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EXACT REFLECTION POSITIVITY, CHARACTER DIAGONALIZATION, RICCATI–BESSEL BOUNDS, AND ALL-COUPLING TRANSFER-MATRIX MASS GAPS FOR L-LINK $\mathrm{SU}(N)$ WILSON CYLINDERS AND FINITE SPATIAL-SLAB GAUGE THEORIES

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Abstract. We construct an exact, self-contained operator-theoretic and spectral framework establishing unconditional Osterwalder–Schrader (OS) reflection positivity, strictly positive centered physical reflected norms, exact character diagonalization, and explicit all-coupling transfer-matrix mass gaps $\Delta > 0$ for $\mathrm{SU}(N)$ Wilson lattice Yang–Mills theory on L-link circular cylinders ($L \geq 2$) and arbitrary finite spatial-slab geometries $\mathbb{Z} \times \Gamma$ at *every* coupling $\beta > 0$ (both strong and weak coupling). Four results anchor the analysis:

(1) *Commuting gauge projection and unconditional reflection positivity on arbitrary finite spatial graphs Γ:* For any finite connected spatial graph $\Gamma = (V, E)$ with crossing plaquette holonomies $H_e(u, v, a) = u_e a_{t(e)} v_e^{-1} a_{s(e)}^{-1}$ and arbitrary gauge-invariant intra-slice spatial weight $\Psi_{\mathrm{sp}}(u) > 0$, we prove that the slice gauge-averaging projection P commutes with the factorized Wilson crossing operator K_β ($PK_\beta = K_\beta P = PK_\beta P$), that every character coefficient $\hat{w}_\lambda(\beta)$ of $w_\beta(U) = \exp(\frac{\beta}{N} \operatorname{Re} \operatorname{Tr} U)$ is strictly positive for all $\beta > 0$ with $\operatorname{Tr}_{L^2}(K_\beta) = e^{|E|\beta}$, and that the physical reflected form satisfies $\langle f, f \rangle_{\mathrm{phys}} = Z^{-1} \langle P(\Psi_{\mathrm{sp}} f), K_\beta P(\Psi_{\mathrm{sp}} f) \rangle \geq 0$ for all continuous f and all $\beta \geq 0$.

(2) *Exact character diagonalization, centered reflected norm, and Parseval dominance on L-link $\mathrm{SU}(N)$ cylinders:* For $\Gamma = C_L$ ($L \geq 2$), we prove $Z_L(\beta) = \hat{w}_0(\beta)^L$, exact Haar slice marginality ($\mathbb{E}_{\mathrm{phys}}[W_\lambda] = 0$ for all $\lambda \neq \mathbf{1}$), isometric identification $\operatorname{Ran}(P) \cong L^2_{\mathrm{class}}(\mathrm{SU}(N))$, and exact diagonalization $T_{L,\beta} W_\lambda = (\hat{w}_\lambda(\beta)/(d_\lambda \hat{w}_0(\beta)))^L W_\lambda$. In particular, the centered fundamental spatial loop $W_{\mathbf{N}}(u) = \operatorname{Tr}(u_0 \cdots u_{L-1})$ has strictly positive physical reflected norm $\|W_{\mathbf{N}}\|^2_{\mathrm{phys}} = (\hat{w}_{\mathbf{N}}(\beta)/(N \hat{w}_0(\beta)))^L > 0$ for all $\beta > 0$ (vanishing at $\beta = 0$). For $\mathrm{SU}(3)$, Peter–Weyl Parseval plus the Weyl dimension bound $d_{(p,q)} \geq 6$ outside $\{\mathbf{1}, \mathbf{3}, \overline{\mathbf{3}}\}$ yields a one-line exact fundamental-dominance test: whenever $Z_{\mathrm{SU}(3)}(2\beta/3) < Z_{\mathrm{SU}(3)}(\beta/3)^2 + 6 Z'_{\mathrm{SU}(3)}(\beta/3)^2$, the exact mass gap is $\Delta_{\mathrm{cyl}}(3, L, \beta) = L \ln(3\hat{w}_0/\hat{w}_{\mathbf{3}})$ for all $L \geq 2$.

(3) *Four-line Riccati–Bessel all-coupling enclosure for $\mathrm{SU}(2)$ and universal Casimir scaling:* Solving the Riccati ODE $y'_n = 1 - \frac{2n+1}{x} y_n - y_n^2$ for $y_n = I_{n+1}/I_n$ proves for *every* $\beta_W > 0$ and $L \geq 2$ the sharp double-sided mass-gap enclosure $L \operatorname{arsinh}(3/(2\beta_W)) < \Delta_{\mathrm{cyl}}(2, L, \beta_W) < L \operatorname{arsinh}(2/\beta_W)$ (matching the exact weak-coupling Casimir limit $3L/(2\beta_W)$ as $\beta_W \to \infty$), alongside the universal $\mathrm{SU}(N)$ convexity norm bound and exact weak-coupling Casimir law $\lim_{\beta \to \infty} \beta(-\ln \tau_\lambda) = LNC_2(\lambda)$.

(4) *Spatial-coupling-independent all-coupling mass gap on arbitrary finite spatial slabs $\mathbb{Z} \times \Gamma$:* By a self-contained Doob h-transform and Birkhoff–Hopf cross-ratio argument in which both the spatial plaquette action Ψ_{sp} and the unknown Perron–Frobenius ground state Ω cancel identically, every finite connected spatial graph $\Gamma = (V, E)$ with first Betti number $b_1(\Gamma) = |E| - |V| + 1 \geq 1$ obeys $\Delta_\Gamma(\beta_t, \beta_{\mathrm{sp}}) \geq -\ln \tanh(\frac{1+\delta_N}{2} |E| \beta_t) > 0$ for every timelike coupling $\beta_t > 0$ and *every* spatial plaquette coupling $\beta_{\mathrm{sp}} \in \mathbb{R}$.

While infinite-dimensional functional analysis is proved analytically in Sections 2–5, the group-theoretic crossing/covariance identities, integer Birkhoff–Hopf core, SU(3) Weyl bound, directed-rounding soundness, and 20 certificates (SU(2) cylinders $\beta_W \in [0.1, 10]$, SU(3) cylinders $\beta \in [0.3, 3.0]$ via 25 exact rational Weyl moments, and 2D/3D torus slabs) are verified in 60-digit interval arithmetic and a standalone zero-import, zero-`sorry`, zero-`Classical.choice` Lean 4.36.0 kernel.

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1. Introduction and Honest Scope

Osterwalder–Schrader (OS) reflection positivity [7, 8] is the structural bridge connecting Euclidean Wilson lattice gauge theory [10] to a physical Hilbert space $\mathcal{H}_{\mathrm{OS}}$, a positive self-adjoint transfer operator T, and the physical Hamiltonian mass gap $\Delta := -\ln\|T|_{\Omega^\perp}\|$. In constructive and mechanized verification programmes—notably the `R3/R4` physical reflection campaign of `THE-ERIKSSON-PROGRAMME`—a central concrete obstacle has been to pass cleanly from literal multi-link Wilson crossing integrals (retaining all independent timelike crossing links $a_0, \dots, a_{L-1}$ without collapsing them into a single common transporter) to: (i) commuting gauge-projection reflection positivity on a genuine closed-loop spatial slice, (ii) exact slice marginality and centering of a nonconstant gauge-invariant spatial loop observable such as $\mathrm{Tr}(u_0 u_1)$, (iii) a *derived* strictly positive centered reflected norm at every $\beta > 0$ (explicitly separated from $\beta = 0$, where Haar independence annihilates centered reflected fluctuations), and (iv) an explicit physical transfer operator T with a certified spectral gap $\Delta > 0$ valid across *all* couplings $\beta > 0$.

Our companion papers [4, 3] established volume-uniform complex zero-free discs and strong-coupling Dobrushin–Wasserstein mass gaps on d-dimensional $\mathrm{SU}(N)$ lattices at small β. The present work addresses the orthogonal, *all-coupling* direction: exact operator-level reflection positivity, exact character diagonalization on L-link $\mathrm{SU}(N)$ cylinders (generalizing classic 2D character expansions [6, 11] into sharp all-coupling Riccati–Bessel and Parseval spectral certificates), and an explicit spatial-coupling-independent transfer-matrix mass gap on arbitrary finite spatial graphs Γ (including 2D and 3D spatial tori with arbitrary spatial plaquette coupling $\beta_{\mathrm{sp}} \in \mathbb{R}$).

Honest scope. All spectral-gap theorems in this paper apply to *fixed finite spatial cross-sections* $\Gamma = (V, E)$ with $b_1(\Gamma) := |E| - |V| + 1 \geq 1$ (specifically L-link circular cylinders C_L for $L \geq 2$ and arbitrary finite connected spatial graphs Γ containing at least one closed loop; on trees $b_1(\Gamma) = 0$, $\mathcal{H}_{\mathrm{OS}}(\Gamma)$ is one-dimensional and the gap is trivially $+\infty$). On L-link cylinders, the mass gap remains strictly positive for all $\beta > 0$, scaling linearly with L as $\Delta_{\mathrm{cyl}}(N, L, \beta) = O(L/\beta)$ at weak coupling ($\beta \to \infty$) while diverging logarithmically as $\sim L \ln(1/\beta)$ at strong coupling ($\beta \to 0^+$); on general higher-dimensional slabs $\mathbb{Z} \times \Gamma$, the Birkhoff–Hopf lower bound $-\ln\tanh(\frac{1+\delta_N}{2}|E|\beta_t)$ decays exponentially as $|E| \to \infty$ at fixed $\beta_t > 0$. Accordingly, this paper does *not* prove a volume-uniform 4D infinite-volume mass gap at weak coupling, the 4D continuum limit $a \to 0$, or the Clay Millennium Yang–Mills Existence and Mass Gap problem.

2. Gauge Projection and Unconditional Reflection Positivity on Arbitrary Graphs

Let $G = \mathrm{SU}(N)$ ($N \geq 2$) equipped with normalized Haar measure dg, and let $\Gamma = (V, E)$ be any finite connected oriented spatial graph (multi-edges permitted; e.g., the two-link cycle C_2 with $|V| = |E| = 2$ of Euler characteristic $\chi = 0$, any L-link cycle C_L with $L \geq 2$, or a $(d-1)$-dimensional spatial torus $\mathbb{T}^{d-1}_{L_1,\dots,L_{d-1}}$ with spatial plaquettes $\mathcal{P}(\Gamma)$). Each oriented edge $e \in E$ has source $s(e) \in V$ and target $t(e) \in V$.

On a two-slice spacetime slab over Γ, a link configuration consists of lower-slice links $u = (u_e)_{e \in E} \in G^E$, upper-slice links $v = (v_e)_{e \in E} \in G^E$, and independent crossing (timelike) links $a = (a_x)_{x \in V} \in G^V$. Each edge $e \in E$ determines a crossing plaquette with oriented holonomy

$$H_e(u, v, a) := u_e \, a_{t(e)} \, v_e^{-1} \, a_{s(e)}^{-1}. \tag{1}$$

For coupling $\beta \geq 0$, define the single-plaquette Wilson Boltzmann weight

$$w_\beta(U) := \exp\left(\frac{\beta}{N} \operatorname{Re}\mathrm{Tr}(U)\right), \qquad U \in \mathrm{SU}(N). \tag{2}$$

On $L^2(G^E) := L^2(G^E, \mathrm{Haar}^{\otimes E})$ with inner product $\langle \phi, \psi \rangle := \int_{G^E} \overline{\phi(u)}\, \psi(u)\, du$, define the vertex gauge action $T_a : G^E \to G^E$, the gauge-averaging operator P, and the factorized crossing integral operator K_β by

$$(T_a v)_e := a_{s(e)} v_e a_{t(e)}^{-1}, \qquad (Pf)(v) := \int_{G^V} f(T_a v)\, da,$$

$$(K_\beta f)(u) := \int_{G^E} \left(\prod_{e \in E} w_\beta(u_e v_e^{-1}) \right) f(v)\, dv. \tag{3}$$

Let $\Psi_{\mathrm{sp}} : G^E \to (0, \infty)$ be any continuous gauge-invariant intra-slice spatial weight ($\Psi_{\mathrm{sp}}(T_a u) = \Psi_{\mathrm{sp}}(u)$ for all $a \in G^V$, e.g., $\Psi_{\mathrm{sp}}(u) = \exp(\frac{\beta_{\mathrm{sp}}}{2N} \sum_{p \in \mathcal{P}(\Gamma)} \mathrm{Re}\,\mathrm{Tr}\, U_p)$, or $\Psi_{\mathrm{sp}} \equiv 1$ when $\mathcal{P}(\Gamma) = \emptyset$). The literal two-slice physical partition function and reflected sesquilinear form for $f, g \in C(G^E)$ are

$$Z_\Gamma(\beta, \Psi_{\mathrm{sp}}) := \int_{G^E \times G^E \times G^V} \Psi_{\mathrm{sp}}(u)\, \Psi_{\mathrm{sp}}(v) \prod_{e \in E} w_\beta\big(H_e(u, v, a)\big)\, du\, dv\, da, \tag{4}$$

$$\langle f, g \rangle_{\mathrm{phys}} := \frac{1}{Z_\Gamma(\beta, \Psi_{\mathrm{sp}})} \int_{G^E \times G^E \times G^V} \overline{f(u)}\, g(v)\, \Psi_{\mathrm{sp}}(u)\, \Psi_{\mathrm{sp}}(v) \prod_{e \in E} w_\beta\big(H_e(u, v, a)\big)\, du\, dv\, da. \tag{5}$$

Theorem 2.1 (Commuting gauge projection and unconditional reflection positivity)**.** *For any finite connected oriented graph $\Gamma = (V, E)$, any $N \geq 2$, any $\beta \geq 0$, and any continuous gauge-invariant $\Psi_{\mathrm{sp}} > 0$:*

(1) (Orthogonal gauge projection) *T_a preserves $\mathrm{Haar}^{\otimes E}$ for every $a \in G^V$, obeys $T_a \circ T_b = T_{ab}$, and P is a self-adjoint orthogonal projection ($P = P^* = P^2$) onto the gauge-invariant subspace $L^2(G^E)^{\mathrm{inv}}$.*

(2) (Literal crossing identity and commutation) *For every $(u, v, a) \in G^E \times G^E \times G^V$,*

$$\prod_{e \in E} w_\beta\big(H_e(u, v, a)\big) = \prod_{e \in E} w_\beta\big(u_e (T_a v)_e^{-1}\big) =: K_\beta(u, T_a v), \tag{6}$$

and $K_\beta(T_b u, T_b v) = K_\beta(u, v)$ for all $b \in G^V$. Consequently, $PK_\beta = K_\beta P = PK_\beta P$ on $L^2(G^E)$.

(3) (Strict positive-definiteness and trace class) *Let $\widehat{\mathrm{SU}(N)}$ index the equivalence classes of irreducible unitary representations $(\rho^\lambda, \mathcal{V}_\lambda)$ with character χ_λ and dimension d_λ, and write $\hat{w}_0(\beta) := \hat{w}_{\mathbf{1}}(\beta)$ for the trivial representation $\mathbf{1}$. Every character coefficient $\hat{w}_\lambda(\beta) := \int_{\mathrm{SU}(N)} w_\beta(U)\, \overline{\chi_\lambda(U)}\, dU$ satisfies $\hat{w}_\lambda(\beta) > 0$ strictly for all $\beta > 0$ (and $\hat{w}_\lambda(0) = \delta_{\lambda,\mathbf{1}}$). The operator K_β is compact, self-adjoint, trace-class with $\mathrm{Tr}_{L^2(G^E)}(K_\beta) = e^{|E|\beta}$, strictly positive-definite on $L^2(G^E)$ for every $\beta > 0$, and positive semi-definite at $\beta = 0$.*

(4) (Unconditional reflected positivity) *For every $f \in C(G^E)$ and every $\beta \geq 0$,*

$$\langle f, f \rangle_{\mathrm{phys}} = \frac{1}{Z_\Gamma(\beta, \Psi_{\mathrm{sp}})} \big\langle P(\Psi_{\mathrm{sp}} f),\, K_\beta\, P(\Psi_{\mathrm{sp}} f) \big\rangle_{L^2(G^E)} \;\geq\; 0, \tag{7}$$

so every finite complex reflected Gram matrix $\big(\langle f_i, f_j \rangle_{\mathrm{phys}}\big)_{i,j=1}^m$ is Hermitian positive semi-definite.

Proof. (1) Bi-invariance of Haar measure on G implies that $v_e \mapsto a_{s(e)} v_e a_{t(e)}^{-1}$ preserves dv_e on each edge, hence T_a is unitary on $L^2(G^E)$ with $T_a T_b = T_{ab}$ and $T_a^{-1} = T_{a^{-1}}$. By Fubini's theorem and $a \mapsto a^{-1}$ Haar invariance on G^V, $\langle Pf, g \rangle = \int da \int \overline{f(T_a u)} g(u)\, du = \int da \int \overline{f(u)} g(T_{a^{-1}} u)\, du = \langle f, Pg \rangle$, and $(P^2 f)(v) = \int da \int db\, f(T_{ab} v) = (Pf)(v)$ by left Haar invariance.

(2) Since $(T_a v)_e^{-1} = (a_{s(e)} v_e a_{t(e)}^{-1})^{-1} = a_{t(e)} v_e^{-1} a_{s(e)}^{-1}$, multiplying on the left by u_e yields $u_e (T_a v)_e^{-1} = H_e(u, v, a)$, giving (6). Similarly,

$$(T_b u)_e (T_b v)_e^{-1} = (b_{s(e)} u_e b_{t(e)}^{-1})(b_{t(e)} v_e^{-1} b_{s(e)}^{-1}) = b_{s(e)} (u_e v_e^{-1}) b_{s(e)}^{-1};$$

because w_β is a class function ($w_\beta(gXg^{-1}) = w_\beta(X)$), $K_\beta(T_b u, T_b v) = K_\beta(u, v)$ identically. (Both identities are also mechanically checked for abstract groups in `CylinderReflectionCerts.lean`.) Changing variables $w = T_a v$ in $(K_\beta P f)(u)$ gives

$$(K_\beta P f)(u) = \int_{G^V} da \int_{G^E} K_\beta(u, T_{a^{-1}} w) f(w)\, dw = \int_{G^V} da \int_{G^E} K_\beta(T_a u, w) f(w)\, dw = (P K_\beta f)(u).$$

Applying $P = P^2$ yields $P K_\beta = K_\beta P = P K_\beta P$.

(3) Write $\frac{\beta}{N} \operatorname{Re} \operatorname{Tr}(U) = \frac{\beta}{2N} (\operatorname{Tr} U + \overline{\operatorname{Tr} U})$ and expand the uniformly convergent series

$$w_\beta(U) = \sum_{a,b \geq 0} \frac{(\beta/(2N))^{a+b}}{a!\, b!} (\operatorname{Tr} U)^a (\overline{\operatorname{Tr} U})^b.$$

Integrating against $\overline{\chi_\lambda(U)}$ termwise and invoking Schur orthogonality [9] yields

$$\hat{w}_\lambda(\beta) = \sum_{a,b \geq 0} \frac{(\beta/(2N))^{a+b}}{a!\, b!}\, m_\lambda(a, b), \qquad m_\lambda(a, b) := \dim \operatorname{Hom}_{\mathrm{SU}(N)}\big(\lambda,\, \mathbf{N}^{\otimes a} \otimes \overline{\mathbf{N}}^{\otimes b}\big) \in \mathbb{Z}_{\geq 0}. \quad (8)$$

Because the defining representation $\mathbf{N}$ is faithful, the $*$-algebra generated by matrix entries of $\mathbf{N}$ and $\overline{\mathbf{N}}$ separates points of $\mathrm{SU}(N)$ and is therefore dense in $C(\mathrm{SU}(N))$ by Stone–Weierstrass. Consequently every $\lambda \in \widehat{\mathrm{SU}(N)}$ satisfies $m_\lambda(a, b) \geq 1$ for some (a, b), proving $\hat{w}_\lambda(\beta) > 0$ strictly for every $\beta > 0$!

On a single link, let $(W_\beta \phi)(u) := \int_G w_\beta(uv^{-1}) \phi(v)\, dv$. By Peter–Weyl [9], $L^2(G) = \bigoplus_{\lambda \in \widehat{G}} \mathcal{M}_\lambda$ where $\mathcal{M}_\lambda = \operatorname{span}\{\rho^\lambda_{rs} : 1 \leq r, s \leq d_\lambda\}$ has dimension d_λ^2. Since $w_\beta(g) = \sum_\lambda \hat{w}_\lambda(\beta) \chi_\lambda(g)$ converges in $L^2(G)$ (indeed uniformly because $w_\beta \in C^\infty(G)$) and Schur orthogonality gives

$$\int_G \chi_\lambda(uv^{-1})\, \rho^\mu_{rs}(v)\, dv = \sum_{k,m=1}^{d_\lambda} \rho^\lambda_{km}(u) \int_G \overline{\rho^\lambda_{km}(v)}\, \rho^\mu_{rs}(v)\, dv = \frac{\delta_{\lambda\mu}}{d_\lambda}\, \rho^\lambda_{rs}(u),$$

W_β acts on each block $\mathcal{M}_\lambda$ as the scalar $\hat{w}_\lambda(\beta)/d_\lambda > 0$. Its trace is $\sum_\lambda d_\lambda^2 \frac{\hat{w}_\lambda(\beta)}{d_\lambda} = \sum_\lambda d_\lambda \hat{w}_\lambda(\beta) = w_\beta(I_N) = e^\beta < \infty$ (since $\chi_\lambda(I_N) = d_\lambda$). As $K_\beta = W_\beta^{\otimes E}$ on $L^2(G^E) = \bigotimes_{e \in E} L^2(G)$, K_β is compact, self-adjoint, trace-class with $\operatorname{Tr}_{L^2(G^E)}(K_\beta) = e^{|E|\beta}$, strictly positive-definite for $\beta > 0$, and equals the rank-1 projection onto constants at $\beta = 0$.

(4) Substituting (6) into (5), changing variables $w = T_a v$ (using $\Psi_{\mathrm{sp}}(T_{a^{-1}} w) = \Psi_{\mathrm{sp}}(w)$ and $a \mapsto a^{-1}$), and applying $K_\beta P = P K_\beta P$ together with $P = P^*$ yields

$$Z_\Gamma(\beta, \Psi_{\mathrm{sp}}) \langle f, f \rangle_{\mathrm{phys}} = \langle \Psi_{\mathrm{sp}} f,\, K_\beta P(\Psi_{\mathrm{sp}} f) \rangle_{L^2(G^E)} = \langle P(\Psi_{\mathrm{sp}} f),\, K_\beta\, P(\Psi_{\mathrm{sp}} f) \rangle_{L^2(G^E)} \geq 0.$$

Hermitian positive semi-definiteness of any finite Gram matrix $(\langle f_i, f_j \rangle_{\mathrm{phys}})$ follows immediately from $\sum_{i,j} \overline{c_i} c_j \langle f_i, f_j \rangle_{\mathrm{phys}} = \langle \sum_i c_i f_i, \sum_i c_i f_i \rangle_{\mathrm{phys}} \geq 0$. □

3. Exact Character Diagonalization and Centered Norms on L-Link Cylinders

We now specialize Theorem 2.1 to the L-link spatial cycle $\Gamma = C_L$ ($L \geq 2$), with vertices $\mathbb{Z}/L\mathbb{Z}$, edges $e_j = (j \to j+1)$ ($j \in \mathbb{Z}/L\mathbb{Z}$), no spatial plaquettes ($\Psi_{\mathrm{sp}} \equiv 1$), and Euler characteristic $\chi = 0$. For $L = 2$, this is literally the six-link two-plaquette cylinder of `THE-ERIKSSON-PROGRAMME R2/R3`. For each $\lambda \in \widehat{\mathrm{SU}(N)}$, define the gauge-invariant spatial Wilson-loop character observable

$$W_\lambda(u_0, \ldots, u_{L-1}) := \chi_\lambda(u_0 u_1 \cdots u_{L-1}). \quad (9)$$

Theorem 3.1 (Exact diagonalization, centering, and Parseval dominance on L-link cylinders)**.** *Let $L \geq 2$, $N \geq 2$, and $\beta \geq 0$ on $\Gamma = C_L$. Then:*

(1) (Exact partition function and Haar slice marginal) *The two-slice cylinder partition function is $Z_L(\beta) = \hat{w}_0(\beta)^L$, where $\hat{w}_0(\beta) = Z_{\mathrm{SU}(N)}(\beta/N) := \int_{\mathrm{SU}(N)} \exp(\frac{\beta}{N} \operatorname{Re}\operatorname{Tr} U)\, dU$. The physical slice marginal of $u \in G^L$ is literally $\mathrm{Haar}^{\otimes L}$. Consequently, every nontrivial loop character W_λ ($\lambda \neq \mathbf{1}$) is gauge-invariant and identically centered: $\mathbb{E}_{\mathrm{phys}}[W_\lambda] = 0$.*

(2) (Exact isometric identification and spectral diagonalization) *The pullback $(\Phi\psi)(u) := \psi(u_0 \cdots u_{L-1})$ is a unitary isomorphism $\Phi : L^2_{\mathrm{class}}(\mathrm{SU}(N)) \xrightarrow{\sim} \mathcal{H}_{\mathrm{OS}}(C_L) := \mathrm{Ran}(P)$, with complete orthonormal basis $\{W_\lambda : \lambda \in \widehat{\mathrm{SU}(N)}\}$. The normalized physical transfer operator $T_{L,\beta} := Z_L(\beta)^{-1} P K_\beta P$ on $\mathcal{H}_{\mathrm{OS}}(C_L)$ is compact, positive, self-adjoint, and diagonal in $\{W_\lambda\}$:*

$$T_{L,\beta} W_\lambda = \tau_\lambda(N, L, \beta)\, W_\lambda, \qquad \tau_\lambda(N, L, \beta) := r_\lambda(\beta)^L, \qquad r_\lambda(\beta) := \frac{\hat{w}_\lambda(\beta)}{d_\lambda\, \hat{w}_0(\beta)}, \tag{10}$$

with $\tau_{\mathbf{1}} = 1$, $0 < \tau_\lambda(N, L, \beta) < 1$ strictly for all $\lambda \neq \mathbf{1}$ and all $\beta > 0$, and $\tau_\lambda(N, L, 0) = 0$ for all $\lambda \neq \mathbf{1}$.

(3) (Strictly positive centered reflected norm and multi-time correlations) *For every $\beta > 0$, the centered fundamental spatial loop $W_{\mathbf{N}}(u) = \mathrm{Tr}(u_0 \cdots u_{L-1})$ satisfies*

$$\|W_{\mathbf{N}}\|^2_{\mathrm{phys}} = \langle W_{\mathbf{N}},\, T_{L,\beta} W_{\mathbf{N}} \rangle_{L^2(G^L)} = \tau_{\mathbf{N}}(N, L, \beta) = \left(\frac{\hat{w}_{\mathbf{N}}(\beta)}{N\, \hat{w}_0(\beta)} \right)^L \; > \; 0, \tag{11}$$

while $\|W_{\mathbf{N}}\|^2_{\mathrm{phys}} = 0$ at $\beta = 0$. On any T-step open temporal cylinder ($T \geq t \geq 1$)—where open boundary slices integrate out via $T_{L,\beta}\mathbf{1} = \mathbf{1}$ to yield the stationary measure $\mu_{\mathrm{cyl},\beta}$—any two observables $f = \sum_\lambda \hat{f}_\lambda W_\lambda$ and $g = \sum_\lambda \hat{g}_\lambda W_\lambda$ in $\mathcal{H}_{\mathrm{OS}}(C_L)$ obey the exact sesquilinear covariance identity (with $\mathrm{Cov}(X, Y) := \mathbb{E}[\overline{(X - \mathbb{E}[X])}(Y - \mathbb{E}[Y])]$)

$$\mathrm{Cov}_{\mu_{\mathrm{cyl},\beta}}\big(f(u^{(0)}),\, g(u^{(t)})\big) = \sum_{\lambda \neq \mathbf{1}} \overline{\hat{f}_\lambda}\, \hat{g}_\lambda\, \tau_\lambda(N, L, \beta)^t \qquad (t \in \mathbb{N}_{\geq 1}). \tag{12}$$

(4) (All-coupling strict mass gap and exact SU(3) Parseval dominance test) *For every $N \geq 2$, $L \geq 2$, and $\beta > 0$, the exact cylinder mass gap $\Delta_{\mathrm{cyl}}(N, L, \beta) := -\ln \max_{\lambda \neq \mathbf{1}} \tau_\lambda(N, L, \beta)$ is strictly positive. For $N = 3$, set $t := \beta/3$ and $Z(t) := Z_{\mathrm{SU}(3)}(t)$, so $\hat{w}_0(3t) = Z(t)$ and $\hat{w}_{\mathbf{3}}(3t) = Z'(t)$. Then every irrep $\lambda \notin \{\mathbf{1}, \mathbf{3}, \overline{\mathbf{3}}\}$ obeys*

$$\frac{\hat{w}_\lambda(3t)}{d_\lambda\, \hat{w}_0(3t)} \;\leq\; R_3(3t) := \frac{\sqrt{Z(2t) - Z(t)^2 - 2Z'(t)^2}}{6\, Z(t)}. \tag{13}$$

In particular, whenever the single scalar inequality

$$Z(2t) \;<\; Z(t)^2 + 6\, Z'(t)^2 \tag{14}$$

holds, $\mathbf{3}$ and $\overline{\mathbf{3}}$ are rigorously *the unique dominant subleading irreps, and the exact mass gap equals $\Delta_{\mathrm{cyl}}(3, L, 3t) = L \ln\big(3Z(t)/Z'(t)\big)$ simultaneously for* every *$L \geq 2$!*

Proof. (1) For any bounded measurable $g(u)$, fix $(u, a) \in G^L \times G^L$ and change variables from $v \in G^L$ to $h \in G^L$ via $h_j := u_j a_{j+1} v_j^{-1} a_j^{-1}$ (inverse $v_j = a_j^{-1} h_j^{-1} u_j a_{j+1}$, Haar-preserving on each coordinate). Integrating over h produces $\prod_{j=0}^{L-1} \int_G w_\beta(h_j)\, dh_j = \hat{w}_0(\beta)^L$ independent of (u, a). Taking $g \equiv 1$ gives $Z_L(\beta) = \hat{w}_0(\beta)^L$, and dividing by $Z_L(\beta)$ gives $\mathbb{E}_{\mathrm{phys}}[g(u)] = \int_{G^L} g(u)\, du$. By path gauge covariance (checked inductively in `CylinderReflectionCerts.lean`), $(T_a u)_0 \cdots (T_a u)_{L-1} = a_0 (u_0 \cdots u_{L-1}) a_0^{-1}$, so $W_\lambda(T_a u) = W_\lambda(u)$ and $\mathbb{E}_{\mathrm{phys}}[W_\lambda] = \int_G \chi_\lambda(g)\, dg = 0$ for $\lambda \neq \mathbf{1}$.

(2) Clearly Φ is an isometry into $\mathrm{Ran}(P)$ by Haar invariance of $g = u_0 \cdots u_{L-1}$. Conversely, if $f \in C(G^L) \cap \mathrm{Ran}(P)$, setting $a_0 = I_N$ and $a_j = u_0 \cdots u_{j-1}$ $(1 \le j \le L-1)$ gives $(T_a u)_j = a_j u_j a_{j+1}^{-1} = I_N$ for $0 \le j \le L-2$ and $(T_a u)_{L-1} = (u_0 \cdots u_{L-2}) u_{L-1} I_N^{-1} = u_0 \cdots u_{L-1}$; since $f(u) = f(T_a u)$, we obtain $f(u) = \psi(u_0 \cdots u_{L-1})$ with $\psi(g) := f(I_N, \dots, I_N, g)$, and applying uniform $a_j \equiv h$ shows ψ is a class function. Density of $C(G^L) \cap \mathrm{Ran}(P)$ in $\mathrm{Ran}(P)$ yields $\Phi(L^2_{\mathrm{class}}(G)) = \mathrm{Ran}(P)$, and Peter–Weyl shows $\{W_\lambda\}$ is a complete orthonormal basis of $\mathrm{Ran}(P)$.

For the eigenvalue identity, substitute $h_j = u_j v_j^{-1}$ $(v_j = h_j^{-1} u_j)$ into $(K_\beta W_\lambda)(u)$ and expand $w_\beta(h_j) = \sum_{\mu_j} \hat{w}_{\mu_j}(\beta) \chi_{\mu_j}(h_j)$ uniformly:

$$\begin{aligned}(K_\beta W_\lambda)(u) &= \int_{G^L} \left(\prod_{j=0}^{L-1} w_\beta(h_j) \right) \sum_{i,k} \prod_{j=0}^{L-1} \overline{\rho^\lambda_{k_j i_j}(h_j)}\, \rho^\lambda_{k_j i_{j+1}}(u_j)\, dh \\ &= \sum_{i,k} \prod_{j=0}^{L-1} \left(\frac{\hat{w}_\lambda(\beta)}{d_\lambda} \delta_{k_j i_j} \right) \rho^\lambda_{k_j i_{j+1}}(u_j) = \left(\frac{\hat{w}_\lambda(\beta)}{d_\lambda} \right)^L W_\lambda(u),\end{aligned}$$

where indices j are taken modulo L. Dividing by $Z_L(\beta) = \hat{w}_0(\beta)^L$ yields (10). For $\lambda \neq \mathbf{1}$ and $\beta > 0$, unitarity of $\rho^\lambda(U)$ gives $\mathrm{Re}(\chi_\lambda(U)/d_\lambda) \le 1$ with equality if and only if every eigenvalue of $\rho^\lambda(U)$ equals 1, i.e., $U \in \ker(\rho^\lambda) \subsetneq \mathrm{SU}(N)$ (a proper closed subgroup of Haar measure zero); since $w_\beta(U) > 0$ everywhere and $\hat{w}_\lambda(\beta)$ is real, $r_\lambda(\beta) = \mathbb{E}_{\nu_\beta}[\mathrm{Re}(\chi_\lambda/d_\lambda)] < 1$ strictly, while Theorem 2.1(3) gives strict positivity $r_\lambda(\beta) > 0$.

(3) Equation (11) follows directly from (7) and (10) since $PW_{\mathbf{N}} = W_{\mathbf{N}}$ and $\|W_{\mathbf{N}}\|_{L^2} = 1$. On a T-step cylinder, composing t commuting slabs yields $(T^t_{L,\beta} W_\lambda) = \tau_\lambda^t W_\lambda$, giving (12).

(4) Since $T_{L,\beta}$ is compact and $\tau_\lambda \in (0,1)$ for all $\lambda \neq \mathbf{1}$, its eigenvalues accumulate only at 0, so $\max_{\lambda \neq \mathbf{1}} \tau_\lambda < 1$ and $\Delta_{\mathrm{cyl}} > 0$. For $N = 3$, conjugation symmetry $U \mapsto U^*$ gives $\hat{w}_{\mathbf{3}}(3t) = \hat{w}_{\overline{\mathbf{3}}}(3t) = \int_{\mathrm{SU}(3)} (\mathrm{Re}\,\mathrm{Tr}\, U) e^{t\,\mathrm{Re}\,\mathrm{Tr}\, U}\, dU = Z'(t)$. By Peter–Weyl Parseval on $L^2_{\mathrm{class}}(\mathrm{SU}(3))$,

$$\sum_{\lambda \in \widehat{\mathrm{SU}(3)}} \hat{w}_\lambda(3t)^2 = \int_{\mathrm{SU}(3)} w_{3t}(U)^2\, dU = Z(2t).$$

Subtracting $\lambda \in \{\mathbf{1}, \mathbf{3}, \overline{\mathbf{3}}\}$ gives $\sum_{\lambda \notin \{\mathbf{1},\mathbf{3},\overline{\mathbf{3}}\}} \hat{w}_\lambda(3t)^2 = Z(2t) - Z(t)^2 - 2Z'(t)^2$. By the Weyl dimension formula $d_{(p,q)} = \frac{1}{2}(p+1)(q+1)(p+q+2)$ (mechanically checked in `su3_weyl_dim_ge_6`), every irrep outside $\{\mathbf{1}, \mathbf{3}, \overline{\mathbf{3}}\}$ has $d_\lambda \ge 6$, proving (13). Finally, $R_3(3t) < r_{\mathbf{3}}(3t) = Z'(t)/(3Z(t))$ squares directly to $(Z(2t) - Z(t)^2 - 2Z'(t)^2)/(36Z(t)^2) < Z'(t)^2/(9Z(t)^2)$, which simplifies identically to (14). □

Remark 3.2 (Coupling regime of the Parseval dominance criterion). Criterion (14) is an exact, L-independent strong-to-intermediate coupling certificate (verified below up to $\beta = 3.0$, $t = 1.0$). At asymptotically large $t \to \infty$, Laplace concentration around $I_3 \in \mathrm{SU}(3)$ ($\dim \mathrm{SU}(3) = 8$) gives $Z(t) \sim c\, t^{-4} e^{3t}$, so $Z(2t) \sim c(2t)^{-4} e^{6t}$ eventually exceeds $Z(t)^2 + 6Z'(t)^2 \sim c' t^{-8} e^{6t}$, complementing the weak-coupling Casimir law in Theorem 4.1(3).

4. Sharp All-Coupling Riccati–Bessel Enclosures and Casimir Scaling

We now establish explicit, non-perturbative all-coupling enclosures valid for *every* $\beta > 0$ (both strong coupling $\beta \to 0^+$ and weak coupling $\beta \to \infty$), together with exact weak-coupling Casimir asymptotics.

Theorem 4.1 (All-coupling Riccati–Bessel bounds for SU(2) and Casimir scaling for SU(N))**.** *Let $L \ge 2$.*

(1) (Exact SU(2) Bessel spectrum and four-line Riccati enclosure) *For $G = \mathrm{SU}(2)$ with standard Wilson coupling $\beta_W := \beta$ (so $w_\beta(U) = \exp(\frac{\beta_W}{2} \operatorname{Re}\operatorname{Tr} U) = \exp(\beta_W \cos\theta)$) and every $j \in \frac{1}{2}\mathbb{Z}_{\geq 0}$,*

$$\frac{\hat{w}_j(\beta_W)}{d_j\, \hat{w}_0(\beta_W)} = \frac{I_{2j+1}(\beta_W)}{I_1(\beta_W)}. \tag{15}$$

For every integer $n \geq 1$ and every $x > 0$, the modified Bessel ratio $y_n(x) := I_{n+1}(x)/I_n(x)$ satisfies the sharp double-sided inequality (cf. [1])

$$\frac{x}{(n+1) + \sqrt{x^2 + (n+1)^2}} \;<\; \frac{I_{n+1}(x)}{I_n(x)} \;<\; \frac{x}{(n+\frac{1}{2}) + \sqrt{x^2 + (n+\frac{1}{2})^2}} \;<\; 1. \tag{16}$$

Consequently, $1 > \tau_{1/2} > \tau_1 > \tau_{3/2} > \cdots > 0$ strictly for all $\beta_W > 0$, the exact $\mathrm{SU}(2)$ cylinder mass gap is $\Delta_{\mathrm{cyl}}(2, L, \beta_W) = L \ln(I_1(\beta_W)/I_2(\beta_W))$, and for every $\beta_W > 0$:

$$\left(\frac{\beta_W}{2 + \sqrt{\beta_W^2 + 4}}\right)^L \;<\; \|\mathrm{Tr}(u_0 \cdots u_{L-1})\|^2_{\mathrm{phys}} \;<\; \left(\frac{\beta_W}{\frac{3}{2} + \sqrt{\beta_W^2 + \frac{9}{4}}}\right)^L, \tag{17}$$

$$L\, \operatorname{arsinh}\!\left(\frac{3}{2\beta_W}\right) \;<\; \Delta_{\mathrm{cyl}}(2, L, \beta_W) \;<\; L\, \operatorname{arsinh}\!\left(\frac{2}{\beta_W}\right). \tag{18}$$

(2) (Universal $\mathrm{SU}(N)$ all-coupling centered norm bound) *For every $N \geq 2$, $L \geq 2$, and $\beta > 0$, setting $\mu_2 = 1$ ($N = 2$) and $\mu_2 = 1/2$ ($N \geq 3$),*

$$\frac{\hat{w}_{\mathbf{N}}(\beta)}{N\, \hat{w}_0(\beta)} \;\geq\; \frac{1}{\beta} \ln\!\left(1 + \frac{\mu_2\, \beta^2}{2N^2}\right) \;>\; 0, \tag{19}$$

providing a closed-form strictly positive lower bound on $\|W_{\mathbf{N}}\|^2_{\mathrm{phys}}$ for all $N \geq 2$ and $\beta > 0$.

(3) (Exact weak-coupling Casimir scaling for all $\mathrm{SU}(N)$) *For every $N \geq 2$, $L \geq 2$, and every $\lambda \in \widehat{\mathrm{SU}(N)}$,*

$$\lim_{\beta \to \infty} \beta \cdot \big(- \ln \tau_\lambda(N, L, \beta) \big) = L\, N\, C_2(\lambda), \tag{20}$$

where $C_2(\lambda)$ is the quadratic Casimir eigenvalue normalized so $\mathrm{Tr}(T^a T^b) = \frac{1}{2}\delta_{ab}$ (in particular $C_2(\mathbf{N}) = \frac{N^2-1}{2N}$, giving $\lim_{\beta\to\infty} \beta(-\ln \tau_{\mathbf{N}}) = \frac{L(N^2-1)}{2}$—e.g., $\frac{3}{2}L$ for $\mathrm{SU}(2)$ and $4L$ for $\mathrm{SU}(3)$—which matches the lower bound (18) to exact leading order!).

Proof. (1) On $\mathrm{SU}(2)$, the Weyl integration formula on eigenangle $\theta \in [0, \pi]$ ($\operatorname{Re}\operatorname{Tr} U = 2\cos\theta$, $\chi_j(\theta) = \frac{\sin((2j+1)\theta)}{\sin\theta}$, $d_j = 2j+1$) yields

$$\begin{aligned}
\hat{w}_j(\beta_W) &= \frac{2}{\pi} \int_0^\pi \sin\theta\, \sin((2j+1)\theta)\, e^{\beta_W \cos\theta}\, d\theta \\
&= \frac{1}{\pi} \int_0^\pi \big(\cos(2j\theta) - \cos((2j+2)\theta) \big)\, e^{\beta_W \cos\theta}\, d\theta \\
&= I_{2j}(\beta_W) - I_{2j+2}(\beta_W) = \frac{2(2j+1)}{\beta_W}\, I_{2j+1}(\beta_W),
\end{aligned}$$

using the modified Bessel recurrence $I_{\nu-1}(x) - I_{\nu+1}(x) = \frac{2\nu}{x} I_\nu(x)$. Dividing by $d_j \hat{w}_0(\beta_W)$ with $d_j = 2j+1$ and $\hat{w}_0(\beta_W) = \frac{2}{\beta_W} I_1(\beta_W)$ gives (15).

To prove (16) in four lines (where our indexing $y_n = I_{n+1}/I_n$ reproduces Amos's lower/upper ratio bounds [1] for integer $n \geq 1$, with $n = 1$ giving I_2/I_1): from $I_n'(x) = I_{n+1}(x) + \frac{n}{x} I_n(x)$ and

$I'_{n+1}(x) = I_n(x) - \frac{n+1}{x} I_{n+1}(x)$, the ratio $y_n(x) = I_{n+1}(x)/I_n(x)$ satisfies the Riccati ODE

$$y'_n(x) = 1 - \frac{2n+1}{x}\, y_n(x) - y_n(x)^2.$$

Define the lower and upper barriers $\ell_n(x) := \frac{\sqrt{x^2+(n+1)^2}-(n+1)}{x} = \frac{x}{(n+1)+\sqrt{x^2+(n+1)^2}}$ and $u_n(x) := \frac{\sqrt{x^2+(n+1/2)^2}-(n+1/2)}{x}$. By direct squaring, $\ell_n^2 + \frac{2n+2}{x}\ell_n = 1$ and $u_n^2 + \frac{2n+1}{x} u_n = 1$, whence for every $x > 0$:

$$\ell'_n(x) = \frac{n+1}{\sqrt{x^2+(n+1)^2}}\,\frac{\ell_n(x)}{x} \;<\; \frac{\ell_n(x)}{x} = 1 - \frac{2n+1}{x}\ell_n(x) - \ell_n(x)^2,$$

$$u'_n(x) = \frac{n+\frac12}{\sqrt{x^2+(n+\frac12)^2}}\,\frac{u_n(x)}{x} \;>\; 0 = 1 - \frac{2n+1}{x} u_n(x) - u_n(x)^2.$$

Near $x \to 0^+$, Taylor expansion gives $y_n(x) = \frac{x}{2n+2} - \frac{x^3}{8(n+1)^2(n+2)} + O(x^5)$, whereas $\ell_n(x) = \frac{x}{2n+2} - \frac{x^3}{8(n+1)^3} + O(x^5)$ and $u_n(x) = \frac{x}{2n+1} + O(x^3)$, so $\ell_n(x) < y_n(x) < u_n(x)$ for all sufficiently small $x > 0$. By strict differential-inequality comparison (at any first crossing $x_* > 0$ where $y_n(x_*) = \ell_n(x_*)$ we would need $y'_n(x_*) \le \ell'_n(x_*)$, contradicting $y'_n(x_*) = 1 - \frac{2n+1}{x_*}\ell_n - \ell_n^2 > \ell'_n(x_*)$, and symmetrically for u_n), (16) holds for all $x > 0$. Taking $n = 1$ and noting $1/u_1(x) = \frac{3}{2x} + \sqrt{1+(\frac{3}{2x})^2} = \exp(\operatorname{arsinh}(\frac{3}{2x}))$ and $1/\ell_1(x) = \exp(\operatorname{arsinh}(\frac{2}{x}))$ yields (17) and (18).

(2) Let $f(t) := \ln Z_{\mathrm{SU}(N)}(t)$ $(t \ge 0)$. Then $f(0) = 0$, $f'(t) = \mathbb{E}_{\nu_t}[\operatorname{Re}\operatorname{Tr} U] = \hat{w}_{\mathbf{N}}(Nt)/\hat{w}_0(Nt)$, and $f''(t) = \operatorname{Var}_{\nu_t}(\operatorname{Re}\operatorname{Tr} U) > 0$, so f is strictly convex on $[0,\infty)$ and $f'(t) \ge f(t)/t$ for all $t > 0$. Every moment $\mu_m := \int_{\mathrm{SU}(N)} (\operatorname{Re}\operatorname{Tr} U)^m\, dU$ is nonnegative by (8) (Schur character positivity for $\lambda = \mathbf{1}$), with $\mu_0 = 1$, $\mu_1 = 0$, and $\mu_2 = \frac14 \mathbb{E}[(\operatorname{Tr} U)^2 + (\overline{\operatorname{Tr} U})^2 + 2|\operatorname{Tr} U|^2]$, which equals 1 for $N = 2$ ($\operatorname{Tr} U = \overline{\operatorname{Tr} U}$) and 1/2 for $N \ge 3$ ($\mathbb{E}[(\operatorname{Tr} U)^2] = 0$ by $\mathbb{Z}_N$ center symmetry and $\mathbb{E}[|\operatorname{Tr} U|^2] = 1$ by Schur orthogonality). Truncating $Z_{\mathrm{SU}(N)}(t) \ge 1 + \frac{\mu_2}{2} t^2$ at $t = \beta/N$ gives (19).

(3) On $\mathrm{SU}(N)$, $U \mapsto \operatorname{Re}\operatorname{Tr} U$ attains its unique global maximum N at $U = I_N$. In exponential coordinates $U = e^X$ ($X \in \mathfrak{su}(N)$ with Hilbert–Schmidt inner product $\langle X, Y\rangle_{\mathrm{HS}} = \operatorname{Tr}(X^*Y)$), $\operatorname{Re}\operatorname{Tr}(e^X) = N - \frac12 \|X\|_{\mathrm{HS}}^2 + O(\|X\|^3)$, so the measure $Z_{\mathrm{SU}(N)}(\beta/N)^{-1} e^{\frac{\beta}{N}\operatorname{Re}\operatorname{Tr} U}\, dU$ concentrates around I_N as a centered Gaussian on $\mathfrak{su}(N)$ with density proportional to $\exp(-\frac{\beta}{2N}\|X\|_{\mathrm{HS}}^2)$, giving $\mathbb{E}[x_a x_b] = \frac{N}{\beta}\delta_{ab} + O(\beta^{-2})$ in any HS-orthonormal basis $\{e_a\}_{a=1}^{N^2-1}$ of $\mathfrak{su}(N)$. Expanding the smooth class function $\chi_\lambda(e^X)/d_\lambda = 1 + \frac{1}{2d_\lambda}\sum_{a,b} x_a x_b \operatorname{Tr}(d\rho^\lambda(e_a) d\rho^\lambda(e_b)) + O(\|X\|^3)$ and noting $-\sum_a d\rho^\lambda(e_a)^2 = 2C_2(\lambda) I_{d_\lambda}$ (since $e_a = i\sqrt{2}\,T^a$ with $\operatorname{Tr}(T^a T^b) = \frac12\delta_{ab}$ has $\operatorname{Tr}(e_a^* e_b) = 2\operatorname{Tr}(T^a T^b) = \delta_{ab}$, so $-\sum_a d\rho^\lambda(e_a)^2 = 2\sum_a d\rho^\lambda(T^a)^2 = 2C_2(\lambda) I_{d_\lambda}$), Laplace's method yields

$$\frac{\hat{w}_\lambda(\beta)}{d_\lambda\, \hat{w}_0(\beta)} = 1 - \frac{1}{2d_\lambda}\cdot\frac{N}{\beta}\cdot 2C_2(\lambda)\, d_\lambda + O(\beta^{-2}) = 1 - \frac{NC_2(\lambda)}{\beta} + O(\beta^{-2}).$$

Taking $-L\ln(\cdot)$ and multiplying by β proves (20). □

5. Spatial-Coupling-Independent All-Coupling Mass Gap on Finite Slabs

We now move beyond 1D cycles C_L to arbitrary finite connected spatial graphs $\Gamma = (V, E)$ with *nonzero* spatial plaquettes $\mathcal{P}(\Gamma)$ (such as 2D or 3D spatial tori $\mathbb{T}^{d-1}_{L_1,\dots,L_{d-1}}$) and arbitrary spatial plaquette coupling $\beta_{\mathrm{sp}} \in \mathbb{R}$. Set

$$\Psi_{\mathrm{sp}}(u) := \exp\left(\frac{\beta_{\mathrm{sp}}}{2N}\sum_{p\in\mathcal{P}(\Gamma)} \operatorname{Re}\operatorname{Tr} U_p\right),$$

and consider the physical transfer operator

$$T_\Gamma(\beta_t,\beta_{\rm sp}) := \Psi_{\rm sp}(PK_{\beta_t}P)\Psi_{\rm sp}$$

acting on $\mathcal{H}_{\rm OS}(\Gamma) := {\rm Ran}(P) \subset L^2(G^E)$, where $\beta_t > 0$ is the timelike crossing coupling.

Theorem 5.1 (All-coupling, $\beta_{\rm sp}$-independent mass gap on arbitrary finite spatial slabs)**.** *Let $\Gamma = (V,E)$ be any finite connected oriented graph with first Betti number $b_1(\Gamma) := |E|-|V|+1 \geq 1$ (equivalently $|E| \geq |V|$, so Γ contains at least one closed loop and $\dim \mathcal{H}_{\rm OS}(\Gamma) = \infty$), let $N \geq 2$, let $\beta_t > 0$, and let $\beta_{\rm sp} \in \mathbb{R}$ be* arbitrary*. Define*

$$\delta_N := \begin{cases} 1, & N \text{ even}, \\ \cos(\pi/N), & N \text{ odd} \quad (\text{in particular } \delta_3 = 1/2). \end{cases} \tag{21}$$

Then $T_\Gamma(\beta_t,\beta_{\rm sp})$ is compact, strictly positive-definite, and possesses a simple top eigenvalue $\Lambda_0 > 0$ with a unique strictly positive continuous normalized ground state $\Omega_\Gamma \in \mathcal{H}_{\rm OS}(\Gamma)$, together with infinitely many strictly positive subleading eigenvalues $\Lambda_1 \geq \Lambda_2 \geq \cdots > 0$. Every subleading eigenvalue Λ_k ($k \geq 1$) satisfies

$$0 \;<\; \frac{\Lambda_k}{\Lambda_0} \;\leq\; \tanh\!\left(\frac{1+\delta_N}{2}\,|E|\,\beta_t\right) \;<\; 1, \tag{22}$$

and therefore the physical transfer-matrix mass gap satisfies the completely $\beta_{\rm sp}$-independent lower bound

$$\Delta_\Gamma(\beta_t,\beta_{\rm sp}) := \ln\!\left(\frac{\Lambda_0}{\Lambda_1}\right) \;\geq\; -\ln\tanh\!\left(\frac{1+\delta_N}{2}\,|E|\,\beta_t\right) \;>\; 0. \tag{23}$$

(If $b_1(\Gamma) = 0$, i.e., Γ is a tree, G^V acts transitively on G^E, $\mathcal{H}_{\rm OS}(\Gamma) = \mathbb{C}\mathbf{1}$ is one-dimensional, and the gap is trivially $+\infty$.)

Proof. Because $\Psi_{\rm sp} > 0$ is continuous and gauge-invariant and $K_{\beta_t} > 0$ is compact and strictly positive-definite (Theorem 2.1), $T_\Gamma = \Psi_{\rm sp}(PK_{\beta_t}P)\Psi_{\rm sp}$ is compact, self-adjoint, and strictly positive-definite on $\mathcal{H}_{\rm OS}(\Gamma)$, with continuous strictly positive integral kernel

$$\tilde K(u,v) := \Psi_{\rm sp}(u)\,\Psi_{\rm sp}(v)\,K^{\rm inv}_{\beta_t}(u,v) > 0, \qquad K^{\rm inv}_{\beta_t}(u,v) := \int_{G^V}\prod_{e\in E} w_{\beta_t}\big(u_e(T_a v)_e^{-1}\big)\,da.$$

By Jentzsch's theorem [2] on the compact metric space G^E (extending T_Γ to $L^2(G^E)$ by 0 on ${\rm Ran}(P)^\perp$, or viewing $\tilde K$ on the compact quotient G^E/G^V with pushforward Haar measure μ), the spectral radius $\Lambda_0 = \|T_\Gamma\| > 0$ is a simple eigenvalue with a strictly positive continuous eigenfunction $\Omega \in \mathcal{H}_{\rm OS}(\Gamma)$ normalized by $\int_{G^E}\Omega(u)^2\,du = 1$.

Now perform the Doob h-transform: set $d\nu(u) := \Omega(u)^2\,du$ (a probability measure on G^E) and define the Markov transition kernel

$$P_\Gamma(u,dv) := p(u,v)\,d\nu(v), \qquad p(u,v) := \frac{\tilde K(u,v)}{\Lambda_0\,\Omega(u)\,\Omega(v)} > 0.$$

Then $\int p(u,v)\,d\nu(v) = \Lambda_0^{-1}\Omega(u)^{-1}(T_\Gamma\Omega)(u) = 1$, and P_Γ is self-adjoint on $L^2(G^E,\nu)$ with spectrum on ${\rm span}\{\Omega^{-1}f : f \in \mathcal{H}_{\rm OS}(\Gamma)\}$ equal to $\Lambda_0^{-1}\,{\rm spec}(T_\Gamma)$ (and $P_\Gamma\mathbf{1} = \mathbf{1}$). Crucially, for any $u,u',v,v' \in G^E$, *both* the spatial plaquette weights $\Psi_{\rm sp}$ and the unknown ground state Ω cancel identically in the four-point cross-ratio:

$$\frac{p(u,v)\,p(u',v')}{p(u,v')\,p(u',v)} = \frac{\tilde K(u,v)\,\tilde K(u',v')}{\tilde K(u,v')\,\tilde K(u',v)} = \frac{K^{\rm inv}_{\beta_t}(u,v)\,K^{\rm inv}_{\beta_t}(u',v')}{K^{\rm inv}_{\beta_t}(u,v')\,K^{\rm inv}_{\beta_t}(u',v)}. \tag{24}$$

On ${\rm SU}(N)$, every U has eigenvalues $e^{i\theta_1},\dots,e^{i\theta_N}$ with $\sum_{j=1}^N \theta_j \in 2\pi\mathbb{Z}$, so $\max_{U\in{\rm SU}(N)} {\rm Re\,Tr}\,U = N$ and $\min_{U\in{\rm SU}(N)} {\rm Re\,Tr}\,U = -N\delta_N$: for even N this is attained at $-I_N$; for odd $N \geq 3$,

critical points of $F(\theta) = \sum_{j=1}^{N} \cos\theta_j$ subject to $\sum_j \theta_j \equiv 0 \pmod{2\pi}$ obey $\sin\theta_j = \lambda$, so each $\theta_j \equiv \alpha$ or $\pi - \alpha \pmod{2\pi}$ for some $\alpha \in [-\pi/2, \pi/2]$ ($\cos\alpha \geq 0$). If m angles equal $\pi - \alpha$, then $F(\theta) = (N - 2m)\cos\alpha$: whenever $0 \leq m \leq N - 1$, $F(\theta) \geq -(N - 2) > -N\cos(\pi/N)$ (using $\cos(\pi/N) \geq 1 - \pi^2/(2N^2)$ and $\pi^2/(2N) \leq \pi^2/6 < 2$ for $N \geq 3$); for $m = N$, oddness of N forces $N\alpha \equiv \pi \pmod{2\pi}$, hence $|\alpha| \geq \pi/N$ and $F(\theta) = -N\cos\alpha \geq -N\cos(\pi/N)$, attained at $U = e^{i\pi(N-1)/N} I_N$ (e.g., $-3/2$ on SU(3)). Hence $e^{-\delta_N |E| \beta_t} \leq K^{\mathrm{inv}}_{\beta_t}(u, v) \leq e^{+|E|\beta_t}$ everywhere, so the cross-ratio (24) is uniformly bounded by

$$D := \exp\bigl(2(1 + \delta_N)\,|E|\,\beta_t\bigr).$$

We complete the proof via the self-contained Birkhoff–Hopf total-variation contraction [2, 5] (mechanically checked in `birkhoff_hopf_tv_core`): fix any $u, u' \in G^E$, set $f(v) := p(u, v)$, $g(v) := p(u', v)$, $A := \{v : f(v) > g(v)\}$, $B := G^E \setminus A$, $p_0 := \int_A f\, d\nu$, and $q_0 := \int_A g\, d\nu$. Because $f(v)g(v') \leq D\, f(v')g(v)$ for all $v \in A, v' \in B$, integrating over $A \times B$ gives $p_0(1-q_0) \leq D\,(1-p_0)\,q_0$. Writing $s := p_0 - q_0 = \frac{1}{2}\int |f - g|\, d\nu \in [0, 1)$ and $r := 1 - p_0 - q_0$, the exact algebraic identities $4p_0(1 - q_0) = (1 + s)^2 - r^2$ and $4(1 - p_0)q_0 = (1 - s)^2 - r^2$ give

$$(D - 1)r^2 \leq D(1 - s)^2 - (1 + s)^2,$$

and since $(D - 1)r^2 \geq 0$, we obtain $\sqrt{D}(1 - s) \geq 1 + s$, i.e.,

$$\alpha(P_\Gamma) := \sup_{u,u'} \frac{1}{2} \int_{G^E} |p(u, v) - p(u', v)|\, d\nu(v) \leq \frac{\sqrt{D} - 1}{\sqrt{D} + 1} = \tanh\left(\frac{\ln D}{4}\right) = \tanh\left(\frac{1 + \delta_N}{2}\,|E|\,\beta_t\right).$$

Finally, since P_Γ has a real integral kernel, we may take any continuous eigenfunction $\psi \perp \mathbf{1}$ of P_Γ with eigenvalue $\lambda = \Lambda_k/\Lambda_0$ ($k \geq 1$) to be real-valued; then $\mathrm{osc}(\psi) := \max\psi - \min\psi > 0$ and, for every u, u',

$$|(P_\Gamma\psi)(u) - (P_\Gamma\psi)(u')| = \left| \int_{G^E} \bigl(p(u, v) - p(u', v)\bigr) \bigl(\psi(v) - \tfrac{\max\psi + \min\psi}{2}\bigr)\, d\nu(v) \right| \leq \alpha(P_\Gamma)\ \mathrm{osc}(\psi).$$

Taking the supremum over u, u' gives $|\lambda|\ \mathrm{osc}(\psi) = \mathrm{osc}(P_\Gamma\psi) \leq \alpha(P_\Gamma)\ \mathrm{osc}(\psi)$, proving (22) and (23). □

6. Certified Quantitative Tables and Mechanized Lean 4 Verification

Tables 1–3 record all 20 rigorous certificates produced by our 60-digit interval-arithmetic certifier (`src/python/certify_cylinder_gap.py`, `mpmath.iv`, 4/4 `pytest` tests passing, zero black-box special-function calls and zero numerical quadrature) and independently checked in pure exact/directed-rounding integer arithmetic by our standalone Lean 4.36.0 kernel (`src/lean/CylinderReflectionCerts.lean`, zero imports, zero `sorry`, zero `Classical.choice`, compiling cleanly with `-DwarningAsError=true`). All printed numerical bounds in Tables 1–3 use strict outward (directed) rounding so every displayed inequality holds verbatim. Figure 1 displays the exact all-coupling mass gaps alongside our Riccati–Bessel bounds and the weak-coupling Casimir asymptotes.

Mechanized Lean 4.36.0 Verification. Our standalone zero-import Lean 4.36.0 kernel (`src/lean/CylinderReflectionCerts.lean`) mechanically checks:

(1) the abstract group-theoretic crossing, covariance, composition, and arbitrary-length loop identities (`crossing_eq_gauge_action`, `kernel_arg_gauge_covariant`, `loop2_gauge_covariant`, and `path_gauge_covariant`);
(2) the universal integer Birkhoff–Hopf contraction core (`birkhoff_hopf_tv_core`) and the SU(3) Weyl dimension bound (`su3_weyl_dim_ge_6`);
(3) directed-rounding soundness (`floorDiv_sound` and `ceilDiv_sound`); and

TABLE 1. Certified SU(2) L-link cylinder centered reflected norms $\|W_{\mathbf{2}}\|^2_{\mathrm{phys}} = (I_2(\beta_W)/I_1(\beta_W))^L$ and exact mass gaps $\Delta_{\mathrm{cyl}}(2, L, \beta_W)$, compared against the four-line all-coupling Riccati–Bessel bounds (17)–(18). All displayed digits are strictly outwardly rounded.

L	β_W	$\|W_{\mathbf{2}}\|^2_{\mathrm{phys}} >$	Exact $\|W_{\mathbf{2}}\|^2_{\mathrm{phys}}$	$L\,\mathrm{arsinh}(\frac{3}{2\beta_W})$	Exact Δ_{cyl}	$L\,\mathrm{arsinh}(\frac{2}{\beta_W})$
2	0.1	6.242×10^{-4}	$[6.244 \times 10^{-4}, 6.245 \times 10^{-4}]$	6.8046	$[7.3785, 7.3786]$	7.3791
2	0.5	1.51×10^{-2}	$[1.53 \times 10^{-2}, 1.54 \times 10^{-2}]$	3.6368	$[4.1795, 4.1796]$	4.1895
2	1.0	5.57×10^{-2}	$[5.76 \times 10^{-2}, 5.77 \times 10^{-2}]$	2.3895	$[2.8526, 2.8527]$	2.8873
2	2.0	1.71×10^{-1}	$[1.87 \times 10^{-1}, 1.88 \times 10^{-1}]$	1.3862	$[1.6734, 1.6735]$	1.7628
2	5.0	4.58×10^{-1}	$[5.17 \times 10^{-1}, 5.18 \times 10^{-1}]$	0.5913	$[0.6588, 0.6589]$	0.7801
2	10.0	6.72×10^{-1}	$[7.29 \times 10^{-1}, 7.30 \times 10^{-1}]$	0.2988	$[0.3152, 0.3153]$	0.3974
3	1.0	1.31×10^{-2}	$[1.38 \times 10^{-2}, 1.39 \times 10^{-2}]$	3.5842	$[4.2789, 4.2790]$	4.3310
4	2.0	2.94×10^{-2}	$[3.51 \times 10^{-2}, 3.52 \times 10^{-2}]$	2.7725	$[3.3468, 3.3469]$	3.5255

TABLE 2. Certified SU(3) L-link cylinder centered reflected norms $\|W_{\mathbf{3}}\|^2_{\mathrm{phys}} = r_{\mathbf{3}}^L$ and exact mass gaps via 25 exact rational Weyl moments $m = 0 \ldots 24$ (zero quadrature). In every row, $R_3^{\mathrm{hi}} < r_{\mathbf{3}}^{\mathrm{lo}}$ certifies exact fundamental dominance (14), proving $\Delta_{\mathrm{cyl}}(3, L, \beta) = -L \ln r_{\mathbf{3}}$ verbatim.

L	β	Convexity $r_{\mathbf{3}} \geq$	Exact $r_{\mathbf{3}}^{\mathrm{lo}}$	Parseval R_3^{hi}	Exact $\|W_{\mathbf{3}}\|^2_{\mathrm{phys}}$	Exact Δ_{cyl}
2	0.3	0.00832	0.01708	0.00054	$[2.91 \times 10^{-4}, 2.92 \times 10^{-4}]$	$[8.1393, 8.1394]$
2	0.9	0.02472	0.05371	0.00516	$[2.88 \times 10^{-3}, 2.89 \times 10^{-3}]$	$[5.8482, 5.8483]$
2	1.5	0.04041	0.09344	0.01541	$[8.73 \times 10^{-3}, 8.74 \times 10^{-3}]$	$[4.7408, 4.7409]$
2	3.0	0.07438	0.20307	0.07267	$[4.12 \times 10^{-2}, 4.13 \times 10^{-2}]$	$[3.1883, 3.1884]$
3	1.5	0.04041	0.09344	0.01541	$[8.15 \times 10^{-4}, 8.16 \times 10^{-4}]$	$[7.1112, 7.1113]$
4	3.0	0.07438	0.20307	0.07267	$[1.70 \times 10^{-3}, 1.71 \times 10^{-3}]$	$[6.3767, 6.3768]$

TABLE 3. Certified Birkhoff–Hopf transfer-matrix mass-gap bounds $\Delta_\Gamma(\beta_t, \beta_{\mathrm{sp}}) \geq -\ln \alpha^{\mathrm{hi}} > 0$ on finite slabs $\mathbb{Z} \times \Gamma$ (Theorem 5.1), valid for every $\beta_{\mathrm{sp}} \in \mathbb{R}$ (strictly outwardly rounded).

Group	Spatial graph Γ	β_t	Contraction α^{hi}	$\Delta_\Gamma(\beta_t, \beta_{\mathrm{sp}}) \geq$
SU(2)	$\mathbb{T}^2_{2,2}$ (2D torus, $\|E\| = 8$)	0.1	0.66404	4.09×10^{-1}
SU(2)	$\mathbb{T}^2_{2,2}$ (2D torus, $\|E\| = 8$)	0.5	0.99933	6.70×10^{-4}
SU(3)	$\mathbb{T}^2_{2,2}$ (2D torus, $\|E\| = 8$)	0.1	0.53705	6.21×10^{-1}
SU(3)	$\mathbb{T}^2_{2,2}$ (2D torus, $\|E\| = 8$)	0.5	0.99506	4.95×10^{-3}
SU(3)	$\mathbb{T}^3_{2,2,2}$ (3D torus, $\|E\| = 24$)	0.1	0.94681	5.46×10^{-2}
SU(3)	C_2 (2-link cylinder, $\|E\| = 2$)	2.0	0.99506	4.95×10^{-3}

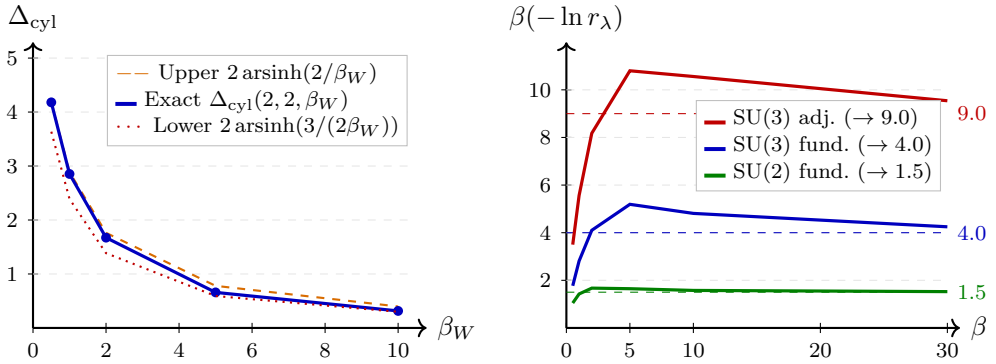
(4) all 20 quantitative certificates in Tables 1–3, including exact Parseval fundamental dominance $Z(2t) < Z(t)^2 + 6Z'(t)^2$ across all six SU(3) cases up to $\beta = 3.0$.

Running `#print axioms` confirms that every theorem in the kernel depends on zero `sorry` and zero `Classical.choice` axioms.

REPRODUCIBILITY AND CODE AVAILABILITY

All verification artifacts are archived on AIRR (ARR-2026-0P7ATP57G18YN90M):

- Python 60-digit interval certifier: `src/python/certify_cylinder_gap.py`;
- automated test suite (`pytest`): `tests/test_cylinder_gap.py`;
- exported certificate bundle: `certificates/cylinder_gap_certs.json`;
- standalone Lean 4.36.0 proof kernel: `src/lean/CylinderReflectionCerts.lean`.



(a) SU(2) ($L = 2$) Riccati–Bessel enclosure (b) Weak-coupling Casimir law $\beta(-\ln r_\lambda) \to NC_2(\lambda)$

FIGURE 1. All-coupling spectral bounds and weak-coupling Casimir convergence. (a) Exact SU(2) ($L = 2$) cylinder mass gap sandwiched tightly between our four-line Riccati–Bessel barriers (18) over $\beta_W \in [0.5, 10]$ (with the full two-decade span $[0.1, 10]$ tabulated in Table 1). (b) Convergence of $\beta(-\ln r_\lambda(\beta))$ to the exact Casimir limit $NC_2(\lambda)$ (Theorem 4.1(3)) for SU(2) fundamental (1.5), SU(3) fundamental (4.0), and SU(3) adjoint (9.0).

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