Sharp Logarithmic Loss for Endpoint Observations in a Discrete Coexact Model

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Abstract

We determine the asymptotic order of the minimum energy invisible to a fixed canonical partition in a finite coexact vector-field model. On an even three-dimensional torus with counting norm and checkerboard source weights $H = cn^2$, $a = c/48$, the full endpoint cell observations leave a nonzero kernel whose minimum Rayleigh quotient has order $c/\log n$. Explicit finite upper and lower bounds are proved. The upper witness is real, lies in the actual image of the forward curl adjoint, and annihilates every original cell mean. It follows that finite penalties on these observations cannot restore a size-uniform target c. The proof combines the complete restricted Fourier spectrum, an original-row range theorem, finite Green estimates, optimization over complementary singleton coefficients, and frequency-band removal localized to the lowest nonzero component sector. We also give an exact conditional-cost identity for the remaining rows of moving partitions. The best canonical partition, including a quadratic moving cut, remains unresolved. All arguments are finite and analytical. Green-function logarithms and Schur minimization are classical; external priority for the precise model-specific theorem is unestablished. No physical Yang–Mills gap or continuum conclusion is asserted.

1 Main result and relation to earlier work

Our main conclusion is a two-sided $c/\log n$ estimate for the full observation kernel of one admissible fixed partition. The distinction between a fixed partition and the best partition is essential. A weak lower bound alone would not refute a uniform gap; here a real invisible field supplies the matching upper order. Conversely, failure of this fixed partition does not establish failure for moving partitions.

This manuscript is a proposed linked revision of the programme working paper *Spectral Reduction and Logarithmic Observation Response in a Discrete Coexact Model*, AIRR version 1 [2]. That version established the spectrum, range, and direct-response bounds but explicitly left optimized cancellation open. The present revision retains their proofs and adds the full endpoint-kernel theorem and conditional residual cost. The earlier version remains a distinct immutable artifact; its assessments are not assessments of this revision.

Checkerboard weights couple two Fourier frequencies, while the actual curl-adjoint image imposes two transverse polarizations and removes the harmonic zero mode. Neither a scalar block alone nor injectivity on low modes decides high-mode cancellation. Sections 3–5 retain all those restrictions and the original rows. The response estimates in Sections 6–7 supply

finite constants. Section 8 then handles complementary coefficients and constructs a full-cell witness; Section 9 isolates the remaining problem for moving cuts.

Two-dimensional logarithmic Green functions are classical. Kenyon's Theorem 7.3 treats critical planar graphs [4]; Lawler and Limic develop continuous-time walks and the two-dimensional potential kernel, including Theorem 4.4.4 [5]. Their graph, volume, and operator normalizations are not identified with our finite coexact compression. Schur minimization is also classical; the primary abstract of Corach, Maestripieri, and Stojanoff explicitly relates shorted operators and projections [3]. Every finite comparison used here is proved below. These bounded primary-source comparisons do not establish priority for the precise endpoint theorem; neither the logarithmic mechanism nor the general minimization algebra is claimed as new.

The earlier programme working paper *Coverage and coexactness in checkerboard slice observations* [1] concerns coverage, projection issues and an ambient-field lower bound. Those results are not presented as new here. Remark 6.3 distinguishes its domain from the present actual-coexact lower bound.

2 The original model and observations

Let $n \geq 4$ be even and $X = (\mathbf{Z}/n\mathbf{Z})^3$. All physical norms and adjoints use counting measure. For $u : X \to \mathbf{R}^3$, put

$$\Delta_j u(x) = u(x + e_j) - u(x), \qquad (d_1 u)_{j\ell} = \Delta_j u_\ell - \Delta_\ell u_j \quad (j < \ell).$$

The norm on two-forms counts each pair $j < \ell$ once. Our actual domain is $U = \operatorname{im} d_1^*$, not an abstract divergence-free space with freely added constants. We prove its exact Fourier description in Section 3.

There are two original channels, indexed by $\varepsilon \in \{-1, +1\}$. Write $\chi(x) = (-1)^{x_1+x_2+x_3}$ and, initially, let $H \geq a > 0$. Set $A_0 = (H + a)/2$, $B_0 = (H - a)/2$, and

$$b_\varepsilon(x) = A_0 + \varepsilon B_0 \chi(x), \qquad Q_\varepsilon(u) = \sum_{x,j,r} b_\varepsilon(x) |\Delta_j u_r(x)|^2. \tag{1}$$

Thus $\varepsilon = -1$ has strong weight H on odd source vertices; the other channel exchanges the two parities. Write A_ε for the self-adjoint operator of this form *restricted to* U. We do not assert that the ambient weighted-gradient operator preserves U. The two-channel form and domain are the original direct sum.

For each $i \in \mathbf{Z}_n$ choose $\tau_i \in \{0, \dots, n-1\}$. The original nonwrapping cells are

$$\{i\} \times I \times J, \quad I \in \{[0, \tau_i - 1], \{\tau_i\}, [\tau_i + 1, n-1]\}, \quad J \in \{\{0\}, [1, n-1]\},$$

with empty intervals omitted. Every spatial component and channel is observed separately, with

$$(Cu)_{r,B} = \frac{1}{\sqrt{n^3|B|}} \sum_{x \in B} u_r(x). \tag{2}$$

No channel rotation, adapted mean or projected row replaces these observations. Unless stated otherwise, proofs are per channel; direct sums give the same constants for both channels.

Our quantitative specialization is

$$H = cn^2, \qquad a = c/48, \qquad c > 0. \tag{3}$$

When $f(i) = i + \tau_i \pmod n$ is a permutation, denote by T the genuine singleton submap

$$(Tu)_i = n^{-3/2} u(i, \tau_i, 0). \tag{4}$$

The quadratic cut $\tau_i = 2i^2 \pmod n$ has this property for dyadic n: $f(i) = f(j)$ implies $(i-j)(1+2(i+j)) = 0 \pmod n$, and the second factor is odd and hence invertible.

For the endpoint cut $\tau_i = 0$, let C_{end} denote all four original cells on each slice, in every spatial component. Put

$$\lambda_{\mathrm{end}}(n) = \min_{0 \neq w \in U \cap \ker C_{\mathrm{end}}} \frac{Q(w)}{\|w\|^2}, \qquad N = n/2, \qquad h_N = \sum_{j=1}^{N} \frac{1}{j}.$$

For $n \geq 16$ this kernel is nonzero: $\dim U = 2(n^3 - 1) > 12n$, the number of scalar observation rows. Define

$$K_n = \frac{48}{13} h_N + \frac{1}{6n^2 - 1}, \tag{5}$$
$$b_{\mathrm{end}}(n) = \frac{h_N}{16\pi^2} - \frac{17}{2} - 32(2\pi + 8)^2 - \frac{256 K_n^2}{48n^2 - 1}.$$

Theorem 2.1 (Sharp endpoint-kernel order)**.** *Under* (3)*, for even $n \geq 16$ with $b_{\mathrm{end}}(n) > 0$,*

$$\frac{c}{K_n} \leq \lambda_{\mathrm{end}}(n) \leq \frac{c\pi^2}{2n^2} + \frac{16c}{13 b_{\mathrm{end}}(n)}. \tag{6}$$

In particular $\lambda_{\mathrm{end}}(n) = \Theta(c/\log n)$ along even sizes, with positive comparison constants independent of n and c. The same bounds hold for both original channels and their direct sum. For all sufficiently large even sizes there is a nonzero real field annihilating every endpoint observation with $Q(w) < c\|w\|^2$.

The lower bound is proved in Corollary 6.2. The upper construction, including its realness and all original cell cancellations, is proved in Section 8. No exact asymptotic constant or small-size threshold is claimed. The theorem concerns this fixed endpoint cut, not the best canonical partition.

3 Complete spectrum on the actual coexact domain

Use the unitary convention

$$u(x) = n^{-3/2} \sum_k \widehat{u}(k) \mathrm{e}^{2\pi i k \cdot x/n}, \qquad g_j(k) = \mathrm{e}^{2\pi i k_j/n} - 1.$$

Complexification is used for calculation; real multiplicities are recovered without doubling the complex dimension.

Lemma 3.1 (The actual image)**.** *At $k \neq 0$, the Fourier image of d_1^* is $g(k)^\perp$ of dimension two; at $k = 0$ it is zero. Consequently* $\dim_{\mathbf{R}} U = 2(n^3 - 1)$.

Proof. For a skew array $W_{j\ell} = -W_{\ell j}$, $(d_1^* W)_\ell = \sum_j \overline{g_j} W_{j\ell}$, which is orthogonal to g. Conversely, for $v \perp g$ and $D = |g|^2 > 0$, set

$$W_{j\ell} = \frac{g_j v_\ell - g_\ell v_j}{D}.$$

Its adjoint image is $v_\ell - g_\ell(g^* v)/D = v_\ell$. At zero all symbols vanish. The formula respects conjugation at negative frequencies, so real fields have real preimages. Counting the nonzero frequencies proves the dimension and excludes harmonic constants. □

Let $q = (n/2, n/2, n/2)$ and consider an unordered pair $\{k, k+q\}$. Put $h = g(k+q)$, $D = |g|^2$, $D' = |h|^2$ and $\Gamma = g^* h$. Direct calculation gives

$$D' = 12 - D, \qquad \Gamma = 2i \sum_j \sin(2\pi k_j/n). \tag{7}$$

Theorem 3.2 (All restricted eigenvalues)**.** *For a nonexceptional pair, both frequencies being nonzero, the four eigenvalues per channel are*

$$\begin{aligned} \lambda_{1,\pm} &= 6A_0 \pm \sqrt{A_0^2 (D-6)^2 + B_0^2 |\Gamma|^2}, \\ \lambda_{2,\pm} &= 6A_0 \pm \sqrt{A_0^2 (D-6)^2 + B_0^2 |\Gamma|^4 / (DD')}. \end{aligned} \tag{8}$$

Repeated roots retain their multiplicities. The pair $\{0, q\}$ has only the q plane and its two eigenvalues are $12A_0$. These blocks exhaust U: their total multiplicity is $4(n^3/2 - 1) + 2 = 2(n^3 - 1)$ per channel. Both original channels have the same spectrum.

Proof. For coefficients $x = \widehat{u}(k)$, $y = \widehat{u}(k+q)$, the original source-weighted form is exactly

$$A_0 D |x|^2 + A_0 D' |y|^2 + 2\varepsilon B_0 \operatorname{Re}(\Gamma x^* y).$$

Indeed multiplication by χ shifts frequency by q, and summing the three forward differences gives $g^* h$ with no extra volume factor. Choose isometries $E, F : \mathbf{C}^2 \to \mathbf{C}^3$ onto $g^\perp, h^\perp$. The compressed block has diagonal $A_0 D I_2, A_0 D' I_2$ and off-diagonal $\varepsilon B_0 \Gamma E^* F$.

The singular values of $E^* F$ are 1 and $\rho = |\Gamma|/\sqrt{DD'}$. The planes have a common unit vector, and their projections $P_g = I - gg^*/D$, $P_h = I - hh^*/D'$ satisfy $\operatorname{tr}(P_g P_h) = 1 + |\Gamma|^2/(DD')$. This proves the two singular values, including coincident or orthogonally normal planes. Singular-value decomposition and absorption of the coupling phase give two scalar two-by-two blocks with coupling moduli $B_0|\Gamma|$ and $B_0|\Gamma|\rho$. Their eigenvalues are (8).

At zero the first coefficient is absent by Lemma 3.1; at q the remaining plane has $D' = 12$. Changing the channel sign is conjugation by a sign on the second coefficient. Complex conjugation takes each block to its negative-frequency block and commutes with the real form, so the real eigenspace dimensions equal those of the complexification. The count above is therefore also the real count, not twice that count. □

Proposition 3.3 (The whole low space)**.** *Under* (3)*, let* $L = 1_{(0,c)}(A_\varepsilon)U$ *and* $F = U \ominus L$*. Then*

$$\dim L = n - 2, \qquad A_L \le (c/8)I, \qquad A_F \ge (16c/3)I.$$

The space L *consists exactly of the low branches at nonexceptional diagonal pairs* $k = (t, t, t)$*; both transverse polarizations are retained.*

Proof. First keep only the strong source weight, setting $a = 0$. Put $S = \sum_j \mathrm{e}^{2\pi i k_j/n}$. Formula (8) gives the attained smallest coexact pair eigenvalue $H(3 - |S|)$. It vanishes exactly when the three phases agree. At such a nonexceptional diagonal pair both couplings have the same singular value and the zero multiplicity is two. The exceptional pair contributes no zero mode. Thus the actual strong kernel has dimension $n - 2$ per channel.

If the residues are not all equal, at least two pairwise differences are nonzero. The identity

$$9 - |S|^2 = 4 \sum_{j<\ell} \sin^2\left(\frac{\pi(k_j - k_\ell)}{n}\right)$$

and division by $3+|S| \le 6$ show $H(3-|S|) \ge (4H/3)\sin^2(\pi/n) \ge 16H/(3n^2)$. Here $\sin(\pi/n) \ge 2/n$ is the concavity chord on $[0, \pi/2]$. Adding the positive weak form preserves this lower bound.

At a diagonal nonexceptional pair, $\theta = 2\pi t/n$, the two roots, each with multiplicity two, are

$$\mu_\pm = 3(H + a) \pm 3\sqrt{(H + a)^2 - 4Ha\sin^2\theta}.$$

They satisfy $0 < \mu_- \le 6a = c/8$ and $\mu_+ \ge 6H$. The exceptional eigenvalue is $6(H + a) \ge 6H$. There are $n/2 - 1$ nonexceptional diagonal pairs, proving the count and the complete separation. In particular no other low direction has been silently omitted. □

4 Original rows and a paid range theorem

For a cell $B = \{i\} \times I \times J$, direct substitution gives

$$(Cu)_{r,B} = \frac{1}{n^3\sqrt{|I||J|}} \sum_k \widehat{u}_r(k)\mathrm{e}^{2\pi i k_1 i/n} \left(\sum_{j\in I} \mathrm{e}^{2\pi i k_2 j/n}\right)\left(\sum_{\ell\in J} \mathrm{e}^{2\pi i k_3 \ell/n}\right). \tag{9}$$

Each pair member has its own three factors. The zero coefficient is absent on U. Finite sums, rather than a geometric quotient at zero, fix the normalization without removable singularities.

Let N_d be the actual diagonal Fourier space: its fields have the form $u(x) = v(x_1+x_2+x_3)$ with mean-zero v and $v_1+v_2+v_3 = 0$. This follows from Lemma 3.1. Thus $\dim N_d = 2(n-1)$ and $L \subset N_d$. Put $O = U \ominus N_d$. The following result permits every original low observation to be cancelled by actual high modes, including row dependencies.

Theorem 4.1 (Range equality and explicit lift)**.** *For every original cut, whether or not* f *is a permutation,*

$$\operatorname{ran} C_O = \operatorname{ran}(C|_U) = \operatorname{ran} C_F.$$

There is a linear $R_d : N_d \to O$ *such that*

$$C_O R_d = -C_{N_d}, \qquad \|R_d\|^2 \le K_n^{\mathrm{lift}} := \frac{n^5}{16(n-3)}.$$

Proof. For any original row coefficient α, put $g = C^*_{\mathrm{amb}}\alpha$, where the ambient map uses precisely (2). For fixed x_1, x_2, g is constant in x_3 on $1, \dots, n-1$, because that is a single long third-coordinate cell. Let $v = P_U g$. Since $U^\perp = \ker d_1$, $d_1 g = d_1 v$. Every component of this curl is independent of x_3 on $1, \dots, n-2$; its third difference therefore vanishes on $1, \dots, n-3$.

If $C^*_O \alpha = 0$, then $v \in N_d$. Each component of $d_1 v$ depends only on $s = x_1 + x_2 + x_3$. Equality of its values at $x_3 = 1, 2$, for all x_1, x_2, forces equal values at adjacent residues of s. It is constant, and its sum is zero because it is a periodic curl. Hence $d_1 v = 0$ and $v \in U \cap U^\perp = 0$. We have proved equality of the annihilators of $\operatorname{ran} C_O$ and $\operatorname{ran}(C|_U)$. Finite-dimensional orthogonal complementation gives the first range equality; $O \subset F \subset U$ gives the other.

For the quantitative claim decompose $v = d + o \in N_d \oplus O$. On the slab $1 \le x_3 \le n-3$, $\Delta_3 d_1 d = -\Delta_3 d_1 o$. The first expression depends only on s, so its squared norm is equally distributed among the n third-coordinate slices. Since the curl symbol has squared norm at most 12 and $|g_3|^2 \le 4$,

$$\|\Delta_3 d_1 d\|^2 \le \frac{48n}{n-3}\|o\|^2.$$

At a nonzero diagonal frequency $z = \mathrm{e}^{2\pi i t/n}$ the same squared symbol on d is $3|z-1|^4$. Using $|z-1|^2 \ge 16/n^2$ gives $\|\Delta_3 d_1 d\|^2 \ge 768 n^{-4}\|d\|^2$. Consequently

$$\|C^*_{N_d}\alpha\|^2 \le K_n^{\mathrm{lift}} \|C^*_O \alpha\|^2. \tag{10}$$

Define $S(C^*_O\alpha) = C^*_{N_d}\alpha$. Inequality (10) makes this well-defined on its range, with norm at most $\sqrt{K_n^{\mathrm{lift}}}$. Extend it by zero on the orthogonal complement. Then $C_O S^* = C_{N_d}$, and $R_d = -S^*$ proves the claim. This is a proved factorization, not an assumed inverse or a full-row-rank assertion. □

5 The exact shifted cancellation criterion

Under (3), set $B = A_F - cI$, $D_L = cI - A_L$ and

$$M = C_F B^{-1} C_F^*, \qquad S_C = \operatorname{ran}(C|_U).$$

Proposition 3.3 gives $B \ge 13c/3$ and $7c/8 \le D_L < cI$. Theorem 4.1 implies $\ker M = S_C^\perp$. Thus M is invertible on S_C, and $M^\dagger$ denotes that inverse extended by zero.

Theorem 5.1 (Exact cost, with dependent rows)**.** *For every $l \in L$,*

$$\min_{f \in F:\ C_F f = -C_L l} \langle f, Bf\rangle = \langle C_L l, M^\dagger C_L l\rangle, \tag{11}$$

$$f_* = -B^{-1} C_F^* M^\dagger C_L l.$$

Consequently $Q(w) \ge c\|w\|^2$ for all $w \in U \cap \ker C$ if and only if

$$C_L^* M^\dagger C_L - D_L \ge 0 \quad \text{on } L. \tag{12}$$

Its dimension is $n-2$ per original channel, or $2(n-2)$ on their direct sum.

Proof. Let $y = C_L l \in S_C$. Then $C_F f_* = -MM^\dagger y = -y$. For every other feasible $f = f_* + h$, $C_F h = 0$ and $\langle h, Bf_* \rangle = -\langle C_F h, M^\dagger y \rangle = 0$. Expanding the square proves the minimum. For $w = l + f$ the spectral splitting gives

$$Q(w) - c\|w\|^2 = -\langle l, D_L l \rangle + \langle f, Bf \rangle.$$

Use the attained minimum in both directions. Every l is feasible by Theorem 4.1, so no extra range assumption hides a bad direction. □

The lift also pays an explicit but inadequate upper bound:

$$\langle R_d l, BR_d l \rangle \le (12H - c) K_n^{\text{lift}} \|l\|^2,$$

because $A \le 12HI$ and $R_d l \in O \subset F$. This grows like cn^6 and does not decide (12). For later use the corresponding dual identity is

$$\langle y, M^\dagger y \rangle = \sup_\alpha \{2\,\mathrm{Re}\langle \alpha, y \rangle - \langle \alpha, M\alpha \rangle\}, \qquad y \in S_C. \tag{13}$$

It follows by completing the square on S_C; components in $S_C^\perp$ pair and respond to zero.

Lemma 5.2 (Singleton frame)**.** *If f is a permutation, $T^*T = n^{-5} I$ as a form on all of N_d. In particular this holds on L and on the strong kernel.*

Proof. For $u(x) = v(s)$, $\|u\|^2 = n^2 \sum_s |v(s)|^2$, whereas

$$\|Tu\|^2 = n^{-3} \sum_i |v(f(i))|^2 = n^{-3} \sum_s |v(s)|^2.$$

□

6 Whole singleton response and an invisible-field bound

Define $R = P_F B^{-1} P_F$ extended by zero and $M_T = TRT^*$. Put $N = n/2$, $h_N = \sum_{j=1}^N 1/j$ and

$$K_n = \frac{48}{13} h_N + \frac{1}{6n^2 - 1}. \tag{14}$$

Theorem 6.1 (Finite whole-response bounds)**.** *For every permutation cut and either original channel,*

$$\frac{h_N}{24\pi^2 cn^5} \le \|M_T\| \le \frac{K_n}{cn^5}. \tag{15}$$

No fixed constant bounds $cn^5 \|M_T\|$ for all such sizes. The same blanket scalar bound fails for the entire original-row response M.

Proof. Write $k = (t + r, t + s, t)$ and $\delta(r, s) = 3 - |1 + \mathrm{e}^{2\pi i r/n} + \mathrm{e}^{2\pi i s/n}|$. For $g = T^*\alpha$,

$$\widehat{g}(t + r, t + s, t) = n^{-3} \sum_i \alpha_i \mathrm{e}^{-2\pi i \{tf(i) + ri + s\tau_i\}/n}.$$

Since f is a permutation, orthogonality in t gives the exact identity

$$\sum_t |\widehat{g}(t+r,t+s,t)|^2 = n^{-5}\|\alpha\|^2. \tag{16}$$

The pair shift q changes only t by $n/2$. Every transverse (r,s) therefore contains exactly $n/2$ unordered pairs, and only $(0,0)$ is diagonal. Off that fibre, Proposition 3.3 and its strong-form calculation give

$$(A-cI)^{-1} \le \frac{1}{cn^2\delta - c} I \le \frac{16}{13cn^2\delta} I.$$

The last step uses $cn^2\delta \ge 16c/3$. On the diagonal high space the bound is $1/[c(6n^2-1)]$. Projections cannot increase the input norm. Thus (16) gives

$$\langle \alpha, M_T \alpha \rangle \le \frac{\|\alpha\|^2}{cn^5} \left\{ \frac{16}{13n^2} \sum_{(r,s)\neq 0} \frac{1}{\delta} + \frac{1}{6n^2-1} \right\}. \tag{17}$$

For unique centered residues in $[-N, N-1]^2$, the sine identity in the proof of Proposition 3.3 implies

$$\delta \ge \frac{2}{3}\{\sin^2(\pi r/n) + \sin^2(\pi s/n)\} \ge \frac{8}{3n^2}(r^2+s^2).$$

At most $8j$ centered sites have $\max(|r|,|s|) = j$. Therefore

$$\sum_{(r,s)\neq 0} \frac{1}{\delta} \le 3n^2 h_N, \qquad \sum_{(r,s)\neq 0} \frac{1}{\delta} \ge \frac{n^2 h_N}{4\pi^2}. \tag{18}$$

For the lower inequality retain $r = j$, $1 \le s \le j$, $1 \le j \le N$. These are distinct residues and $\delta \le 2 - \cos(2\pi r/n) - \cos(2\pi s/n) \le 4\pi^2 j^2/n^2$. The upper inequality and (17) prove the upper norm.

For the lower norm, even-sum translations preserve U, the source form, F and R. They are transitive on each vertex parity. The spatial trace of $R(x,x)$ therefore has two values. The sampled vertex parity is that of $f(i)$, so exactly half the singleton points have each parity. Compared with $n^3/2$ vertices of each parity, the row normalization gives

$$\operatorname{tr} M_T = n^{-5} \operatorname{tr}_F B^{-1}. \tag{19}$$

On each non-diagonal pair one genuine strong-minimizing polarization has energy $cn^2\delta$. The weak form is at most $12a = c/4$ on a unit vector. Rayleigh's principle supplies a full eigenvalue $\lambda \le cn^2\delta + c/4$. It belongs to F; hence $(\lambda - c)^{-1} \ge (cn^2\delta)^{-1}$. There are exactly $n/2$ such pairs per transverse fibre. Consequently

$$\operatorname{tr}_F B^{-1} \ge \frac{1}{2cn} \sum_{(r,s)\neq 0} \frac{1}{\delta} \ge \frac{nh_N}{8\pi^2 c}.$$

The largest eigenvalue of a PSD operator on $3n$ rows is at least its trace divided by $3n$. This proves the lower half of (15). Since $h_N \ge \log(N+1)$, the scaled norm diverges. Finally, inserting singleton coefficients by zeros is an isometry J with $T = J^*C$, so $M_T = J^*MJ$ and $\|M\| \ge \|M_T\|$. No independent-row assumption is used in this compression argument. □

Corollary 6.2 (A paid logarithmic kernel bound)**.** *For every $w \in U \cap \ker C$ under a permutation cut,*

$$Q(w) \geq \frac{c}{K_n}\|w\|^2 \geq \frac{c}{4(1+\log n)}\|w\|^2. \tag{20}$$

The same coefficient holds for the original two-channel direct sum.

Proof. For $l \in L$, insert $\alpha = Tl$ into the singleton coordinates of the full rowspace. Its pairing with $y = C_L l$ is $n^{-5}\|l\|^2$, and Theorem 6.1 bounds its response by $K_n\|l\|^2/(cn^{10})$. Optimize (13) along $t\alpha$ at $t = cn^5/K_n$ to obtain $\langle C_L l, M^\dagger C_L l\rangle \geq (c/K_n)\|l\|^2$. For an invisible $w = l + f$, the high part is feasible and

$$Q(w) = \langle l, A_L l\rangle + \langle f, Bf\rangle + c\|f\|^2 \geq (c/K_n)\|l\|^2 + c\|f\|^2 \geq (c/K_n)\|w\|^2,$$

since $K_n > 1$. Put $L_0 = 1+\log n$. The inequalities $h_N \leq 1+\log(n/2) \leq L_0$ and $1/(6n^2-1) \leq 1/95 < (4/13)L_0$ give $K_n < 4L_0$. The proofs are channelwise, so summation preserves the coefficient. □

Remark 6.3 (Coefficient and domain comparison)*.* The coefficient c/K_n strictly exceeds the prior working-paper benchmark $c/[5(1+\log n)]$ [1] under the same permutation-cut hypothesis. That estimate allowed ambient fields; (20) is on the actual coexact domain with a permutation cut. The numerical improvement does not extend the larger ambient domain, change logarithmic order, or establish the target c. In particular c/K_n is inadequate to dominate $D_L \geq 7c/8$ in (12).

7 Logarithmic response persists on useful low data

Let $E = T_L L$ and let P_E be the ordinary singleton row-space projection. Lemma 5.2 gives $P_E = n^5 T_L T_L^*$ and $\dim E = n-2$.

Theorem 7.1 (Useful-space obstruction)**.** *For every even $n \geq 4$ and permutation cut,*

$$\frac{h_N/(16\pi^2) - 1/2 - 4K_n/(48n^2-1)}{cn^5} \leq \|P_E M_T P_E\| \leq \frac{K_n}{cn^5}. \tag{21}$$

For even $n \geq 16$ the lower bound implies

$$cn^5\|P_E M_T P_E\| \geq \frac{h_N}{32\pi^2} - 1. \tag{22}$$

Thus a uniform response bound fails even for unaugmented tests $\alpha = T_L l$. The lower expressions need not be positive at small sizes.

7.1 A finite component Green estimate

Let K be the actual strong kernel. For strong weight on odd vertices, the strong scalar graph has N components, the paired layers $(s = 2j-1, s = 2j)$, $s = x_1 + x_2 + x_3$. On each layer use coordinates $y = (x_1, x_2)$; x_3 is determined. The three strong edges from an odd vertex have target coordinates $y, y+e_1, y+e_2$ in the even layer. The unit-weight scalar component Laplacian D is degree three on $2n^2$ vertices, with transverse Fourier matrix

$$\begin{pmatrix} 3 & -S(r,s) \\ -\overline{S(r,s)} & 3 \end{pmatrix}, \qquad S(r,s) = 1 + \mathrm{e}^{2\pi i r/n} + \mathrm{e}^{2\pi i s/n}.$$

Its eigenvalues are $\delta = 3 - |S|$ and $\delta_+ = 3 + |S|$. Only $\delta(0,0)$ vanishes, proving connectedness. Interchanging source parity gives the same graph with paired layers $(2j, 2j+1)$.

Lemma 7.2 (Component Green bounds)**.** *For $G = D^\dagger$, the diagonal value g_0 is constant and*

$$g_0 \geq \frac{h_N}{8\pi^2}, \qquad G(x,y) \geq -1 \quad \textit{for all } x, y.$$

Proof. Translations within layers and layer exchange with reflection act transitively on vertices. The diagonal spectral formula is

$$g_0 = \frac{1}{2n^2} \left\{ \sum_{(r,s)\neq 0} \frac{1}{\delta} + \sum_{r,s} \frac{1}{\delta_+} \right\}.$$

The lower lattice sum in (18) gives the diagonal bound. For the off-diagonal bound use the finite heat kernel $p_t = \exp(-tD)$. It is entrywise nonnegative, since $D = 3I - \mathrm{Adj}$ and $\exp(t\mathrm{Adj})$ has a nonnegative power series. Hence the integral representation

$$G(x,y) = \int_0^\infty \{p_t(x,y) - (2n^2)^{-1}\}\, dt$$

is bounded below by $-1/2$ on $0 \leq t \leq n^2$. For the tail, $p_t - P_{\mathrm{const}}$ is PSD with constant diagonal. Cauchy–Schwarz bounds each entry by that diagonal, its trace divided by $2n^2$. The absolute tail is therefore at most

$$\frac{1}{2n^2} \sum_{\lambda \neq 0} \frac{\mathrm{e}^{-n^2\lambda}}{\lambda}.$$

For the δ branch use $\delta \geq 8(r^2+s^2)/(3n^2)$ and the at-most $8j$ shell count. Its contribution is at most

$$\frac{3}{2} \sum_{j \geq 1} \frac{\mathrm{e}^{-(8/3)j^2}}{j} \leq \frac{3}{2} \sum_{j \geq 1} (1/8)^j = \frac{3}{14}.$$

Here $\mathrm{e}^{8/3} > 8$ follows already from four nonnegative Taylor terms. The n^2 high-branch eigenvalues are at least 3, giving a contribution at most $\mathrm{e}^{-3n^2}/6 \leq 1/6$. Thus $G(x,y) \geq -1/2 - 3/14 - 1/6 = -37/42 \geq -1$. No positivity of the Green function or an isolated low band was assumed. □

7.2 The scalar component inside the actual coexact space

Put $V = (1,1,1)^\perp$. The strong kernel K consists exactly of component constants $v_j \in V$ with $\sum_j v_j = 0$. Such fields have mean-zero diagonal Fourier coefficients in V, so Lemma 3.1 puts them in U. Their dimension $2(N-1) = n-2$ equals the full strong-kernel count. Every permutation singleton map selects one point in each s layer, hence one point o_j, e_j in each layer of each component. The space $E_0 = T_K K$ consists of equal pair values in V, with component mean zero.

The next construction prevents an ambient scalar inverse from being mistaken for an actual coexact inverse. At transverse $(r,s) \neq 0$, let $z = (\mathrm{e}^{2\pi i r/n}, \mathrm{e}^{2\pi i s/n}, 1)$ and $W_{r,s} = \{(1,1,1), z\}^\perp$, a line in V. For frequencies $k = (t+r, t+s, t)$ and $k+q$, both $g(k)$ and $g(k+q)$ are

linear combinations of z and $(1,1,1)$. Thus $W_{r,s}$ lies in both actual coexact planes, and is independent of t. Its two coefficient directions form an invariant first-polarization subspace of the strong block, with eigenvalues $H\delta, H\delta_+$. Retaining this line in the transverse Fourier transform of each two-layer component gives actual coexact fields. The projector onto the line depends only on transverse frequency, so it acts inside each component and does not mix components.

Let R_0 be this selected strong inverse, zero on the entire diagonal transverse fibre. It is H^{-1} times the component scalar inverse with spatial projector $P_{W_{r,s}}$. On every non-diagonal actual block,

$$0 < B = A - cI \le A_H - 3cI/4 \le A_H,$$

because $0 \le A_a \le cI/4$. Positive matrix inversion gives $B^{-1} \ge A_H^{-1} \ge R_0$. The last comparison keeps an invariant polarization of the strong inverse. All such blocks lie in F, while the remaining diagonal part of R is positive. Extending by zero gives $R \ge R_0$ globally. This comparison never inverts the strong kernel and never compares inverses on the rotated diagonal low modes.

For the normalized pair vector $p_j = (\delta_{o_j} + \delta_{e_j})/\sqrt{2}$, taking spatial trace against P_V removes the polarization orientation: $\operatorname{tr}(P_V P_{W_{r,s}}) = 1$. The omitted diagonal transverse high branch would contribute zero: its layer vector is $(1,-1)/\sqrt{2}$, constant in transverse coordinates, and pairs to zero with p_j. The scalar zero branch is already zero in G. Therefore the local useful-pair trace is

$$n^{-3}H^{-1}\langle p_j, Gp_j\rangle = \frac{g_0 + \operatorname{Re} G(o_j, e_j)}{cn^5} \ge \frac{h_N/(8\pi^2) - 1}{cn^5}. \tag{23}$$

There is no factor two from V: only one common polarization was retained per transverse frequency. The other positive contributions are omitted in a lower estimate, not removed from the original model.

The component mean-zero projection is $I_N - N^{-1}11^*$, with diagonal $1 - 1/N$. Since R_0 is componentwise, summing (23) and using $R \ge R_0$ proves

$$\operatorname{tr}(P_{E_0} M_T) \ge \frac{N-1}{cn^5}\left(\frac{h_N}{8\pi^2} - 1\right). \tag{24}$$

This is a useful strong-kernel compression trace, not the blanket trace in (19). Arbitrary separation of o_j, e_j is allowed.

7.3 Positive-weight rotation and conclusion

We must not identify L with K. On each nonexceptional diagonal pair the strong scalar block has eigenvalues $0, 6H$, tensored with V. The weak perturbation has norm at most $c/4$, and the true low eigenvalue is at most $c/8$. For its unit eigenvector $l = l_0 + l_1$ in the strong low/high basis, the high equation gives

$$(6H - \lambda)l_1 = -P_{\text{high}} A_a l, \qquad \|l_1\| \le \frac{c/4}{6cn^2 - c/8} =: \epsilon_n = \frac{2}{48n^2 - 1}.$$

The norm difference of two rank-one projections is the sine of their angle, as follows by writing the projections in the strong orthonormal basis. Tensoring with V and taking the orthogonal

sum over pairs gives $\|P_L - P_K\| \le \epsilon_n$. The exceptional and non-diagonal low projections are both zero.

By Lemma 5.2, $n^{5/2}T$ is an isometry on all N_d, so $\|P_E - P_{E_0}\| \le \epsilon_n$. Both ranks are $d = n-2$; their difference has rank at most $2d$ and trace norm at most $2d\epsilon_n$. Using Theorem 6.1,

$$|\operatorname{tr}((P_E - P_{E_0})M_T)| \le \frac{2d\epsilon_n K_n}{cn^5}.$$

Combine this with (24), divide by $d = 2(N-1)$ and use PSD norm at least average eigenvalue. The result is the lower half of (21); the upper half follows by orthogonal compression.

For $n \ge 16$, $D_n = 48n^2 - 1 \ge 12287$ and $(192/13)/D_n \le 1/(32\pi^2)$, since $\pi < 4$ implies $6144\pi^2 < 98304 < 13 \cdot 12287$. Also $4/[(6n^2-1)D_n] < 1/2$. Formula (14) therefore gives $4K_n/D_n \le h_N/(32\pi^2) + 1/2$, which proves (22). Since $h_N \ge \log(N+1)$, the scaled useful norm diverges.

The derivation is valid after complexification; the original response and useful projection are real, with the same traces and eigenvalues. A maximizing direction can be chosen real. Moreover the isometry $n^{5/2}T_L : L \to E$ proves the exact identity

$$\sup_{l \ne 0} \frac{cn^{10}\langle T_L l, M_T T_L l\rangle}{\|l\|^2} = cn^5 \|P_E M_T P_E\|.$$

This is a large dual response, not a physical Rayleigh quotient.

8 Sharp loss for all endpoint observations

We now fix $\tau_i = 0$ for every i and write C_{end} for the *full* map (2). Its cells on each first-coordinate slice are the four products of $\{0\}$ and $[1, n-1]$. The singleton map is evaluation on $(i, 0, 0)$. Throughout this section $n \ge 16$ is even. We prove Theorem 2.1 by retaining a low-frequency sector while cancelling every endpoint cell, including the long rectangles.

8.1 A selected response and its cross terms

Let $S = n^{3/2}T$ be unnormalized evaluation and $W = V^n$ in singleton row space. Using the actual common-polarization inverse R_0 constructed in Section 7, define

$$\mathcal{K} = S(HR_0)S^*.$$

It annihilates $W^\perp$. On W it is a direct sum of two-point blocks, one for each strong component. Write these real blocks as

$$\mathcal{K}_j = \begin{pmatrix} A_0' & B_j' \\ (B_j')^t & A_0' \end{pmatrix}.$$

The primes distinguish these matrices from the scalar source weights. For $\theta = (r, s) \ne 0$, put $z = (\mathrm{e}^{ia}, \mathrm{e}^{ib}, 1)$, $a = 2\pi r/n$, $b = 2\pi s/n$, and let $P(\theta)$ project onto $\{1, z\}^\perp$. With $S_\theta = 1 + \mathrm{e}^{ia} + \mathrm{e}^{ib}$ and $\Lambda_\theta = 9 - |S_\theta|^2$, the component Fourier formula is

$$A_0' = n^{-2} \sum_{\theta \ne 0} \frac{3P(\theta)}{\Lambda_\theta}, \qquad B_j' = n^{-2} \sum_{\theta \ne 0} \frac{\zeta_j(\theta) S_\theta P(\theta)}{\Lambda_\theta}, \qquad |\zeta_j| = 1.$$

Conjugating S_θ when exchanging layers gives the same bounds. The evaluation phase ζ_j records the original two sample points.

Lemma 8.1 (Polarization average and bounded cross block)**.** *On W, $\mathcal{K} \le K_n I$ and*

$$\mathcal{K} \ge \frac{1-n^{-2}}{12} I.$$

After changing to pair-plus and pair-minus coordinates, its local blocks X_j, Y_j, Z_j obey

$$\operatorname{tr} X_j \ge \frac{h_N}{8\pi^2} - 1, \qquad \|Z_j\| \le 2\pi.$$

Proof. The exact polarization average is

$$n^{-2} \sum_{\theta \ne 0} P(\theta) = \frac{1-n^{-2}}{2} P_V. \tag{25}$$

Indeed conjugate frequencies make the sum real. Permute the three coordinates of z and divide by its last coordinate: this bijects the nonzero transverse frequencies and conjugates P by that spatial permutation. A real matrix commuting with all three-coordinate permutations has constant diagonal and constant off-diagonal entries. It annihilates 1, so is scalar on V. Its trace is n^2-1, proving (25). Since the two eigenvalues of each nonzero component matrix lie in $(0,6]$, its inverse is at least $I/6$. Evaluation on one point in each layer and (25) give the floor. The upper bound follows from $R_0 \le R$ and Theorem 6.1, since $Hn^3 = cn^5$.

The plus block is $X_j = A_0' + (B_j' + (B_j')^t)/2$. Its spatial trace is $g_0 + G(o_j, e_j)$, by (23) after removing its normalization. The zero transverse optical branch contributes zero to the plus vector. Lemma 7.2 proves the trace bound.

The cross block is $Z_j = ((B_j')^t - B_j')/2$ up to a sign. To bound it, choose centered angles $a, b \in [-\pi, \pi)$ and set $\rho = |a| + |b|$. A spanning vector for the polarization line is

$$w = (\mathrm{e}^{-ib} - 1,\ 1 - \mathrm{e}^{-ia},\ \mathrm{e}^{-ia} - \mathrm{e}^{-ib}), \qquad |w|^2 = \Lambda_\theta.$$

Its entries have real amplitudes times a common factor $-2i$ and phases $\mathrm{e}^{-ib/2}, \mathrm{e}^{-ia/2}, \mathrm{e}^{-i(a+b)/2}$. Every phase difference is at most $\rho/2$. The Frobenius norm of the entrywise imaginary part of $ww^*/|w|^2$ is therefore at most $\rho/2$. Because B_j' is real, transposition gives

$$\|Z_j\| \le \frac{3}{2} n^{-2} \sum_{\theta \ne 0} \frac{\rho}{\Lambda_\theta}.$$

The chord bound gives $\Lambda_\theta \ge 4(a^2+b^2)/\pi^2$. On a centered-residue shell $m = \max(|r|, |s|)$, $\rho/(a^2+b^2) \le n/(\pi m)$; there are at most $8m$ sites and N shells. Consequently $n^{-2} \sum \rho/\Lambda_\theta \le \pi$. Thus $\|Z_j\| \le 3\pi/2 \le 2\pi$. The phase ζ_j has modulus one, so no bound on the distance between sample points was assumed. □

8.2 Removing line averages inside the original domain

Let Π_j be the ambient Fourier projector onto $k_j \in \{0, N\}$, for $j = 2, 3$, and put

$$\Pi = \Pi_2 + \Pi_3 - \Pi_2\Pi_3, \qquad \Xi = I - \Pi, \qquad R' = \Xi R \Xi, \qquad R_0' = \Xi R_0 \Xi.$$

These are actual frequency projectors, not projections of observation rows. The Fourier description of U shows that they preserve U. They retain or remove a whole pair $\{k, k+q\}$, so

commute with the restricted A. All low pairs are diagonal with coordinates outside $\{0, N\}$, hence $\Xi|_L = I$. Therefore R' is precisely the shifted inverse on $F' = \Xi F$, and $0 \le R'_0 \le R' \le R$. Every field in $\operatorname{ran}\Xi$ has zero ordinary and alternating full-line sums in both the second and third coordinates.

Lemma 8.2 (A floor after removal)**.** *Set $\mathcal{K}' = S(HR'_0)S^*$. On W,*

$$I/32 \le \mathcal{K}' \le \mathcal{K} \le K_n I. \tag{26}$$

Both selected responses annihilate $W^\perp$, while the actual response $\widehat{M}' = cn^5 TR'T^*$ satisfies $\mathcal{K}' \le \widehat{M}' \le K_n I$ on the full row space.

Proof. Let Q_0 be the support projector of R_0: it retains the common polarization on both layers for each nonzero transverse frequency. Then $HR_0 \ge Q_0/6$. In the original dictionary $k = (t+r, t+s, t)$, each Π_j retains a whole pair for every transverse frequency, so commutes with Q_0 and R_0. Equation (25) and one sample in each graph layer give $SQ_0S^* = (1 - n^{-2})I_W/2$. Also $S\Pi_j S^* = 2I/n$ on the full spatial row space: the band kernel vanishes unless the other two coordinates coincide, and distinct selected points have distinct x_1. Its coincident diagonal is $2/n$. Since $0 \le \Pi Q_0 \le \Pi_2 + \Pi_3$,

$$\mathcal{K}' \ge \frac{1}{6} S\Xi Q_0 \Xi S^* \ge \frac{1 - 8/n - n^{-2}}{12} I_W \ge \frac{1}{32} I_W.$$

The scalar expression increases for $n \ge 16$ and the last inequality already holds at 16. All upper comparisons follow by positivity and Theorem 6.1. In particular the floor has been proved after removal; it does not follow from a trace estimate alone. □

8.3 The lowest sector loses only a bounded response

Translation by two in x_1 preserves the endpoint sample line and all operators just defined. The complex row sector $\mathcal{H}_1$ consists of

$$\alpha_i = n^{-1/2}\big(a\mathrm{e}^{2\pi i i/n} + b\mathrm{e}^{2\pi i(1+N)i/n}\big), \qquad a, b \in \mathbf{C}^3.$$

Its squared norm is $|a|^2 + |b|^2$, and its intersection with W has dimension four. In paired-component coordinates this is the nonconstant character $\mathrm{e}^{2\pi ij/N}$. The space $E \cap \mathcal{H}_1$ has dimension two: it comes exactly from the diagonal low pair $(1, 1, 1)$ and its q partner, with both polarizations. Rationalizing its eigenvalue gives

$$\mu_-(1) = \frac{12Ha\sin^2(2\pi/n)}{H + a + \sqrt{(H+a)^2 - 4Ha\sin^2(2\pi/n)}} \le 6a\sin^2(2\pi/n) \le \frac{c\pi^2}{2n^2}. \tag{27}$$

The denominator is at least $2H$, since the square root is at least $H - a$.

Lemma 8.3 (Sector loss)**.** *For $\Delta = \mathcal{K} - \mathcal{K}' \ge 0$, $\|\Delta|_{\mathcal{H}_1}\| \le 8$.*

Proof. For $g = S^*\alpha$, its unitary torus Fourier coefficients are a/n at $k_1 = 1$ and b/n at $k_1 = 1 + N$, independently of k_2, k_3. Represent each unordered checkerboard pair uniquely by $k_1 = 1$. Its datum has squared norm $(|a|^2 + |b|^2)/n^2$, with no extra pair factor. On a removed pair at least one of k_2, k_3 is 0 or N; this pair cannot be diagonal. The norm of HR_0

there is at most $1/\delta$. For k_3 fixed choose centered residues $r = 1 - k_3$ and $t = k_2 - k_3$. Then $|r| \geq 1$, t runs over all residues, and $\delta \geq 8(r^2 + t^2)/(3n^2)$ by the sine bound above. The same argument applies with k_2 fixed, after exchanging the coordinates. Each of the four removed lines contributes at most

$$\frac{1}{n^2} \sum_t \frac{1}{\delta} \leq \frac{3}{8} \sum_{t \in \mathbf{Z}} \frac{1}{1+t^2} \leq \frac{3}{8}(1 + 2 \cdot 2) = \frac{15}{8}.$$

Here $\sum_{t \geq 1} t^{-2} \leq 1 + \int_1^\infty x^{-2}\, dx = 2$. Counting intersections twice only enlarges the positive bound. The four-line total is $15/2 \leq 8$. Polarization projections decrease the quadratic response, and commutation identifies the removed response with exactly Δ. Restriction to W preserves the bound. □

8.4 Optimizing all complementary singleton coefficients

For positive definite G on W and a subspace $E' \subset W$, write

$$\mathcal{S}_{E'}(G) = (P_{E'} G^{-1} P_{E'}|_{E'})^{-1}.$$

This is the minimum energy with fixed $P_{E'}\alpha = y$: solving the multiplier equation $G\alpha \in E'$ gives both this formula and attainment. It is not the inverse of the direct compression of G.

For the endpoint cut the selected two points in each component have graph coordinates $(2j-1, 0), (2j, 0)$. Their relative displacement is the same in every component. Thus $\mathcal{K}$ has the same local block in each component, and the bounds of Lemma 8.1 hold in $\mathcal{H}_1 \cap W$. The strong useful space there is the pair-plus copy of V. Its mean-zero condition is automatic in this nonconstant character; no additional common-mean minimization is needed.

Write the blocks of $\mathcal{K}'$ in pair-plus/minus coordinates as X', Y', Z'. Lemmas 8.2 and 8.3 give

$$\operatorname{tr} X' \geq \frac{h_N}{8\pi^2} - 17, \qquad \|Z'\| \leq 2\pi + 8, \qquad Y' \geq I/32.$$

Minimizing the minus variable gives the Hermitian Schur complement

$$S_0' = X' - Z'(Y')^{-1}(Z')^*.$$

Since V has dimension two,

$$\operatorname{tr} S_0' \geq \frac{h_N}{8\pi^2} - 17 - 64(2\pi + 8)^2. \tag{28}$$

All adjoints in this complex sector are Hermitian.

The positive-weight angle bound in Section 7 restricts to this sector: $\|P_E - P_{E_0}\| \leq \epsilon_n = 2/(48n^2 - 1)$. Both ranks here are two. Principal-plane rotations give a unitary $O : E_0 \to E$, extended to the sector, with $\|O - I\| \leq 2\epsilon_n$. The two compressed inverses of $\mathcal{K}'$ differ in norm by at most $4\epsilon_n \|(\mathcal{K}')^{-1}\| \leq 128\epsilon_n$. Their inverses have norm at most K_n. The inverse-difference identity therefore yields

$$\|O^* \mathcal{S}_E(\mathcal{K}') O - S_0'\| \leq \frac{256 K_n^2}{48n^2 - 1}. \tag{29}$$

Translation symmetry decomposes the full minimization into orthogonal sectors, so the other characters provide no hidden cross terms.

Now allow every original singleton coefficient, including those outside W. The floor implies $\ker \widehat{M}' \subset W^{\perp}$ and $E \subset \operatorname{ran} \widehat{M}'$. Hence

$$\Sigma' = (P_E(\widehat{M}')^{\dagger} P_E|_E)^{-1}$$

is positive definite. It equals the attained minimum of $\langle \alpha, \widehat{M}'\alpha \rangle$ for $P_E\alpha = y$. Indeed, with $\lambda = \Sigma' y$, the vector $\alpha = (\widehat{M}')^{\dagger}\lambda$ is feasible and every feasible perturbation is orthogonal to λ; completing the square proves minimality. The inequality $\widehat{M}' \geq \mathcal{K}'$ gives $\Sigma' \geq \mathcal{S}_E(\mathcal{K}')$, and choosing $\alpha = y$ gives $\Sigma' \leq K_n I$. Combining (28) and (29), and dividing the trace by two, proves

$$\|\Sigma'|_{E \cap \mathcal{H}_1}\| \geq b_{\mathrm{end}}(n). \tag{30}$$

This optimizes all complementary singleton coefficients, rather than equating a large direct response with a large optimized response.

8.5 A real field invisible to every endpoint cell

Suppose $b_{\mathrm{end}}(n) > 0$. Choose a unit top eigenvector y of the sector response, with eigenvalue $\beta \geq b_{\mathrm{end}}(n)$. Let $J = n^{5/2} T_L$, $l = J^* y$, $M' = TR'T^*$, and

$$f = -R'T^*(M')^{\dagger} T_L l.$$

The range inclusion just proved and square completion imply $T(l+f) = 0$, $\|l\| = 1$ and $\langle f, Bf \rangle = c/\beta$. Both fields lie in $\operatorname{ran} \Xi$ and the actual domain U, in sector one. Since $B \geq 13c/3$, $\|f\|^2 \leq 3/(13\beta)$. Spectral orthogonality and (27) give

$$Q(l+f) \leq \frac{c\pi^2}{2n^2} + \frac{16c}{13\beta}, \qquad \|l+f\|^2 \geq 1. \tag{31}$$

The construction so far may be complex. Sectors 1 and $N-1$ are distinct orthogonal characters. All operators are real and preserve their conjugation pairing. Consequently $\sqrt{2}\,\mathrm{Re}(l+f)$ has exactly the same norm and energy, and its real low part has norm one. This gives a nonzero real field without changing any zero observation.

On each first-coordinate slice its point cell is zero by $T(l+f) = 0$. Both full coordinate line sums vanish by Ξ. Subtracting the point from either full line makes each point-by-long cell zero. The whole plane has sum zero as well, so subtracting the two lines and adding back the point makes the long-by-long cell zero. The original row normalizations only multiply these zeros. Thus $C_{\mathrm{end}}(l+f) = 0$, including every spatial component. This proves the upper bound in Theorem 2.1.

The lower bound is Corollary 6.2, since $f(i) = i$ is a permutation. The direct sum of channels has the same lower bound and admits the one-channel upper witness by setting the other channel to zero. Finally $h_N/\log n \to 1$, $K_n/\log n \to 48/13$ and $b_{\mathrm{end}}(n)/\log n \to 1/(16\pi^2)$. Hence

$$\liminf_{n\to\infty} \frac{\lambda_{\mathrm{end}}(n)\log n}{c} \geq \frac{13}{48}, \qquad \limsup_{n\to\infty} \frac{\lambda_{\mathrm{end}}(n)\log n}{c} \leq \frac{256\pi^2}{13},$$

along even sizes. These bounds prove the order, not an exact leading constant. The sufficient thresholds are extremely conservative and do not assert a practical small-size counterexample. Any finite penalty on these same observations vanishes on the constructed field; it cannot restore the target c at all sufficiently large sizes.

9 The remaining rows and their exact conditional cost

For an arbitrary permutation cut, write the original map as $C = (T, P)$, where P contains all remaining original normalized rows. This notation does not change their meanings or delete dependent rows. In this section P_F and P_L denote restrictions of this observation map, not spectral projections. Put

$$M_T = T_F B^{-1} T_F^*, \quad f_T(l) = -B^{-1} T_F^* M_T^\dagger T_L l, \quad R_T = B^{-1} - B^{-1} T_F^* M_T^\dagger T_F B^{-1}.$$

Then define $K_P = P_F R_T P_F^*$ and $r(l) = P_L l + P_F f_T(l)$.

Proposition 9.1 (Paid residual cost)**.** *For every $l \in L$, $r(l) \in \operatorname{ran} K_P$, and the attained full cancellation cost equals*

$$\min_{C_F f = -C_L l} \langle f, Bf \rangle = \langle T_L l, M_T^\dagger T_L l \rangle + \langle r(l), K_P^\dagger r(l) \rangle. \tag{32}$$

Consequently its comparison with $\langle l, (cI - A_L)l \rangle$ is exactly the full-kernel target-c criterion.

Proof. Set $X = T_F B^{-1/2}$ and $\Pi_T = I - X^*(XX^*)^\dagger X$. The finite spectral inverse shows that Π_T is the orthogonal projector onto $\ker X$, and $R_T = B^{-1/2} \Pi_T B^{-1/2}$. The range theorem provides a full cancelling f. The difference $z = f - f_T(l)$ has $T_F z = 0$, $P_F z = -r(l)$. With $Y = P_F B^{-1/2}$, this proves $r(l) \in \operatorname{ran}(Y \Pi_T) = \operatorname{ran}(Y \Pi_T Y^*) = \operatorname{ran} K_P$; the second equality follows by comparing orthogonal kernels. The minimizer of this residual problem is $z_* = -R_T P_F^* K_P^\dagger r(l)$, with cost $\langle r(l), K_P^\dagger r(l) \rangle$, since $R_T B R_T = R_T$. Whitening by $B^{1/2}$ and completing the square proves minimality. Also $\langle f_T(l), Bz \rangle = 0$ for every $z \in \ker T_F$, as $B f_T(l) = -T_F^* M_T^\dagger T_L l$. The costs therefore add, proving (32). Theorem 5.1 gives the last assertion. □

On $\operatorname{ran} \Xi$ from the preceding section, all long third-coordinate rows are fixed multiples of their point-plane partners: for any original second-coordinate interval I,

$$(Cw)_{I,[1,n-1]} = -\frac{(Cw)_{I,\{0\}}}{\sqrt{n-1}}.$$

If also $Tw = 0$, the prefix and suffix sums on the plane $x_3 = 0$ are negatives, since the full second-coordinate line sum is zero. Thus full invisibility is equivalent to vanishing of the actual planar prefix sums $\sum_{j<\tau_i} w(i, j, 0)$. At endpoint cuts these impose no extra condition. For moving cuts they can be substantive. Neither their conditional cost nor a suitable rank deficiency is proved here for $\tau_i = 2i^2 \pmod n$.

10 Limits and the remaining optimization question

The endpoint theorem proves a matching logarithmic order for the full original observation kernel, not merely a large dual compression. It therefore closes the fixed-cut question left open in version 1. The quadratic moving cut and the optimization over canonical partitions remain open: their genuine planar prefixes may raise the conditional cost in (32). A generic large direct response cannot decide that cost, and nominal row counts do not establish a rank defect.

The results concern a finite counting-norm family with prescribed source weights. No finite-size computation is a proof premise, no formal verification is claimed, and no continuum limit or physical Yang–Mills mass gap follows. The finite thresholds are conservative. Establishing external originality or broad importance would require a more extensive comparison than the bounded primary-source context given here.

Model assessments and review context

At preparation of this revision on 12 November 2026, the antecedent research sources underwent separate involved analytical source audits in the same Codex/OpenAI programme context. Those sessions shared programme history; no provider independence, blind review or human peer review is claimed. The exact backend model identifiers were unavailable, and no more specific identities are inferred. The source audits concern the finite analytical statements, not an exact PDF of this manuscript.

For this new revision at that preparation stage, separate review of its consolidated source, remote reproducible production, visual PDF inspection and an exact-PDF scientific assessment had not been completed. Any assessment supplied in deposit metadata is a separate hash-bound artifact with its own actual model identity, context and objections. No score, stars, acceptance decision or assessment of an earlier artifact is transferred to this version. The full original reports, objections, corrections and context belong in the accompanying evidence package. AI-assisted preparation and involved source checks are not represented as independent peer review or publication acceptance. Lluis Eriksson, the listed author and programme owner, is also the owner, creator and administrator of AIRR, the intended deposit destination. This author–platform relationship is not independent editorial or human review. No affiliation or completed personal review is invented.

A Source and build provenance

This revision consolidates the accepted analytical spectrum, original-row range and cost, whole-response and useful-response records underlying version 1, together with the optimized response, conditional cost, line-average removal and sharp endpoint results. The accompanying `source-map289B.json` binds the statements to exact raw source bytes and records omissions. We include the complete proof chain needed for the endpoint theorem. The broader dimension-density and general-cut negative-subspace corollaries of the research notes are omitted, rather than asserted without their longer proofs. The exact conditional-cost identity is retained to state precisely what the moving-cut problem needs. No previously published result or earlier assessment is relabelled as new. The build system injects the exact frozen source commit before inputting this file. Its value for this rendering is

`446124a566483e6f1a8d499d801ce60e0abbb9d8`

Build logs, reproducibility checks, PDF page-inspection reports and any exact-PDF assessments are separate hash-bound artifacts. This provenance statement does not substitute for those records or claim that a particular review or publication decision has occurred.

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