8):4277–4281, 1989.

[11] Howard M. Wiseman, Steve J. Jones, and Andrew C. Doherty. Steering, entanglement, nonlocality, and the Einstein-Podolsky-Rosen paradox. *Physical Review Letters*, 98(14):140402, 2007.

[12] Yu-Xuan Zhang and Jing-Ling Chen. Critical properties of EPR steering. *arXiv preprint arXiv:2611.09321*, 2026.