Spectral Reduction and Logarithmic Observation Response in a Discrete Coexact Model

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Abstract

We study a finite counting-norm vector-field model on an even three-dimensional torus, with checkerboard source weights and componentwise rectangular cell means. On the actual image of the forward curl adjoint, we derive the complete two-polarization Fourier spectrum, including the missing zero mode and the exceptional frequency pair. For weights $H = cn^2$ and $a = c/48$, the entire subtarget spectral space has dimension $n-2$ per channel. The original high modes realize every observation realized by the full coexact space; an explicit polynomial lifting estimate and an exact shifted cancellation-cost criterion follow. For permutation singleton cuts, we prove finite logarithmic upper and lower bounds for the high-response operator, and show that the logarithmic growth persists after compression to useful positive-weight low data. This rules out two uniform response estimates, without deciding the optimized cancellation problem. A logarithmic lower bound for invisible coexact fields is also proved. All arguments are finite and analytical. Two-dimensional Green-function logarithms are classical; originality of the precise model-specific comparisons is unestablished. No continuum or physical Yang–Mills spectral-gap conclusion is asserted.

1 Scope and relation to classical logarithms

Checkerboard weights couple two Fourier frequencies, while a curl-adjoint domain imposes two transverse polarizations and removes the harmonic zero mode. Keeping these restrictions and the original observation rows together is essential: neither a scalar block alone nor injectivity of observations on low modes settles cancellation by high modes.

The argument below proceeds from the complete restricted spectrum to an original-row range theorem, then to a finite cancellation criterion and quantitative response estimates. The final obstruction concerns direct response bounds, including their compression to useful low data. It does not determine the inverse problem after correlated dual directions are optimized.

Logarithmic Green functions in two dimensions are established mathematics. Kenyon's Theorem 7.3 gives logarithmic Green-function asymptotics in the critical planar graph setting [2]; Lawler and Limic develop continuous-time walks and the two-dimensional potential kernel, including Theorem 4.4.4 [3]. Their graph, volume and operator normalizations are not identified with the finite component graph or the coexact compression used here. We derive every finite normalization and tail estimate needed below. These bounded primary-source comparisons do not establish priority for the precise finite statements, and the classical logarithmic mechanism is not claimed as new.

The prior programme working paper *Coverage and coexactness in checkerboard slice observations* [1] treats coverage counterexamples, coexact projection issues and a logarithmic ambient-field bound. We do not present those findings as new here or inherit that artifact's assessments. The present argument centers on the whole actual restricted spectrum, exact high-mode range and cost, and the two response obstructions. Its improved numerical kernel coefficient has a restricted actual-coexact domain, as made explicit in Remark 6.3.

2 The original model and observations

Let $n \geq 4$ be even and $X = (\mathbf{Z}/n\mathbf{Z})^3$. All physical norms and adjoints use counting measure. For $u : X \to \mathbf{R}^3$, put

$$\Delta_j u(x) = u(x + e_j) - u(x), \qquad (d_1 u)_{j\ell} = \Delta_j u_\ell - \Delta_\ell u_j \quad (j < \ell).$$

The norm on two-forms counts each pair $j < \ell$ once. Our actual domain is $U = \operatorname{im} d_1^*$, not an abstract divergence-free space with freely added constants. We prove its exact Fourier description in Section 3.

There are two original channels, indexed by $\varepsilon \in \{-1, +1\}$. Write $\chi(x) = (-1)^{x_1+x_2+x_3}$ and, initially, let $H \geq a > 0$. Set $A_0 = (H + a)/2$, $B_0 = (H - a)/2$, and

$$b_\varepsilon(x) = A_0 + \varepsilon B_0 \chi(x), \qquad Q_\varepsilon(u) = \sum_{x,j,r} b_\varepsilon(x) |\Delta_j u_r(x)|^2. \tag{1}$$

Thus $\varepsilon = -1$ has strong weight H on odd source vertices; the other channel exchanges the two parities. Write A_ε for the self-adjoint operator of this form *restricted to* U. We do not assert that the ambient weighted-gradient operator preserves U. The two-channel form and domain are the original direct sum.

For each $i \in \mathbf{Z}_n$ choose $\tau_i \in \{0, \dots, n-1\}$. The original nonwrapping cells are

$$\{i\} \times I \times J, \quad I \in \{[0, \tau_i - 1], \{\tau_i\}, [\tau_i + 1, n-1]\}, \quad J \in \{\{0\}, [1, n-1]\},$$

with empty intervals omitted. Every spatial component and channel is observed separately, with

$$(Cu)_{r,B} = \frac{1}{\sqrt{n^3|B|}} \sum_{x \in B} u_r(x). \tag{2}$$

No channel rotation, adapted mean or projected row replaces these observations. Unless stated otherwise, proofs are per channel; direct sums give the same constants for both channels.

Our quantitative specialization is

$$H = cn^2, \qquad a = c/48, \qquad c > 0. \tag{3}$$

When $f(i) = i + \tau_i \pmod n$ is a permutation, denote by T the genuine singleton submap

$$(Tu)_i = n^{-3/2} u(i, \tau_i, 0). \tag{4}$$

The quadratic cut $\tau_i = 2i^2 \pmod n$ has this property for dyadic n: $f(i) = f(j)$ implies $(i - j)(1 + 2(i + j)) = 0 \pmod n$, and the second factor is odd and hence invertible.

3 Complete spectrum on the actual coexact domain

Use the unitary convention

$$u(x) = n^{-3/2} \sum_k \widehat{u}(k) \mathrm{e}^{2\pi i k\cdot x/n}, \qquad g_j(k) = \mathrm{e}^{2\pi i k_j/n} - 1.$$

Complexification is used for calculation; real multiplicities are recovered without doubling the complex dimension.

Lemma 3.1 (The actual image)**.** *At $k \neq 0$, the Fourier image of d_1^* is $g(k)^\perp$ of dimension two; at $k = 0$ it is zero. Consequently* $\dim_{\mathbf{R}} U = 2(n^3 - 1)$.

Proof. For a skew array $W_{j\ell} = -W_{\ell j}$, $(d_1^* W)_\ell = \sum_j \overline{g_j} W_{j\ell}$, which is orthogonal to g. Conversely, for $v \perp g$ and $D = |g|^2 > 0$, set

$$W_{j\ell} = \frac{g_j v_\ell - g_\ell v_j}{D}.$$

Its adjoint image is $v_\ell - g_\ell(g^* v)/D = v_\ell$. At zero all symbols vanish. The formula respects conjugation at negative frequencies, so real fields have real preimages. Counting the nonzero frequencies proves the dimension and excludes harmonic constants. □

Let $q = (n/2, n/2, n/2)$ and consider an unordered pair $\{k, k+q\}$. Put $h = g(k+q)$, $D = |g|^2$, $D' = |h|^2$ and $\Gamma = g^* h$. Direct calculation gives

$$D' = 12 - D, \qquad \Gamma = 2i \sum_j \sin(2\pi k_j/n). \tag{5}$$

Theorem 3.2 (All restricted eigenvalues)**.** *For a nonexceptional pair, both frequencies being nonzero, the four eigenvalues per channel are*

$$\begin{aligned} \lambda_{1,\pm} &= 6A_0 \pm \sqrt{A_0^2 (D-6)^2 + B_0^2 |\Gamma|^2}, \\ \lambda_{2,\pm} &= 6A_0 \pm \sqrt{A_0^2 (D-6)^2 + B_0^2 |\Gamma|^4/(DD')}. \end{aligned} \tag{6}$$

Repeated roots retain their multiplicities. The pair $\{0, q\}$ has only the q plane and its two eigenvalues are $12A_0$. These blocks exhaust U: their total multiplicity is $4(n^3/2 - 1) + 2 = 2(n^3 - 1)$ per channel. Both original channels have the same spectrum.

Proof. For coefficients $x = \widehat{u}(k)$, $y = \widehat{u}(k+q)$, the original source-weighted form is exactly

$$A_0 D |x|^2 + A_0 D' |y|^2 + 2\varepsilon B_0 \operatorname{Re}(\Gamma x^* y).$$

Indeed multiplication by χ shifts frequency by q, and summing the three forward differences gives $g^* h$ with no extra volume factor. Choose isometries $E, F : \mathbf{C}^2 \to \mathbf{C}^3$ onto $g^\perp, h^\perp$. The compressed block has diagonal $A_0 D I_2, A_0 D' I_2$ and off-diagonal $\varepsilon B_0 \Gamma E^* F$.

The singular values of $E^* F$ are 1 and $\rho = |\Gamma|/\sqrt{DD'}$. The planes have a common unit vector, and their projections $P_g = I - gg^*/D$, $P_h = I - hh^*/D'$ satisfy $\operatorname{tr}(P_g P_h) = 1 + |\Gamma|^2/(DD')$. This proves the two singular values, including coincident or orthogonally normal

planes. Singular-value decomposition and absorption of the coupling phase give two scalar two-by-two blocks with coupling moduli $B_0|\Gamma|$ and $B_0|\Gamma|\rho$. Their eigenvalues are (6).

At zero the first coefficient is absent by Lemma 3.1; at q the remaining plane has $D' = 12$. Changing the channel sign is conjugation by a sign on the second coefficient. Complex conjugation takes each block to its negative-frequency block and commutes with the real form, so the real eigenspace dimensions equal those of the complexification. The count above is therefore also the real count, not twice that count. □

Proposition 3.3 (The whole low space)**.** *Under* (3)*, let* $L = 1_{(0,c)}(A_\varepsilon)U$ *and* $F = U \ominus L$*. Then*

$$\dim L = n - 2, \qquad A_L \le (c/8)I, \qquad A_F \ge (16c/3)I.$$

The space L consists exactly of the low branches at nonexceptional diagonal pairs $k = (t, t, t)$*; both transverse polarizations are retained.*

Proof. First keep only the strong source weight, setting $a = 0$. Put $S = \sum_j \mathrm{e}^{2\pi i k_j/n}$. Formula (6) gives the attained smallest coexact pair eigenvalue $H(3 - |S|)$. It vanishes exactly when the three phases agree. At such a nonexceptional diagonal pair both couplings have the same singular value and the zero multiplicity is two. The exceptional pair contributes no zero mode. Thus the actual strong kernel has dimension $n - 2$ per channel.

If the residues are not all equal, at least two pairwise differences are nonzero. The identity

$$9 - |S|^2 = 4 \sum_{j<\ell} \sin^2\left(\frac{\pi(k_j - k_\ell)}{n}\right)$$

and division by $3+|S| \le 6$ show $H(3-|S|) \ge (4H/3)\sin^2(\pi/n) \ge 16H/(3n^2)$. Here $\sin(\pi/n) \ge 2/n$ is the concavity chord on $[0, \pi/2]$. Adding the positive weak form preserves this lower bound.

At a diagonal nonexceptional pair, $\theta = 2\pi t/n$, the two roots, each with multiplicity two, are

$$\mu_\pm = 3(H + a) \pm 3\sqrt{(H + a)^2 - 4Ha\sin^2\theta}.$$

They satisfy $0 < \mu_- \le 6a = c/8$ and $\mu_+ \ge 6H$. The exceptional eigenvalue is $6(H + a) \ge 6H$. There are $n/2 - 1$ nonexceptional diagonal pairs, proving the count and the complete separation. In particular no other low direction has been silently omitted. □

4 Original rows and a paid range theorem

For a cell $B = \{i\} \times I \times J$, direct substitution gives

$$(Cu)_{r,B} = \frac{1}{n^3\sqrt{|I||J|}} \sum_k \widehat{u}_r(k)\mathrm{e}^{2\pi i k_1 i/n} \left(\sum_{j\in I} \mathrm{e}^{2\pi i k_2 j/n}\right)\left(\sum_{\ell\in J} \mathrm{e}^{2\pi i k_3 \ell/n}\right). \tag{7}$$

Each pair member has its own three factors. The zero coefficient is absent on U. Finite sums, rather than a geometric quotient at zero, fix the normalization without removable singularities.

Let N_d be the actual diagonal Fourier space: its fields have the form $u(x) = v(x_1+x_2+x_3)$ with mean-zero v and $v_1+v_2+v_3 = 0$. This follows from Lemma 3.1. Thus $\dim N_d = 2(n-1)$ and $L \subset N_d$. Put $O = U \ominus N_d$. The following result permits every original low observation to be cancelled by actual high modes, including row dependencies.

Theorem 4.1 (Range equality and explicit lift)**.** *For every original cut, whether or not f is a permutation,*

$$\operatorname{ran} C_O = \operatorname{ran}(C|_U) = \operatorname{ran} C_F.$$

There is a linear $R_d : N_d \to O$ such that

$$C_O R_d = -C_{N_d}, \qquad \|R_d\|^2 \le K_n^{\mathrm{lift}} := \frac{n^5}{16(n-3)}.$$

Proof. For any original row coefficient α, put $g = C_{\mathrm{amb}}^* \alpha$, where the ambient map uses precisely (2). For fixed x_1, x_2, g is constant in x_3 on $1, \dots, n-1$, because that is a single long third-coordinate cell. Let $v = P_U g$. Since $U^\perp = \ker d_1$, $d_1 g = d_1 v$. Every component of this curl is independent of x_3 on $1, \dots, n-2$; its third difference therefore vanishes on $1, \dots, n-3$.

If $C_O^* \alpha = 0$, then $v \in N_d$. Each component of $d_1 v$ depends only on $s = x_1 + x_2 + x_3$. Equality of its values at $x_3 = 1, 2$, for all x_1, x_2, forces equal values at adjacent residues of s. It is constant, and its sum is zero because it is a periodic curl. Hence $d_1 v = 0$ and $v \in U \cap U^\perp = 0$. We have proved equality of the annihilators of $\operatorname{ran} C_O$ and $\operatorname{ran}(C|_U)$. Finite-dimensional orthogonal complementation gives the first range equality; $O \subset F \subset U$ gives the other.

For the quantitative claim decompose $v = d + o \in N_d \oplus O$. On the slab $1 \le x_3 \le n-3$, $\Delta_3 d_1 d = -\Delta_3 d_1 o$. The first expression depends only on s, so its squared norm is equally distributed among the n third-coordinate slices. Since the curl symbol has squared norm at most 12 and $|g_3|^2 \le 4$,

$$\|\Delta_3 d_1 d\|^2 \le \frac{48n}{n-3} \|o\|^2.$$

At a nonzero diagonal frequency $z = \mathrm{e}^{2\pi i t/n}$ the same squared symbol on d is $3|z-1|^4$. Using $|z-1|^2 \ge 16/n^2$ gives $\|\Delta_3 d_1 d\|^2 \ge 768 n^{-4} \|d\|^2$. Consequently

$$\|C_{N_d}^* \alpha\|^2 \le K_n^{\mathrm{lift}} \|C_O^* \alpha\|^2. \tag{8}$$

Define $S(C_O^* \alpha) = C_{N_d}^* \alpha$. Inequality (8) makes this well-defined on its range, with norm at most $\sqrt{K_n^{\mathrm{lift}}}$. Extend it by zero on the orthogonal complement. Then $C_O S^* = C_{N_d}$, and $R_d = -S^*$ proves the claim. This is a proved factorization, not an assumed inverse or a full-row-rank assertion. □

5 The exact shifted cancellation criterion

Under (3), set $B = A_F - cI$, $D_L = cI - A_L$ and

$$M = C_F B^{-1} C_F^*, \qquad S_C = \operatorname{ran}(C|_U).$$

Proposition 3.3 gives $B \ge 13c/3$ and $7c/8 \le D_L < cI$. Theorem 4.1 implies $\ker M = S_C^\perp$. Thus M is invertible on S_C, and $M^\dagger$ denotes that inverse extended by zero.

Theorem 5.1 (Exact cost, with dependent rows)**.** *For every $l \in L$,*

$$\min_{f \in F:\ C_F f = -C_L l} \langle f, Bf \rangle = \langle C_L l, M^\dagger C_L l \rangle, \tag{9}$$

$$f_* = -B^{-1} C_F^* M^\dagger C_L l.$$

Consequently $Q(w) \geq c\|w\|^2$ for all $w \in U \cap \ker C$ if and only if

$$C_L^* M^\dagger C_L - D_L \geq 0 \quad \text{on } L. \tag{10}$$

Its dimension is $n-2$ per original channel, or $2(n-2)$ on their direct sum.

Proof. Let $y = C_L l \in S_C$. Then $C_F f_* = -MM^\dagger y = -y$. For every other feasible $f = f_* + h$, $C_F h = 0$ and $\langle h, Bf_* \rangle = -\langle C_F h, M^\dagger y \rangle = 0$. Expanding the square proves the minimum. For $w = l + f$ the spectral splitting gives

$$Q(w) - c\|w\|^2 = -\langle l, D_L l \rangle + \langle f, Bf \rangle.$$

Use the attained minimum in both directions. Every l is feasible by Theorem 4.1, so no extra range assumption hides a bad direction. □

The lift also pays an explicit but inadequate upper bound: $\langle R_d l, BR_d l \rangle \leq (12H - c) K_n^{\text{lift}} \|l\|^2$, because $A \leq 12HI$ and $R_d l \in O \subset F$. This grows like cn^6 and does not decide (10). For later use the corresponding dual identity is

$$\langle y, M^\dagger y \rangle = \sup_\alpha \{2\,\mathrm{Re}\langle \alpha, y \rangle - \langle \alpha, M\alpha \rangle\}, \qquad y \in S_C. \tag{11}$$

It follows by completing the square on S_C; components in $S_C^\perp$ pair and respond to zero.

Lemma 5.2 (Singleton frame)**.** *If f is a permutation, $T^*T = n^{-5} I$ as a form on all of N_d. In particular this holds on L and on the strong kernel.*

Proof. For $u(x) = v(s)$, $\|u\|^2 = n^2 \sum_s |v(s)|^2$, whereas $\|Tu\|^2 = n^{-3} \sum_i |v(f(i))|^2 = n^{-3} \sum_s |v(s)|^2$. □

6 Whole singleton response and an invisible-field bound

Define $R = P_F B^{-1} P_F$ extended by zero and $M_T = TRT^*$. Put $N = n/2$, $h_N = \sum_{j=1}^N 1/j$ and

$$K_n = \frac{48}{13} h_N + \frac{1}{6n^2 - 1}. \tag{12}$$

Theorem 6.1 (Finite whole-response bounds)**.** *For every permutation cut and either original channel,*

$$\frac{h_N}{24\pi^2 c n^5} \leq \|M_T\| \leq \frac{K_n}{cn^5}. \tag{13}$$

No fixed constant bounds $cn^5 \|M_T\|$ for all such sizes. The same blanket scalar bound fails for the entire original-row response M.

Proof. Write $k = (t + r, t + s, t)$ and $\delta(r, s) = 3 - |1 + \mathrm{e}^{2\pi i r/n} + \mathrm{e}^{2\pi i s/n}|$. For $g = T^* \alpha$,

$$\widehat{g}(t + r, t + s, t) = n^{-3} \sum_i \alpha_i \mathrm{e}^{-2\pi i \{t f(i) + r i + s \tau_i\}/n}.$$

Since f is a permutation, orthogonality in t gives the exact identity

$$\sum_t |\widehat{g}(t + r, t + s, t)|^2 = n^{-5} \|\alpha\|^2. \tag{14}$$

The pair shift q changes only t by $n/2$. Every transverse (r,s) therefore contains exactly $n/2$ unordered pairs, and only $(0,0)$ is diagonal. Off that fibre, Proposition 3.3 and its strong-form calculation give

$$(A-cI)^{-1} \le \frac{1}{cn^2\delta - c} I \le \frac{16}{13cn^2\delta} I.$$

The last step uses $cn^2\delta \ge 16c/3$. On the diagonal high space the bound is $1/[c(6n^2-1)]$. Projections cannot increase the input norm. Thus (14) gives

$$\langle \alpha, M_T \alpha \rangle \le \frac{\|\alpha\|^2}{cn^5} \left\{ \frac{16}{13n^2} \sum_{(r,s)\neq 0} \frac{1}{\delta} + \frac{1}{6n^2-1} \right\}. \tag{15}$$

For unique centered residues in $[-N, N-1]^2$, the sine identity in the proof of Proposition 3.3 implies

$$\delta \ge \frac{2}{3}\{\sin^2(\pi r/n) + \sin^2(\pi s/n)\} \ge \frac{8}{3n^2}(r^2+s^2).$$

At most $8j$ centered sites have $\max(|r|,|s|) = j$. Therefore

$$\sum_{(r,s)\neq 0} \frac{1}{\delta} \le 3n^2 h_N, \qquad \sum_{(r,s)\neq 0} \frac{1}{\delta} \ge \frac{n^2 h_N}{4\pi^2}. \tag{16}$$

For the lower inequality retain $r = j$, $1 \le s \le j$, $1 \le j \le N$. These are distinct residues and $\delta \le 2 - \cos(2\pi r/n) - \cos(2\pi s/n) \le 4\pi^2 j^2/n^2$. The upper inequality and (15) prove the upper norm.

For the lower norm, even-sum translations preserve U, the source form, F and R. They are transitive on each vertex parity. The spatial trace of $R(x,x)$ therefore has two values. The sampled vertex parity is that of $f(i)$, so exactly half the singleton points have each parity. Compared with $n^3/2$ vertices of each parity, the row normalization gives

$$\operatorname{tr} M_T = n^{-5} \operatorname{tr}_F B^{-1}. \tag{17}$$

On each non-diagonal pair one genuine strong-minimizing polarization has energy $cn^2\delta$. The weak form is at most $12a = c/4$ on a unit vector. Rayleigh's principle supplies a full eigenvalue $\lambda \le cn^2\delta + c/4$. It belongs to F; hence $(\lambda - c)^{-1} \ge (cn^2\delta)^{-1}$. There are exactly $n/2$ such pairs per transverse fibre. Consequently

$$\operatorname{tr}_F B^{-1} \ge \frac{1}{2cn} \sum_{(r,s)\neq 0} \frac{1}{\delta} \ge \frac{nh_N}{8\pi^2 c}.$$

The largest eigenvalue of a PSD operator on $3n$ rows is at least its trace divided by $3n$. This proves the lower half of (13). Since $h_N \ge \log(N+1)$, the scaled norm diverges. Finally, inserting singleton coefficients by zeros is an isometry J with $T = J^*C$, so $M_T = J^*MJ$ and $\|M\| \ge \|M_T\|$. No independent-row assumption is used in this compression argument. □

Corollary 6.2 (A paid logarithmic kernel bound)**.** *For every $w \in U \cap \ker C$ under a permutation cut,*

$$Q(w) \ge \frac{c}{K_n}\|w\|^2 \ge \frac{c}{4(1+\log n)}\|w\|^2. \tag{18}$$

The same coefficient holds for the original two-channel direct sum.

Proof. For $l \in L$, insert $\alpha = Tl$ into the singleton coordinates of the full rowspace. Its pairing with $y = C_L l$ is $n^{-5}\|l\|^2$, and Theorem 6.1 bounds its response by $K_n\|l\|^2/(cn^{10})$. Optimize (11) along $t\alpha$ at $t = cn^5/K_n$ to obtain $\langle C_L l, M^\dagger C_L l\rangle \geq (c/K_n)\|l\|^2$. For an invisible $w = l + f$, the high part is feasible and

$$Q(w) = \langle l, A_L l\rangle + \langle f, Bf\rangle + c\|f\|^2 \geq (c/K_n)\|l\|^2 + c\|f\|^2 \geq (c/K_n)\|w\|^2,$$

since $K_n > 1$. Put $L_0 = 1 + \log n$. The inequalities $h_N \leq 1 + \log(n/2) \leq L_0$ and $1/(6n^2 - 1) \leq 1/95 < (4/13)L_0$ give $K_n < 4L_0$. The proofs are channelwise, so summation preserves the coefficient. □

Remark 6.3 (Coefficient and domain comparison). The coefficient c/K_n strictly exceeds the prior working-paper benchmark $c/[5(1+\log n)]$ [1] under the same permutation-cut hypothesis. That estimate allowed ambient fields; (18) is on the actual coexact domain with a permutation cut. The numerical improvement does not extend the larger ambient domain, change logarithmic order, or establish the target c. In particular c/K_n is inadequate to dominate $D_L \geq 7c/8$ in (10).

7 Logarithmic response persists on useful low data

Let $E = T_L L$ and let P_E be the ordinary singleton row-space projection. Lemma 5.2 gives $P_E = n^5 T_L T_L^*$ and $\dim E = n - 2$.

Theorem 7.1 (Useful-space obstruction)**.** *For every even $n \geq 4$ and permutation cut,*

$$\frac{h_N/(16\pi^2) - 1/2 - 4K_n/(48n^2 - 1)}{cn^5} \leq \|P_E M_T P_E\| \leq \frac{K_n}{cn^5}. \tag{19}$$

For even $n \geq 16$ the lower bound implies

$$cn^5\|P_E M_T P_E\| \geq \frac{h_N}{32\pi^2} - 1. \tag{20}$$

Thus a uniform response bound fails even for unaugmented tests $\alpha = T_L l$. The lower expressions need not be positive at small sizes.

7.1 A finite component Green estimate

Let K be the actual strong kernel. For strong weight on odd vertices, the strong scalar graph has N components, the paired layers $(s = 2j - 1, s = 2j)$, $s = x_1 + x_2 + x_3$. On each layer use coordinates $y = (x_1, x_2)$; x_3 is determined. The three strong edges from an odd vertex have target coordinates $y, y + e_1, y + e_2$ in the even layer. The unit-weight scalar component Laplacian D is degree three on $2n^2$ vertices, with transverse Fourier matrix

$$\begin{pmatrix} 3 & -S(r,s) \\ -\overline{S(r,s)} & 3 \end{pmatrix}, \qquad S(r,s) = 1 + \mathrm{e}^{2\pi i r/n} + \mathrm{e}^{2\pi i s/n}.$$

Its eigenvalues are $\delta = 3 - |S|$ and $\delta_+ = 3 + |S|$. Only $\delta(0,0)$ vanishes, proving connectedness. Interchanging source parity gives the same graph with paired layers $(2j, 2j + 1)$.

Lemma 7.2 (Component Green bounds)**.** *For $G = D^\dagger$, the diagonal value g_0 is constant and*

$$g_0 \geq \frac{h_N}{8\pi^2}, \qquad G(x,y) \geq -1 \quad \textit{for all } x, y.$$

Proof. Translations within layers and layer exchange with reflection act transitively on vertices. The diagonal spectral formula is

$$g_0 = \frac{1}{2n^2} \left\{ \sum_{(r,s)\neq 0} \frac{1}{\delta} + \sum_{r,s} \frac{1}{\delta_+} \right\}.$$

The lower lattice sum in (16) gives the diagonal bound. For the off-diagonal bound use the finite heat kernel $p_t = \exp(-tD)$. It is entrywise nonnegative, since $D = 3I - \mathrm{Adj}$ and $\exp(t\mathrm{Adj})$ has a nonnegative power series. Hence the integral representation

$$G(x,y) = \int_0^\infty \{p_t(x,y) - (2n^2)^{-1}\}\, dt$$

is bounded below by $-1/2$ on $0 \leq t \leq n^2$. For the tail, $p_t - P_{\mathrm{const}}$ is PSD with constant diagonal. Cauchy–Schwarz bounds each entry by that diagonal, its trace divided by $2n^2$. The absolute tail is therefore at most

$$\frac{1}{2n^2} \sum_{\lambda\neq 0} \frac{\mathrm{e}^{-n^2\lambda}}{\lambda}.$$

For the δ branch use $\delta \geq 8(r^2+s^2)/(3n^2)$ and the at-most $8j$ shell count. Its contribution is at most

$$\frac{3}{2} \sum_{j\geq 1} \frac{\mathrm{e}^{-(8/3)j^2}}{j} \leq \frac{3}{2} \sum_{j\geq 1} (1/8)^j = \frac{3}{14}.$$

Here $\mathrm{e}^{8/3} > 8$ follows already from four nonnegative Taylor terms. The n^2 high-branch eigenvalues are at least 3, giving a contribution at most $\mathrm{e}^{-3n^2}/6 \leq 1/6$. Thus $G(x,y) \geq -1/2 - 3/14 - 1/6 = -37/42 \geq -1$. No positivity of the Green function or an isolated low band was assumed. □

7.2 The scalar component inside the actual coexact space

Put $V = (1,1,1)^\perp$. The strong kernel K consists exactly of component constants $v_j \in V$ with $\sum_j v_j = 0$. Such fields have mean-zero diagonal Fourier coefficients in V, so Lemma 3.1 puts them in U. Their dimension $2(N-1) = n-2$ equals the full strong-kernel count. Every permutation singleton map selects one point in each s layer, hence one point o_j, e_j in each layer of each component. The space $E_0 = T_K K$ consists of equal pair values in V, with component mean zero.

The next construction prevents an ambient scalar inverse from being mistaken for an actual coexact inverse. At transverse $(r,s) \neq 0$, let $z = (\mathrm{e}^{2\pi i r/n}, \mathrm{e}^{2\pi i s/n}, 1)$ and $W_{r,s} = \{(1,1,1), z\}^\perp$, a line in V. For frequencies $k = (t+r, t+s, t)$ and $k+q$, both $g(k)$ and $g(k+q)$ are linear combinations of z and $(1,1,1)$. Thus $W_{r,s}$ lies in both actual coexact planes, and is independent of t. Its two coefficient directions form an invariant first-polarization subspace of the strong block, with eigenvalues $H\delta, H\delta_+$. Retaining this line in the transverse Fourier

transform of each two-layer component gives actual coexact fields. The projector onto the line depends only on transverse frequency, so it acts inside each component and does not mix components.

Let R_0 be this selected strong inverse, zero on the entire diagonal transverse fibre. It is H^{-1} times the component scalar inverse with spatial projector $P_{W_{r,s}}$. On every non-diagonal actual block,

$$0 < B = A - cI \le A_H - 3cI/4 \le A_H,$$

because $0 \le A_a \le cI/4$. Positive matrix inversion gives $B^{-1} \ge A_H^{-1} \ge R_0$. The last comparison keeps an invariant polarization of the strong inverse. All such blocks lie in F, while the remaining diagonal part of R is positive. Extending by zero gives $R \ge R_0$ globally. This comparison never inverts the strong kernel and never compares inverses on the rotated diagonal low modes.

For the normalized pair vector $p_j = (\delta_{o_j} + \delta_{e_j})/\sqrt{2}$, taking spatial trace against P_V removes the polarization orientation: $\operatorname{tr}(P_V P_{W_{r,s}}) = 1$. The omitted diagonal transverse high branch would contribute zero: its layer vector is $(1,-1)/\sqrt{2}$, constant in transverse coordinates, and pairs to zero with p_j. The scalar zero branch is already zero in G. Therefore the local useful-pair trace is

$$n^{-3}H^{-1}\langle p_j, Gp_j\rangle = \frac{g_0 + \operatorname{Re} G(o_j, e_j)}{cn^5} \ge \frac{h_N/(8\pi^2) - 1}{cn^5}. \tag{21}$$

There is no factor two from V: only one common polarization was retained per transverse frequency. The other positive contributions are omitted in a lower estimate, not removed from the original model.

The component mean-zero projection is $I_N - N^{-1}11^*$, with diagonal $1 - 1/N$. Since R_0 is componentwise, summing (21) and using $R \ge R_0$ proves

$$\operatorname{tr}(P_{E_0} M_T) \ge \frac{N-1}{cn^5}\left(\frac{h_N}{8\pi^2} - 1\right). \tag{22}$$

This is a useful strong-kernel compression trace, not the blanket trace in (17). Arbitrary separation of o_j, e_j is allowed.

7.3 Positive-weight rotation and conclusion

We must not identify L with K. On each nonexceptional diagonal pair the strong scalar block has eigenvalues $0, 6H$, tensored with V. The weak perturbation has norm at most $c/4$, and the true low eigenvalue is at most $c/8$. For its unit eigenvector $l = l_0 + l_1$ in the strong low/high basis, the high equation gives

$$(6H - \lambda)l_1 = -P_{\text{high}} A_a l, \qquad \|l_1\| \le \frac{c/4}{6cn^2 - c/8} =: \epsilon_n = \frac{2}{48n^2 - 1}.$$

The norm difference of two rank-one projections is the sine of their angle, as follows by writing the projections in the strong orthonormal basis. Tensoring with V and taking the orthogonal sum over pairs gives $\|P_L - P_K\| \le \epsilon_n$. The exceptional and non-diagonal low projections are both zero.

By Lemma 5.2, $n^{5/2}T$ is an isometry on all N_d, so $\|P_E - P_{E_0}\| \le \epsilon_n$. Both ranks are $d = n-2$; their difference has rank at most $2d$ and trace norm at most $2d\epsilon_n$. Using Theorem 6.1,

$$|\operatorname{tr}((P_E - P_{E_0})M_T)| \le \frac{2d\epsilon_n K_n}{cn^5}.$$

Combine this with (22), divide by $d = 2(N-1)$ and use PSD norm at least average eigenvalue. The result is the lower half of (19); the upper half follows by orthogonal compression.

For $n \ge 16$, $D_n = 48n^2 - 1 \ge 12287$ and $(192/13)/D_n \le 1/(32\pi^2)$, since $\pi < 4$ implies $6144\pi^2 < 98304 < 13 \cdot 12287$. Also $4/[(6n^2-1)D_n] < 1/2$. Formula (12) therefore gives $4K_n/D_n \le h_N/(32\pi^2) + 1/2$, which proves (20). Since $h_N \ge \log(N+1)$, the scaled useful norm diverges.

The derivation is valid after complexification; the original response and useful projection are real, with the same traces and eigenvalues. A maximizing direction can be chosen real. Moreover the isometry $n^{5/2}T_L : L \to E$ proves the exact identity

$$\sup_{l \ne 0} \frac{cn^{10}\langle T_L l, M_T T_L l\rangle}{\|l\|^2} = cn^5 \|P_E M_T P_E\|.$$

This is a large dual response, not a physical Rayleigh quotient.

8 What the obstructions leave open

Theorems 6.1 and 7.1 exclude uniform scalar response estimates on the complete singleton rowspace and on its direct useful low-data subspace. Adding original prefix rows does not repair a blanket full-row norm estimate, since it retains the singleton compression. It can nevertheless change an optimized dual problem.

In fact, for useful data an added singleton coefficient in $E^\perp$ has zero low pairing but may lower response through cross terms. Correlated prefix or long-cell coefficients may do the same. The corresponding optimized response involves an inverse or Schur complement, which is not determined by a large direct compression. No theorem here asserts that all singleton methods fail, that the best full cancellation cost must lose a logarithm, or that an invisible field has quotient below c. The sign in (10) remains unresolved for the quadratic cut.

The polynomial lift and the logarithmic kernel bound are paid finite estimates, with their inadequate target-c strength explicitly retained. No selected numerical finite-size result is used as a proof premise, and no numerical rank, formal verification or continuum limit is asserted. These statements concern this discrete counting-norm family; they are not a physical Yang–Mills spectral gap or a resolution of the continuum problem. The classical logarithmic mechanism and the model-specific finite comparisons should be distinguished from any future claim of external originality, which remains unestablished here.

Model assessments and review context

At manuscript preparation on 12 November 2026, the antecedent research sources underwent separate involved analytical source audits in the same Codex/OpenAI programme context. Those sessions shared programme history; no provider independence, blind review or human peer review is claimed. The exact backend model identifiers were unavailable, and no more

specific identities are inferred. The source audits concern the finite analytical statements, not an exact PDF of this manuscript.

At that preparation stage, separate review of the consolidated source, remote reproducible production, visual PDF inspection and an exact-PDF scientific assessment had not been completed. Any assessment supplied in deposit metadata is a separate hash-bound artifact with its own actual model identity, context and objections. No score, stars, acceptance decision or assessment of an earlier artifact is transferred to this version. The full original reports, objections, corrections and context belong in the accompanying evidence package. AI-assisted preparation and involved source checks are not represented as independent peer review or publication acceptance. Lluis Eriksson, the listed author and programme owner, is also the owner, creator and administrator of AIRR, the intended deposit destination. This author–platform relationship is not independent editorial or human review. No affiliation or completed personal review is invented.

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References

[1] L. Eriksson, *Coverage and coexactness in checkerboard slice observations*, programme working paper, AIRR, version 1 (2026). `https://airr.science/papers/ARR-2026-451FW8N5MC8KYV5S/versions/v1/`.

[2] R. Kenyon, *The Laplacian and $\bar{\partial}$ operators on critical planar graphs*, arXiv:math-ph/0202018 (2002), especially Theorem 7.3. `https://arxiv.org/abs/math-ph/0202018`.

[3] G. F. Lawler and V. Limic, *Random Walk: A Modern Introduction*, Cambridge University Press, 2010. Author-hosted text, Sections 1.2 and 4.4, especially Theorem 4.4.4. `https://www.math.uchicago.edu/~lawler/srwbook.pdf`.