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Backward Heat-Flow Collapse Times of Real-Rooted Polynomials: Universal Gap–Fisher–Variance Hierarchies, Exact Dyson–Stieltjes Trace Laws, Hermite Phase Transitions, and Hard-Edge Entire-Function Scaling

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Abstract

For a monic degree-d real-rooted polynomial $p \in \mathcal{R}_d$ ($d \geq 2$), the forward heat flow $p_t = e^{-\frac{t}{2}\partial_x^2} p = p \boxplus_d He_d^{(t)}$ preserves real-rootedness for all $t \geq 0$, whereas the backward heat flow $p_{-\tau} = e^{+\frac{\tau}{2}\partial_x^2} p$ ($\tau \geq 0$) loses real-rootedness at a finite *backward heat-flow collapse time* $\tau_*(p) := \sup\{\tau \geq 0 : p_{-\tau} \in \mathcal{R}_d\}$. We develop a unified analytic, spectral, and asymptotic theory of $\tau_*(p)$ and its exact interaction with finite free Fisher information $\Phi_d(p)$.

First, we establish a self-contained instantaneous simplicity lemma and prove a sharp four-term universal collapse-time sandwich valid for every $p \in \mathcal{R}_d$:

$$\frac{\Delta_{\min}(p)^2}{4} \;\leq\; \tau_*(p) \;\leq\; \min\left\{ \frac{d(d-1)}{4\Phi_d(p)},\; \frac{W(p)^2}{8d-12},\; \tau_{\mathrm{skew}}(p) \right\} \;\leq\; \frac{\sigma^2(p)}{d-1},$$

where $\Delta_{\min}(p)$ is the minimal root gap, $W(p) = \lambda_d - \lambda_1$ is the root span, $\sigma^2(p)$ is the empirical root variance, and $\tau_{\mathrm{skew}}(p) = \frac{\sigma^2(p)}{d-1} - \left(\frac{|\mu_3(p)|}{(d-1)(d-2)}\right)^{2/3}$ ($d \geq 3$) is the skewness-sharpened ceiling from iterated differentiation. We further prove derivative monotonicity $\tau_*(p) \leq \tau_*(p')$, finite free superadditivity $\tau_*(p \boxplus_d q) \geq \tau_*(p) + \tau_*(q)$, and *Finite Free Costa Concavity*: the finite free entropy power $N_d(p_t) = |\operatorname{Disc}(p_t)|^{\frac{2}{d(d-1)}}$ is concave along the heat flow ($N_d''(p_t) \leq 0$), with equality if and only if p is an affine Hermite polynomial.

Second, introducing the Dyson–Stieltjes operators $A_{ij} = (\lambda_i - \lambda_j)^{-1}$, $B = A^\top A = -A^2$, the pairwise Laplacian L, and the critical-point Cauchy matrix $C_{im} = (\lambda_i - \mu_m)^{-1}$, we prove four exact algebraic identities: the all-orders commutator collapse $\mathbf{1}_d^\top B^k \mathbf{1}_d = \operatorname{Tr}(B^k)$ ($k \geq 1$), critical-point orthogonality $C^\top C = \operatorname{diag}(\kappa_m)$ with $\Phi_d(p) = \frac{1}{4}\sum_m \kappa_m$, the Frobenius-trace identity $\operatorname{Tr}(L^2) = 2\operatorname{Tr}(BL)$ (equivalently $\|L - B\|_F = \|B\|_F$), and the exact dissipation law $S_2(p) = -\frac{1}{2}\dot{\Phi}_d(p) = \frac{3}{2}\operatorname{Tr}(L^2) - \operatorname{Tr}(B^2) = \frac{5}{2}\sum_{i\neq j}(\lambda_i - \lambda_j)^{-4} - \frac{1}{2}\sum_i w_i^2$. We also derive an exact decomposition of $(1/\Phi_d(p_t))''$ proving local concavity whenever the score vector is monotone (in particular for all Jacobi polynomials).

Third, we solve the exact backward collapse law for all cubics ($d = 3$) and all symmetric quartics $p(x) = (x^2 - a^2)(x^2 - b^2)$ ($d = 4$), revealing a sharp phase transition at the Hermite ratio $b/a = \sqrt{3} + \sqrt{2}$ with universal critical amplitude ratio $\sqrt{2}$ separating outer-pair from inner-pair collision, and proving global concavity of $t \mapsto 1/\Phi_d(p_t)$ (hence $\tau_*(p) \leq S_1(p)/(2S_2(p))$) for every cubic and every symmetric quartic—including non-monotone scores $1 < b/a < \sqrt{5}$. Based on these exact results and large-scale numerical tests, we formulate the Finite Free Toscani Concavity Conjecture for general $d \geq 4$.

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Fourth, for all ultraspherical Jacobi polynomials $P_d^{(\alpha,\alpha)}$ ($\alpha > -1$), we prove exact closed forms $\Phi_d(P_d^{(\alpha,\alpha)}) = \frac{(d-1)d(d+2\alpha+1)(d+2\alpha+2)}{8(\alpha+2)}$ and $\frac{\sigma^2(P_d^{(\alpha,\alpha)})}{d-1} = \frac{1}{2d+2\alpha-1}$, and establish uniform hard-edge entire-function convergence of $(e^{+\frac{t}{2d^4}\partial_x^2} P_d^{(\alpha,\alpha)})(1-\frac{y}{2d^2})/P_d^{(\alpha,\alpha)}(1)$ to $F_\alpha(y;t) = e^{+2t\partial_y^2}[\Gamma(\alpha+1)(y/4)^{-\alpha/2}J_\alpha(\sqrt{y})]$. Combined with Sturm comparison and certified 40-digit interval-arithmetic Laguerre negativity bounds, this proves the sharp two-sided scaling $\tau_*(P_d^{(\alpha,\alpha)}) = \Theta(d^{-4})$ for Chebyshev T_d, Legendre P_d, and Chebyshev U_d, demonstrating a clean three-scale separation $\tau_* = \Theta(d^{-4}) \ll \frac{d(d-1)}{4\Phi_d} = \Theta(d^{-2}) \ll \frac{\sigma^2}{d-1} = \Theta(d^{-1})$ with limiting constants $t^*_{\text{edge}}(-1/2) \approx 25.906417$, $t^*_{\text{edge}}(0) \approx 41.150192$, and $t^*_{\text{edge}}(1/2) \approx 60.013098$.

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1 Introduction

The action of the constant-coefficient heat semigroup $e^{-\frac{t}{2}\partial_x^2}$ on real-rooted polynomials lies at the crossroads of classical complex analysis (Pólya–Schur theory and the de Bruijn–Newman constant [12, 2, 11, 13]), random matrix theory and Dyson Brownian motion [5, 18], and finite free probability [10, 9, 1]. Let $\mathcal{R}_d$ denote the closed subset of monic degree-d polynomials ($d \geq 2$) all of whose roots are real, and let $\mathcal{R}_d^+ \subset \mathcal{R}_d$ be the open subset of polynomials with d strictly simple real roots $\lambda_1 < \lambda_2 < \cdots < \lambda_d$. For any $p \in \mathcal{R}_d$ and $t \in \mathbb{R}$, define the heat-evolved polynomial

$$p_t(x) \;:=\; (e^{-\frac{t}{2}\partial_x^2}p)(x) \;=\; \sum_{m=0}^{\lfloor d/2\rfloor} \frac{(-t/2)^m}{m!}\, p^{(2m)}(x), \tag{1}$$

which solves the backward-signed PDE $\partial_t p_t = -\frac{1}{2}p_t''$ with $p_0 = p$. In the language of Marcus, Spielman, and Srivastava [10], $p_t = p \boxplus_d He_d^{(t)}$ is the finite free additive convolution of p with the scaled probabilists' Hermite polynomial $He_d^{(t)}(x) := e^{-\frac{t}{2}\partial_x^2}x^d = t^{d/2}He_d(x/\sqrt{t})$. For $t > 0$, the flow $p \mapsto p_t$ preserves $\mathcal{R}_d$ and drives the roots apart along the deterministic Dyson–Calogero–Moser system

$$\dot\lambda_i(t) \;=\; \frac{p_t''(\lambda_i(t))}{2\,p_t'(\lambda_i(t))} \;=\; \sum_{j\neq i}\frac{1}{\lambda_i(t)-\lambda_j(t)} \;=:\; s_i(p_t), \qquad i = 1,\ldots,d. \tag{2}$$

Reversing time ($\tau = -t \geq 0$) yields the *backward heat flow* $p_{-\tau} = e^{+\frac{\tau}{2}\partial_x^2}p$, under which roots attract each other according to $\frac{d\lambda_i}{d\tau} = -s_i(p_{-\tau})$ until at least two real roots collide and

subsequently bifurcate into complex conjugate pairs. This motivates our central geometric invariant:

Definition 1.1 (Backward Heat-Flow Collapse Time)**.** For $p \in \mathcal{R}_d$ ($d \geq 2$), the *backward heat-flow collapse time* of p is

$$\tau_*(p) \;:=\; \sup\{\tau \geq 0 : e^{+\frac{\tau}{2}\partial_x^2} p \in \mathcal{R}_d\}. \tag{3}$$

Equivalently, $\tau_*(p)$ is the exact finite-degree analog of the de Bruijn–Newman / heat-flow real-rootedness threshold studied for entire functions by Hall and Ho [6] and for polynomial root evolution by Steinerberger [15, 16], Tao [18], Kabluchko [8], and Hoskins–Kabluchko [7]: it measures the maximal amount of Gaussian / Hermite finite free convolution $He_d^{(\tau)}$ that can be deconvolved from p while preserving real-rootedness.

Despite extensive work on macroscopic limiting root densities under forward heat flow and repeated differentiation [15, 8, 7], sharp *non-asymptotic* two-sided bounds on $\tau_*(p)$, exact operator-algebraic identities for the dissipation of finite free Fisher information $\Phi_d(p) = \sum_{i=1}^d s_i(p)^2$, exact phase diagrams in low degree, and the microscopic edge scaling of $\tau_*(p)$ for classical orthogonal polynomials have remained largely open.

1.1 Summary of Main Results

This paper establishes four main theorems (all proved completely and self-containedly) and formulates one explicit open conjecture backed by exact low-degree theorems and large-scale computation:

1. **Universal Collapse-Time Hierarchy and Finite Free Costa Concavity (Lemma 2.1 and Theorem 2.2).** After proving via induction and Rouché's theorem that $e^{-\frac{t}{2}\partial_x^2}(\mathcal{R}_d) \subset \mathcal{R}_d^+$ for every $t > 0$, we establish the sharp four-term universal sandwich valid for every $p \in \mathcal{R}_d$:

$$\frac{\Delta_{\min}(p)^2}{4} \;\leq\; \tau_*(p) \;\leq\; \min\left\{\frac{d(d-1)}{4\Phi_d(p)},\; \frac{W(p)^2}{8d-12},\; \tau_{\text{skew}}(p)\right\} \;\leq\; \frac{\sigma^2(p)}{d-1},$$

   characterizing every equality case. We prove derivative monotonicity $\tau_*(p) \leq \tau_*(p') \leq \cdots \leq \tau_*(p^{(d-2)}) = \frac{\sigma^2(p)}{d-1}$, finite free superadditivity $\tau_*(p \boxplus_d q) \geq \tau_*(p) + \tau_*(q)$, and *Finite Free Costa Concavity*: the finite free entropy power $N_d(p_t) := |\operatorname{Disc}(p_t)|^{\frac{2}{d(d-1)}}$ satisfies $N_d''(p_t) \leq 0$ for all $t > -\tau_*(p)$ (the exact finite-free analog of Costa's 1985 entropy power concavity theorem [4] and Shlyakhtenko–Tao's free entropy power [14]).

2. **Dyson–Stieltjes Operator Algebra and Exact Trace Laws (Theorem 3.1).** Associating to $p \in \mathcal{R}_d^+$ the skew-symmetric Hilbert matrix $A_{ij} = (\lambda_i - \lambda_j)^{-1}$, its Gramian $B = A^\top A = -A^2$, the inverse-square Laplacian L, and the critical-point Cauchy matrix $C_{im} = (\lambda_i - \mu_m)^{-1}$, we prove:

   - The all-orders commutator collapse $\mathbf{1}_d^\top B^k \mathbf{1}_d = \operatorname{Tr}(B^k)$ for all $k \geq 1$ (whose $k = 1$ case recovers $\Phi_d(p) = \operatorname{Tr}(B) = \operatorname{Tr}(L)$);
   - Critical-point orthogonality $C^\top C = \operatorname{diag}(\kappa_m)$ with $\kappa_m = -p''(\mu_m)/p(\mu_m) > 0$, the residue sum rule $\Phi_d(p) = \frac{1}{4}\sum_{m=1}^{d-1} \kappa_m$, and the exact curvature ODE $\dot\kappa_m = -\frac{1}{2}\kappa_m^2 + \frac{1}{2}\kappa_m(\tilde s_m^2 + 3\tilde w_m)$;
   - The Frobenius-trace law $\operatorname{Tr}(L^2) = 2\operatorname{Tr}(BL)$ (equivalently $\|L - B\|_F = \|B\|_F$) and the exact trace and quartic-sum representations of the Fisher dissipation $S_2(p) = -\frac{1}{2}\dot\Phi_d(p) = s^\top L s = \frac{3}{2}\operatorname{Tr}(L^2) - \operatorname{Tr}(B^2) = \frac{5}{2}\sum_{i\neq j}(\lambda_i - \lambda_j)^{-4} - \frac{1}{2}\sum_i w_i^2$;

- An exact formula for $(1/\Phi_d(p_t))''$ showing unconditionally that $(1/\Phi_d(p_t))''|_{t=0} \le 0$ whenever the score vector s is monotone ($s_1 \le \cdots \le s_d$), which holds for every Jacobi polynomial $P_d^{(\alpha,\beta)}$ ($\alpha, \beta > -1$).

3. **Exact Cubic Law, Symmetric Quartic Hermite Phase Transition, and Toscani Concavity (Theorem 4.1 and Conjecture 4.2).** We compute $\tau_*(p)$ and $\Phi_d(p_t)$ in exact closed form for all cubics ($d = 3$) and all symmetric quartics $p(x) = (x^2 - a^2)(x^2 - b^2)$ ($d = 4$). In the symmetric quartic family, $\tau_*(p)/(\sigma^2(p)/3)$ exhibits a sharp second-order-like collision-channel phase transition at the exact Hermite root ratio $b/a = \sqrt{3}+\sqrt{2}$ with universal critical amplitude ratio $\sqrt{2}$ between the outer-pair collision phase ($1 < b/a < \sqrt{3}+\sqrt{2}$) and the inner-pair collision phase ($b/a > \sqrt{3}+\sqrt{2}$). We prove analytically that $t \mapsto 1/\Phi_d(p_t)$ is concave on $(-\tau_*(p), \infty)$ for all cubics and all symmetric quartics—even in the non-monotone-score regime $1 < b/a < \sqrt{5}$—yielding the sharp tangent upper bound $\tau_*(p) \le S_1(p)/(2S_2(p))$. We conjecture that this Finite Free Toscani Concavity holds for all $p \in \mathcal{R}_d^+$ ($d \ge 4$) without log-concavity assumptions.

4. **Ultraspherical Jacobi Exact Laws and Hard-Edge Entire-Function Scaling (Theorem 5.1).** For every ultraspherical Jacobi polynomial $P_d^{(\alpha,\alpha)}$ ($\alpha > -1$, $d \ge 2$), we prove the exact closed-form identities $\Phi_d(P_d^{(\alpha,\alpha)}) = \frac{(d-1)d(d+2\alpha+1)(d+2\alpha+2)}{8(\alpha+2)}$ and $\frac{\sigma^2(P_d^{(\alpha,\alpha)})}{d-1} = \frac{1}{2d+2\alpha-1}$, and prove via a Weierstrass M-test that the hard-edge rescaled backward heat-flow polynomial converges uniformly on compact subsets of $\mathbb{C}^2$ to the entire function $F_\alpha(y;t) = e^{+2t\partial_y^2}[\Gamma(\alpha+1)(y/4)^{-\alpha/2}J_\alpha(\sqrt{y})]$. Combining Hurwitz's theorem, Sturm comparison for $\alpha \in [-\frac{1}{2}, \frac{1}{2}]$, and 40-digit interval-arithmetic Laguerre negativity certificates $F'_\alpha(y_0;t_0)^2 - F_\alpha(y_0;t_0)F''_\alpha(y_0;t_0) < 0$, we rigorously establish the two-sided hard-edge scaling $\tau_*(P_d^{(\alpha,\alpha)}) = \Theta(d^{-4})$ for Chebyshev T_d, Legendre P_d, and Chebyshev U_d, proving a clean three-scale separation:

$$\tau_*(P_d^{(\alpha,\alpha)}) = \Theta(d^{-4}) \ll \frac{d(d-1)}{4\Phi_d(P_d^{(\alpha,\alpha)})} = \Theta(d^{-2}) \ll \frac{\sigma^2(P_d^{(\alpha,\alpha)})}{d-1} = \Theta(d^{-1}).$$

2 Universal Collapse-Time Hierarchy and Finite Free Costa Concavity

Throughout, for $p(x) = \prod_{i=1}^d (x-\lambda_i) \in \mathcal{R}_d$ with roots ordered $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_d$, we define the empirical mean $\mu(p) := \frac{1}{d}\sum_{i=1}^d \lambda_i$, empirical variance $\sigma^2(p) := \frac{1}{d}\sum_{i=1}^d (\lambda_i - \mu(p))^2$, third central moment $\mu_3(p) := \frac{1}{d}\sum_{i=1}^d (\lambda_i - \mu(p))^3$, minimal root gap $\Delta_{\min}(p) := \min_{1\le j\le d-1}(\lambda_{j+1} - \lambda_j)$, and root span $W(p) := \lambda_d - \lambda_1$. For $p \in \mathcal{R}_d^+$, we define the root score vector $s(p) = (s_1, \ldots, s_d)^\top$ by $s_i := \sum_{j\ne i}(\lambda_i - \lambda_j)^{-1} = \frac{p''(\lambda_i)}{2p'(\lambda_i)}$ and the finite free Fisher information [9]

$$\Phi_d(p) := \sum_{i=1}^d s_i(p)^2, \tag{4}$$

setting $\Phi_d(p) := +\infty$ (and $1/\Phi_d(p) := 0$) for $p \in \mathcal{R}_d \setminus \mathcal{R}_d^+$. We call p an *affine Hermite polynomial* if $p(x) = He_d^{(t_0)}(x - \mu)$ for some $t_0 \ge 0$ and $\mu \in \mathbb{R}$ (where $He_d^{(t_0)}(x) = e^{-\frac{t_0}{2}\partial_x^2}x^d$, with $t_0 = 0$ giving the degenerate polynomial $(x-\mu)^d$), and *nondegenerate affine Hermite* if $t_0 > 0$.

We begin by establishing that forward heat flow instantaneously separates all multiple roots and that $\tau_*(p)$ is finite and realized by a root collision.

Lemma 2.1 (Instantaneous Root Simplicity and Well-Posedness of τ_*)**.** *Let $d \ge 2$.*

(i) For every $q \in \mathcal{R}_d$ and every $t > 0$, we have $q_t = e^{-\frac{t}{2}\partial_x^2} q \in \mathcal{R}_d^+$ (all d roots are real and strictly simple).

(ii) For every $p \in \mathcal{R}_d$, the set $\{t \in \mathbb{R} : p_t \in \mathcal{R}_d\}$ equals the closed half-line $[-\tau_*(p), \infty)$ with $0 \le \tau_*(p) \le \frac{\sigma^2(p)}{d-1} < \infty$. Moreover, $p_{-\tau} \in \mathcal{R}_d^+$ for every $\tau \in [0, \tau_*(p))$, $p_{-\tau_*(p)} \in \mathcal{R}_d \setminus \mathcal{R}_d^+$ possesses at least one multiple real root, and $\lim_{\tau \uparrow \tau_*(p)} \Delta_{\min}(p_{-\tau}) = 0$.

Proof. We first verify that $e^{-\frac{t}{2}\partial_x^2}(\mathcal{R}_d) \subset \mathcal{R}_d$ for all $t \ge 0$. For any $c \in \mathbb{R} \setminus \{0\}$ and $q \in \mathcal{R}_d$, the function $e^{x/c} q(x)$ vanishes at the roots of q and as $x \to -\operatorname{sgn}(c)\infty$, so Rolle's theorem implies that $(1 + c\partial_x) q \in \mathcal{R}_d$. Composing $(1 - \sqrt{\frac{t}{2n}}\partial_x)(1 + \sqrt{\frac{t}{2n}}\partial_x) = 1 - \frac{t}{2n}\partial_x^2$ and letting $n \to \infty$ (since $\mathcal{R}_d$ is closed in the coefficient topology by Hurwitz's theorem) yields $e^{-\frac{t}{2}\partial_x^2} q \in \mathcal{R}_d$.

To prove strict simplicity $q_t \in \mathcal{R}_d^+$ for $t > 0$, we induct on $d \ge 2$. For $d = 2$, writing $q(x) = (x - \mu)^2 - \sigma^2$ gives $q_t(x) = (x - \mu)^2 - (\sigma^2 + t)$, whose roots $\mu \pm \sqrt{\sigma^2 + t}$ are real and distinct since $\sigma^2 \ge 0$ and $t > 0$. Now let $d \ge 3$ and assume the claim holds for degree $d - 1$. Since ∂_x commutes with $e^{-\frac{t}{2}\partial_x^2}$ and $q' \in d \cdot \mathcal{R}_{d-1}$ by Rolle's theorem, the induction hypothesis shows that $q_t' = (q')_t$ has $d - 1$ strictly simple real roots. Consequently, every root x_0 of q_t has multiplicity at most 2 (otherwise $q_t'(x_0) = q_t''(x_0) = 0$). Suppose for contradiction that x_0 is a double root of q_t, so $q_t(x_0) = q_t'(x_0) = 0$ and $c := q_t''(x_0) \ne 0$. For $0 < \varepsilon < t$, we have $q_{t-\varepsilon} = e^{+\frac{\varepsilon}{2}\partial_x^2} q_t \in \mathcal{R}_d$ because $t - \varepsilon > 0$. Expanding $q_{t-\varepsilon}(x_0 + \sqrt{\varepsilon}\, u)$ as a finite polynomial in u and $\sqrt{\varepsilon}$ yields

$$\begin{aligned} q_{t-\varepsilon}(x_0 + \sqrt{\varepsilon}\, u) &= \big(q_t(x_0) + \tfrac{\varepsilon}{2} q_t''(x_0) + O(\varepsilon^2)\big) + \sqrt{\varepsilon}\, u \,\big(q_t'(x_0) + O(\varepsilon)\big) \\ &\quad + \tfrac{\varepsilon}{2} u^2 \big(q_t''(x_0) + O(\varepsilon)\big) + O(\varepsilon^{3/2}) \;=\; \frac{\varepsilon c}{2}(u^2 + 1) + O(\varepsilon^{3/2}) \end{aligned}$$

uniformly on compact subsets of $u \in \mathbb{C}$. Applying Rouché's theorem to $\varepsilon^{-1} q_{t-\varepsilon}(x_0 + \sqrt{\varepsilon}\, u)$ on the disk $|u - i| \le \frac{1}{2}$, for all sufficiently small $\varepsilon > 0$ there is a root u_ε with $|u_\varepsilon - i| < \frac{1}{2}$, hence $\operatorname{Im}(x_0 + \sqrt{\varepsilon}\, u_\varepsilon) > \frac{1}{2}\sqrt{\varepsilon} > 0$, contradicting $q_{t-\varepsilon} \in \mathcal{R}_d$. This proves (i).

For (ii), write $p(x) = x^d + a_1 x^{d-1} + a_2 x^{d-2} + \dots$; then $p_{-\tau}(x) = x^d + a_1 x^{d-1} + (a_2 + \binom{d}{2}\tau) x^{d-2} + \dots$, so by Newton's identities the empirical variance evolves linearly as

$$\sigma^2(p_{-\tau}) \;=\; \frac{d-1}{d^2} a_1^2 - \frac{2}{d}\left(a_2 + \frac{d(d-1)}{2}\tau\right) \;=\; \sigma^2(p) - (d-1)\tau \tag{5}$$

for all $\tau \in \mathbb{R}$ whenever $p_{-\tau} \in \mathcal{R}_d$. Since $\sigma^2(q) \ge 0$ on $\mathcal{R}_d$, every τ with $p_{-\tau} \in \mathcal{R}_d$ satisfies $\tau \le \frac{\sigma^2(p)}{d-1}$, so $\tau_*(p) \le \frac{\sigma^2(p)}{d-1} < \infty$. If $p_{-\tau_1} \in \mathcal{R}_d$, then by the semigroup property $p_{-\tau} = (p_{-\tau_1})_{\tau_1 - \tau} \in \mathcal{R}_d^+$ for every $\tau < \tau_1$ by (i); by closedness of $\mathcal{R}_d$, $p_{-\tau_*(p)} \in \mathcal{R}_d$. Finally, $p_{-\tau_*(p)}$ cannot lie in $\mathcal{R}_d^+$: because $\mathcal{R}_d^+$ is open in the coefficient topology (complex roots occur in conjugate pairs, so simple real roots remain real under small coefficient perturbations), $p_{-\tau_*(p)} \in \mathcal{R}_d^+$ would imply $p_{-\tau_*(p)-\delta} \in \mathcal{R}_d^+$ for small $\delta > 0$, contradicting the maximality of $\tau_*(p)$. Continuity of roots in coefficients therefore gives $\Delta_{\min}(p_{-\tau}) \to \Delta_{\min}(p_{-\tau_*(p)}) = 0$ as $\tau \uparrow \tau_*(p)$. □

We now state and prove our first main theorem.

Theorem 2.2 (Universal Collapse-Time Sandwich, Monotonicity, Superadditivity, and Finite Free Costa Concavity)**.** *Let $d \ge 2$ and $p \in \mathcal{R}_d$. Define $\tau_{\mathrm{skew}}(p) := \frac{\sigma^2(p)}{d-1} - \big(\frac{|\mu_3(p)|}{(d-1)(d-2)}\big)^{2/3}$ for $d \ge 3$ and $\tau_{\mathrm{skew}}(p) := \sigma^2(p)$ for $d = 2$.*

(a) ***Universal four-term sandwich.*** *Every $p \in \mathcal{R}_d$ satisfies*

$$\frac{\Delta_{\min}(p)^2}{4} \;\le\; \tau_*(p) \;\le\; \min\left\{\frac{d(d-1)}{4\Phi_d(p)},\; \frac{W(p)^2}{8d - 12},\; \tau_{\mathrm{skew}}(p)\right\} \;\le\; \frac{\sigma^2(p)}{d-1}. \tag{6}$$

Moreover:

- *For $d = 2$, all five quantities in (6) coincide identically. For $d \geq 3$ and $p \in \mathcal{R}_d^+$, the gap lower bound is strict ($\tau_*(p) > \Delta_{\min}(p)^2/4$), yet the constant $1/4$ is sharp:* $\inf_{p \in \mathcal{R}_d^+} \frac{\tau_*(p)}{\Delta_{\min}(p)^2} = \frac{1}{4}$.
- *For $p \in \mathcal{R}_d^+$, equality holds in $\tau_*(p) \leq \frac{d(d-1)}{4\Phi_d(p)}$ or in $\tau_*(p) \leq \frac{\sigma^2(p)}{d-1}$ if and only if p is a nondegenerate affine Hermite polynomial $He_d^{(t_0)}(x-\mu)$ ($t_0 > 0$).*
- *For $d = 3$, equality holds in $\tau_*(p) \leq \frac{W(p)^2}{8d-12}$ if and only if p is an affine Hermite polynomial $He_3^{(t_0)}(x-\mu)$ ($t_0 \geq 0$, including $(x-\mu)^3$); for $d \geq 4$ and $p \in \mathcal{R}_d^+$, the span inequality is always strict: $\tau_*(p) < \frac{W(p)^2}{8d-12}$.*
- *For $d = 3$, equality holds in $\tau_*(p) = \tau_{\text{skew}}(p)$ identically for all $p \in \mathcal{R}_3$.*

(b) ***Derivative monotonicity and finite free superadditivity.*** *For every $p, q \in \mathcal{R}_d$,*

$$\tau_*(p) \;\leq\; \tau_*(p') \;\leq\; \ldots \;\leq\; \tau_*(p^{(d-2)}) \;=\; \frac{\sigma^2(p)}{d-1}, \qquad \tau_*(p \boxplus_d q) \;\geq\; \tau_*(p) + \tau_*(q). \tag{7}$$

In the superadditivity inequality, equality holds whenever p or q is an affine Hermite polynomial $He_d^{(t)}(x-\mu)$ ($t \geq 0$), and more generally if and only if $p_{-\tau_*(p)} \boxplus_d q_{-\tau_*(q)} \in \mathcal{R}_d \setminus \mathcal{R}_d^+$.

(c) ***Finite Free Costa Concavity.*** *For $p \in \mathcal{R}_d$, define the finite free entropy power $N_d(p_t) := |\operatorname{Disc}(p_t)|^{\frac{2}{d(d-1)}}$ on $(-\tau_*(p), \infty)$. Then*

$$\begin{aligned} \frac{d}{dt} \log |\operatorname{Disc}(p_t)| &= 2\Phi_d(p_t), \\ N_d''(p_t) &= \frac{4}{d(d-1)} N_d(p_t) \left(\dot{\Phi}_d(p_t) + \frac{4}{d(d-1)} \Phi_d(p_t)^2 \right) \;\leq\; 0, \end{aligned} \tag{8}$$

with equality at some $t > -\tau_*(p)$ (and hence for all $t > -\tau_*(p)$) if and only if p is an affine Hermite polynomial $He_d^{(t_0)}(x-\mu)$ ($t_0 \geq 0$, including $(x-\mu)^d$).

Proof. If $p \in \mathcal{R}_d \setminus \mathcal{R}_d^+$, then $\Delta_{\min}(p) = 0$, $\tau_*(p) = 0$, and $1/\Phi_d(p) = 0$, so (6) holds trivially (nonnegativity of $\tau_{\text{skew}}(p)$ follows from (b) below since $\tau_{\text{skew}}(p) = \tau_*(p^{(d-3)}) \geq 0$). Assume henceforth that $p \in \mathcal{R}_d^+$, so $\tau_*(p) > 0$, and let $q_t := p_{-\tau_*(p)+t}$ for $t \in (0, \tau_*(p)]$. By Lemma 2.1, $q_t \in \mathcal{R}_d^+$ with strictly ordered C^∞ root trajectories $\lambda_1(t) < \cdots < \lambda_d(t)$ obeying (2).

Step 1: Gap lower bound and sharpness. For each $j \in \{1, \ldots, d-1\}$, let $g_j(t) := \lambda_{j+1}(t) - \lambda_j(t) > 0$. By (2),

$$\dot{g}_j(t) \;=\; s_{j+1} - s_j \;=\; \frac{2}{g_j(t)} - g_j(t) \sum_{k \notin \{j, j+1\}} \frac{1}{(\lambda_{j+1} - \lambda_k)(\lambda_j - \lambda_k)}. \tag{9}$$

For every $k \notin \{j, j+1\}$, the two differences $\lambda_{j+1} - \lambda_k$ and $\lambda_j - \lambda_k$ have the same sign (both positive if $k < j$, both negative if $k > j+1$), so their product is strictly positive. Hence $\frac{d}{dt}(g_j(t)^2) = 2g_j(t)\dot{g}_j(t) \leq 4$, with strict inequality whenever $d \geq 3$. Since $q_0 = p_{-\tau_*(p)}$ has a multiple root by Lemma 2.1(ii), there exists $j_0 \in \{1, \ldots, d-1\}$ with $\lim_{t \downarrow 0} g_{j_0}(t) = 0$. Integrating $\frac{d}{dt}(g_{j_0}^2) \leq 4$ over $[\varepsilon, \tau_*(p)]$ and letting $\varepsilon \downarrow 0$ yields

$$\Delta_{\min}(p)^2 \;\leq\; g_{j_0}(\tau_*(p))^2 \;=\; \lim_{\varepsilon \downarrow 0} (g_{j_0}(\tau_*(p))^2 - g_{j_0}(\varepsilon)^2) \;\leq\; 4\tau_*(p),$$

with strict inequality for all $d \geq 3$. To prove sharpness of the constant $1/4$ for $d \geq 3$, consider $p_R(x) := x(x-1)\prod_{k=1}^{d-2}(x - kR)$ as $R \to \infty$, which has $\Delta_{\min}(p_R) = 1$ for $R > 2$. Setting $\tilde{p}_R(x) := p_R(x)/\prod_{k=1}^{d-2}(-kR) = x(x-1)\prod_{k=1}^{d-2}(1 - x/(kR))$, we have $\tilde{p}_R \to x(x-1)$ coefficient-wise as $R \to \infty$. Since $e^{+\frac{\tau}{2}\partial_x^2}$ is a fixed finite-dimensional linear operator on polynomials of

degree $\le d$, for any fixed $\tau > \frac{1}{4}$ we have $e^{+\frac{\tau}{2}\partial_x^2}\tilde{p}_R(x) \to x^2 - x + \tau$ uniformly on compact subsets of $\mathbb{C}$. Because $x^2 - x + \tau$ has non-real roots $\frac{1}{2} \pm i\sqrt{\tau - 1/4}$, Hurwitz's theorem implies that $e^{+\frac{\tau}{2}\partial_x^2}p_R \notin \mathcal{R}_d$ for all sufficiently large R, so $\limsup_{R\to\infty} \tau_*(p_R) \le \tau$. Letting $\tau \downarrow \frac{1}{4}$ gives $\lim_{R\to\infty} \tau_*(p_R)/\Delta_{\min}(p_R)^2 = \frac{1}{4}$.

Step 2: Finite Free Costa Concavity and Stam–Fisher upper bound. Along $p_t \in \mathcal{R}_d^+$, differentiating $\log|\operatorname{Disc}(p_t)| = \sum_{i\neq j} \log|\lambda_i(t) - \lambda_j(t)|$ gives

$$\frac{d}{dt}\log|\operatorname{Disc}(p_t)| \;=\; \sum_{i\neq j}\frac{s_i - s_j}{\lambda_i - \lambda_j} \;=\; 2\sum_{i\neq j}\frac{s_i}{\lambda_i - \lambda_j} \;=\; 2\sum_{i=1}^{d} s_i^2 \;=\; 2\Phi_d(p_t).$$

Next, differentiating $s_i(t) = \sum_{j\neq i}(\lambda_i - \lambda_j)^{-1}$ yields $\dot{s}_i = -\sum_{j\neq i}\frac{s_i - s_j}{(\lambda_i - \lambda_j)^2}$, and symmetrizing over $(i,j) \leftrightarrow (j,i)$ gives

$$\dot{\Phi}_d(p_t) \;=\; 2\sum_{i=1}^{d} s_i\dot{s}_i \;=\; -2\sum_{i\neq j}\frac{s_i(s_i - s_j)}{(\lambda_i - \lambda_j)^2} \;=\; -\sum_{i\neq j} x_{ij}^2, \qquad x_{ij} := \frac{s_i - s_j}{\lambda_i - \lambda_j}. \tag{10}$$

Since $\sum_{i\neq j} x_{ij} = 2\Phi_d(p_t)$ has $d(d-1)$ summands, the Cauchy–Schwarz inequality gives

$$-\dot{\Phi}_d(p_t) \;=\; \sum_{i\neq j} x_{ij}^2 \;\ge\; \frac{1}{d(d-1)}\left(\sum_{i\neq j} x_{ij}\right)^2 \;=\; \frac{4}{d(d-1)}\Phi_d(p_t)^2. \tag{11}$$

For $N_d(p_t) = |\operatorname{Disc}(p_t)|^{\frac{2}{d(d-1)}}$, we have $N_d'/N_d = \frac{4}{d(d-1)}\Phi_d$, and differentiating once more produces (8), whose nonpositivity is immediate from (11). Equality in (11) holds at t iff $x_{ij} = c$ is constant for all $i \neq j$, i.e., $s_i = c(\lambda_i - \mu)$ (since $\sum_i s_i = 0$ and $c = \frac{2\Phi_d}{d(d-1)} > 0$). Setting $t_0 := \frac{1}{2c} > 0$, the polynomial $t_0 p_t''(x) - (x-\mu)p_t'(x) + d\,p_t(x)$ has degree $\le d-1$ (its x^d coefficient $-d + d = 0$ cancels) and vanishes at all d distinct roots λ_i, hence vanishes identically; thus $p_t(x) = He_d^{(t_0)}(x-\mu)$ with $t_0 > 0$, so $p = e^{+\frac{t}{2}\partial_x^2}p_t = He_d^{(t_0-t)}(x-\mu)$ with $t_0 - t \ge 0$ (since $p \in \mathcal{R}_d$ and $\tau_*(p_t) = t_0 \ge t$). This proves (c) for all $p \in \mathcal{R}_d$ (for $p \in \mathcal{R}_d \setminus \mathcal{R}_d^+$, Lemma 2.1(i) gives $p_t \in \mathcal{R}_d^+$ for every $t > 0 = -\tau_*(p)$, so the same ODE characterization applies verbatim to p_t and yields $p = (x-\mu)^d$).

Furthermore, (11) rewrites as $\frac{d}{dt}(1/\Phi_d(p_t)) \ge \frac{4}{d(d-1)}$ on $(-\tau_*(p), \infty)$. Integrating over $[-\tau, 0]$ for any $\tau \in (0, \tau_*(p))$ yields

$$\frac{1}{\Phi_d(p)} \;>\; \frac{1}{\Phi_d(p)} - \frac{1}{\Phi_d(p_{-\tau})} \;\ge\; \frac{4}{d(d-1)}\tau,$$

and letting $\tau \uparrow \tau_*(p)$ gives $\tau_*(p) \le \frac{d(d-1)}{4\Phi_d(p)}$, with equality iff p is affine Hermite. The finite free Cramér–Rao inequality $\frac{d(d-1)}{4\Phi_d(p)} \le \frac{\sigma^2(p)}{d-1}$ follows immediately from Cauchy–Schwarz applied to $\sum_{i=1}^d(\lambda_i - \mu)s_i = \frac{1}{2}\sum_{i\neq j}\frac{\lambda_i - \lambda_j}{\lambda_i - \lambda_j} = \frac{d(d-1)}{2}$, with the same Hermite equality condition.

Step 3: Span upper bound. Let $W(t) := \lambda_d(t) - \lambda_1(t)$ on $(-\tau_*(p), \infty)$. By (2),

$$\dot{W}(t) \;=\; s_d - s_1 \;=\; \frac{2}{W(t)} + \sum_{k=2}^{d-1}\left(\frac{1}{\lambda_d - \lambda_k} + \frac{1}{\lambda_k - \lambda_1}\right).$$

For each $k \in \{2, \dots, d-1\}$, let $u_k := \lambda_d - \lambda_k > 0$ and $v_k := \lambda_k - \lambda_1 > 0$, so $u_k + v_k = W(t)$. By the AM–HM inequality $\frac{1}{u_k} + \frac{1}{v_k} = \frac{u_k + v_k}{u_k v_k} \ge \frac{4}{u_k + v_k} = \frac{4}{W(t)}$, with equality iff $\lambda_k = \frac{\lambda_1 + \lambda_d}{2}$. Summing over $k = 2, \dots, d-1$ gives $\dot{W}(t) \ge \frac{4d-6}{W(t)}$, hence $\frac{d}{dt}(W(t)^2) \ge 8d - 12$. Integrating over $[-\tau, 0]$ and letting $\tau \uparrow \tau_*(p)$ yields $W(p)^2 - W(p_{-\tau_*(p)})^2 \ge (8d-12)\tau_*(p)$, so $\tau_*(p) \le \frac{W(p)^2}{8d-12}$. For

$d \geq 4$, simple roots $\lambda_2 < \cdots < \lambda_{d-1}$ cannot all equal the midpoint $\frac{\lambda_1+\lambda_d}{2}$, so the inequality is strict. For $d = 3$, equality holds iff $\lambda_2 = \frac{\lambda_1+\lambda_3}{2}$ and $W(p_{-\tau_*}) = 0$, i.e., p is symmetric and hence affine Hermite $He_3^{(t_0)}(x-\mu)$.

Step 4: Derivative monotonicity, skewness ceiling, and finite free superadditivity. Because ∂_x commutes with $e^{+\frac{\tau}{2}\partial_x^2}$, whenever $p_{-\tau} \in \mathcal{R}_d$ Rolle's theorem yields $(p')_{-\tau} = (p_{-\tau})' \in d \cdot \mathcal{R}_{d-1}$, so $\tau_*(p) \leq \tau_*(p')$. Iterating yields the chain in (7). Translating so $\mu(p) = 0$, Newton's identities give $p(x) = x^d - \frac{d}{2}\sigma^2 x^{d-2} - \frac{d}{3}\mu_3 x^{d-3} + \ldots$. Differentiating $d-2$ times produces the monic quadratic $\frac{2}{d!}p^{(d-2)}(x) = x^2 - \frac{\sigma^2}{d-1}$, whose collapse time is $\frac{\sigma^2}{d-1}$. For $d \geq 3$, differentiating $d-3$ times produces the monic cubic

$$q(x) \;=\; \frac{6}{d!}p^{(d-3)}(x) \;=\; x^3 - \frac{3\sigma^2}{d-1}x - \frac{2\mu_3}{(d-1)(d-2)}.$$

Under backward heat flow, $q_{-\tau}(x) = x^3 + (3\tau - \frac{3\sigma^2}{d-1})x - \frac{2\mu_3}{(d-1)(d-2)}$, which lies in $\mathcal{R}_3$ iff its discriminant $-4(3\tau - \frac{3\sigma^2}{d-1})^3 - 27(\frac{2\mu_3}{(d-1)(d-2)})^2 \geq 0$, i.e., iff $\tau \leq \frac{\sigma^2}{d-1} - (\frac{|\mu_3|}{(d-1)(d-2)})^{2/3} = \tau_{\mathrm{skew}}(p)$. Thus $\tau_*(p) \leq \tau_*(p^{(d-3)}) = \tau_{\mathrm{skew}}(p)$, with equality for all $d = 3$.

Finally, writing $p(x) = P(\partial_x)x^d$ and $q(x) = Q(\partial_x)x^d$ (with $\deg P, \deg Q \leq d$, $P(0) = Q(0) = 1$), the finite free additive convolution is $(p \boxplus_d q)(x) = P(\partial_x)Q(\partial_x)x^d$ [10]. Since constant-coefficient differential operators commute, for all $s, t \in \mathbb{R}$ we have

$$e^{+\frac{s+t}{2}\partial_x^2}(p \boxplus_d q) \;=\; (e^{+\frac{s}{2}\partial_x^2}p) \boxplus_d (e^{+\frac{t}{2}\partial_x^2}q).$$

Setting $s = \tau_*(p)$ and $t = \tau_*(q)$, both p_{-s} and q_{-t} belong to $\mathcal{R}_d$; by the real-rootedness preservation theorem of Marcus–Spielman–Srivastava [10, Theorem 1.3], their convolution $p_{-s} \boxplus_d q_{-t}$ belongs to $\mathcal{R}_d$. Hence $\tau_*(p \boxplus_d q) \geq \tau_*(p) + \tau_*(q)$. By Lemma 2.1(ii), equality holds iff $(p \boxplus_d q)_{-(s+t)} = p_{-s} \boxplus_d q_{-t}$ has a multiple root, i.e., lies in $\mathcal{R}_d \setminus \mathcal{R}_d^+$. In particular, if $q(x) = He_d^{(t_0)}(x-\mu)$ ($t_0 \geq 0$), then $q_{-\tau_*(q)}(x) = (x-\mu)^d$ and $(p \boxplus_d q)_{-(s+t)}(x) = p_{-\tau_*(p)}(x-\mu) \in \mathcal{R}_d \setminus \mathcal{R}_d^+$, so equality holds. □

3 Dyson–Stieltjes Operator Algebra and Exact Trace Laws

To analyze higher-order derivatives of $\Phi_d(p_t)$ and expose the hidden spectral geometry of the Dyson flow (2), we introduce four canonical matrices associated to any $p \in \mathcal{R}_d^+$ with roots $\lambda_1 < \cdots < \lambda_d$ and critical points $\mu_1 < \cdots < \mu_{d-1}$ (roots of p', strictly interlacing $\lambda_1 < \mu_1 < \lambda_2 < \cdots < \mu_{d-1} < \lambda_d$):

- The skew-symmetric Hilbert matrix $A \in \mathbb{R}^{d\times d}$ with $A_{ij} := (\lambda_i - \lambda_j)^{-1}$ ($i \neq j$) and $A_{ii} := 0$, together with its positive semidefinite Gramian $B := A^\top A = -A^2 \succeq 0$.
- The inverse-square pairwise graph Laplacian $L \in \mathbb{R}^{d\times d}$ with $L_{ij} := -(\lambda_i - \lambda_j)^{-2}$ ($i \neq j$) and $L_{ii} := w_i := \sum_{j\neq i}(\lambda_i - \lambda_j)^{-2}$. Note $L = EE^\top \succeq 0$, where $E \in \mathbb{R}^{d\times\binom{d}{2}}$ has column $e_{ij} = (\mathbf{e}_i - \mathbf{e}_j)/(\lambda_i - \lambda_j)$ for $1 \leq i < j \leq d$.
- The critical-point Cauchy matrix $C \in \mathbb{R}^{d\times(d-1)}$ with $C_{im} := (\lambda_i - \mu_m)^{-1}$, and the critical-point curvatures $\kappa_m := -p''(\mu_m)/p(\mu_m) > 0$ ($1 \leq m \leq d-1$).

Writing $D_\lambda := \mathrm{diag}(\lambda_1, \ldots, \lambda_d)$ and $x_{ij} := \frac{s_i - s_j}{\lambda_i - \lambda_j}$ ($i \neq j$), we also define the three Fisher-moment functionals

$$\begin{aligned} S_1(p) &:= \Phi_d(p) = \|s\|_2^2 = \frac{1}{2}\sum_{i\neq j} x_{ij}, \qquad S_3(p) := \frac{1}{2}\sum_{i\neq j} x_{ij}^3, \\ S_2(p) &:= s^\top L s = \frac{1}{2}\sum_{i\neq j} x_{ij}^2 = -\frac{1}{2}\dot{\Phi}_d(p). \end{aligned} \tag{12}$$

Theorem 3.1 (Dyson–Stieltjes Operator Algebra and Exact Dissipation Trace Laws)**.** *For every $d \geq 2$ and every $p \in \mathcal{R}_d^+$, the following identities hold:*

(a) ***All-orders commutator collapse.*** *The commutator $[D_\lambda, A] = D_\lambda A - AD_\lambda$ equals $\mathbf{1}_d\mathbf{1}_d^\top - I_d$. Consequently, for every integer $k \geq 1$,*

$$\mathbf{1}_d^\top B^k \mathbf{1}_d \;=\; \mathrm{Tr}(B^k). \tag{13}$$

In particular, $k = 1$ yields $\Phi_d(p) = \mathbf{1}_d^\top B\mathbf{1}_d = \mathrm{Tr}(B) = \sum_{i\neq j}(\lambda_i - \lambda_j)^{-2} = \mathrm{Tr}(L)$.

(b) ***Critical-point orthogonality and curvature flow.*** *The matrix C satisfies $C^\top \mathbf{1}_d = 0$ and*

$$C^\top C \;=\; \mathrm{diag}(\kappa_1, \ldots, \kappa_{d-1}), \qquad \Phi_d(p) \;=\; \frac{1}{4}\sum_{m=1}^{d-1} \kappa_m. \tag{14}$$

Along the heat flow p_t, the critical values and curvatures obey

$$\frac{d}{dt}\log|p_t(\mu_m(t))| \;=\; \frac{1}{2}\kappa_m(t), \qquad \dot\kappa_m \;=\; -\frac{1}{2}\kappa_m^2 + \frac{1}{2}\kappa_m(\tilde s_m^2 + 3\tilde w_m), \tag{15}$$

where $\tilde s_m := \sum_{n\neq m}(\mu_m - \mu_n)^{-1}$ and $\tilde w_m := \sum_{n\neq m}(\mu_m - \mu_n)^{-2}$ are the critical-point score and inverse-square weight.

(c) ***Frobenius-trace law and exact dissipation formulas.*** *Let $Q := \sum_{i\neq j}(\lambda_i - \lambda_j)^{-4}$ and $T_1 := \sum_{i=1}^d w_i^2 - Q = \sum_{i,j,k\ dist.}(\lambda_i - \lambda_j)^{-2}(\lambda_i - \lambda_k)^{-2}$. Then*

$$\mathrm{Tr}(L^2) \;=\; 2Q + T_1, \qquad \mathrm{Tr}(BL) \;=\; Q + \frac{1}{2}T_1 \;=\; \frac{1}{2}\mathrm{Tr}(L^2), \qquad \mathrm{Tr}(B^2) \;=\; Q + 2T_1, \tag{16}$$

so in particular $\|L - B\|_F = \|B\|_F$, and the Fisher dissipation $S_2(p) = -\frac{1}{2}\dot\Phi_d(p)$ satisfies

$$S_2(p) \;=\; \sum_{i\neq j}\frac{s_i - s_j}{(\lambda_i - \lambda_j)^3} \;=\; \frac{3}{2}\mathrm{Tr}(L^2) - \mathrm{Tr}(B^2) \;=\; \frac{5}{2}\sum_{i\neq j}(\lambda_i - \lambda_j)^{-4} - \frac{1}{2}\sum_{i=1}^d w_i^2. \tag{17}$$

(d) ***Exact reciprocal-Fisher curvature and monotone-score concavity.*** *Along the heat flow p_t, $\dot s = -Ls$, $\dot S_2 = -2(\|Ls\|_2^2 + S_3)$, $\ddot\Phi_d = 4(\|Ls\|_2^2 + S_3)$, and*

$$\frac{d^2}{dt^2}\left(\frac{1}{\Phi_d(p_t)}\right) \;=\; -\frac{4}{S_1^3}\Big[(S_1\|Ls\|_2^2 - S_2^2) + (S_1S_3 - S_2^2)\Big]. \tag{18}$$

Here $S_1\|Ls\|_2^2 - S_2^2 \geq 0$ unconditionally by Cauchy–Schwarz. Whenever the score vector of p is monotone ($s_1 \leq s_2 \leq \cdots \leq s_d$)—in particular, for every Jacobi polynomial $P_d^{(\alpha,\beta)}$ with $\alpha, \beta > -1$—we have $x_{ij} \geq 0$ for all $i \neq j$, hence $S_1S_3 - S_2^2 \geq 0$ and $\frac{d^2}{dt^2}(1/\Phi_d(p_t))\Big|_{t=0} \leq 0$.

Proof. Part (a). For $i \neq j$, $(D_\lambda A - AD_\lambda)_{ij} = (\lambda_i - \lambda_j)A_{ij} = 1$, while for $i = j$, $(D_\lambda A - AD_\lambda)_{ii} = 0$; thus $[D_\lambda, A] = \mathbf{1}_d\mathbf{1}_d^\top - I_d$. Because $B = -A^2$ commutes with A, cyclicity of the trace yields for every $k \geq 1$:

$$0 \;=\; \mathrm{Tr}([A, B^k]D_\lambda) \;=\; \mathrm{Tr}(B^k[D_\lambda, A]) \;=\; \mathrm{Tr}\big(B^k(\mathbf{1}_d\mathbf{1}_d^\top - I_d)\big) \;=\; \mathbf{1}_d^\top B^k\mathbf{1}_d - \mathrm{Tr}(B^k).$$

Since $s = A\mathbf{1}_d$ and $B = A^\top A$, setting $k = 1$ gives $\Phi_d(p) = \|A\mathbf{1}_d\|_2^2 = \mathbf{1}_d^\top B\mathbf{1}_d = \mathrm{Tr}(A^\top A) = \sum_{i\neq j}(\lambda_i - \lambda_j)^{-2} = \mathrm{Tr}(L)$.

Part (b). Since $p'(\mu_m) = 0$ and $p(\mu_m) \neq 0$, we have $(C^\top\mathbf{1}_d)_m = \sum_{i=1}^d(\lambda_i - \mu_m)^{-1} = -p'(\mu_m)/p(\mu_m) = 0$. For $m \neq n$, partial fractions give

$$(C^\top C)_{mn} \;=\; \sum_{i=1}^d \frac{1}{(\lambda_i - \mu_m)(\lambda_i - \mu_n)} \;=\; \frac{1}{\mu_m - \mu_n}\sum_{i=1}^d\left(\frac{1}{\lambda_i - \mu_m} - \frac{1}{\lambda_i - \mu_n}\right) \;=\; 0.$$

For $m = n$, differentiating $-p'(x)/p(x) = \sum_{i=1}^{d}(\lambda_i - x)^{-1}$ at $x = \mu_m$ (where $p'(\mu_m) = 0$) gives $(C^\top C)_{mm} = \sum_{i=1}^{d}(\lambda_i - \mu_m)^{-2} = -p''(\mu_m)/p(\mu_m) = \kappa_m > 0$. To prove $\Phi_d(p) = \frac{1}{4}\sum_m \kappa_m$, consider the rational function $M(z) := \frac{p''(z)^2}{p(z)p'(z)}$. Because $\deg(p''^2) = 2d-4$ and $\deg(pp') = 2d-1$, we have $M(z) = O(|z|^{-3})$ as $|z| \to \infty$, so the sum of all residues of M vanishes. Its poles are simple poles at $\{\lambda_i\}$ and $\{\mu_m\}$ with residues

$$\mathrm{Res}_{z=\lambda_i} M(z) = \left(\frac{p''(\lambda_i)}{p'(\lambda_i)}\right)^2 = 4s_i^2, \qquad \mathrm{Res}_{z=\mu_m} M(z) = \frac{p''(\mu_m)^2}{p(\mu_m)p''(\mu_m)} = -\kappa_m.$$

Summing residues to zero gives $4\Phi_d(p) - \sum_{m=1}^{d-1}\kappa_m = 0$.

For the curvature ODE (15), since $p_t'(\mu_m(t)) = 0$ and $\partial_t p_t = -\frac{1}{2}p_t''$, we have $\frac{d}{dt}p_t(\mu_m(t)) = -\frac{1}{2}p_t''(\mu_m) = \frac{1}{2}\kappa_m p_t(\mu_m)$, giving the first equation of (15). Because p_t' itself solves $\partial_t(p_t') = -\frac{1}{2}(p_t')''$, its roots $\mu_m(t)$ obey $\dot\mu_m = \frac{p_t'''(\mu_m)}{2p_t''(\mu_m)} = \tilde s_m$. Writing $p_t'(x) = (x-\mu_m)r(x)$ with $r(x) = d\prod_{n\neq m}(x-\mu_n)$ gives $p_t''(\mu_m) = r(\mu_m)$, $p_t'''(\mu_m) = 2r'(\mu_m) = 2\tilde s_m r(\mu_m)$, and $p_t^{(4)}(\mu_m) = 3r''(\mu_m) = 3(\tilde s_m^2 - \tilde w_m)r(\mu_m)$. Therefore

$$\frac{\dot\kappa_m}{\kappa_m} = \frac{-\frac{1}{2}p_t^{(4)}(\mu_m) + p_t'''(\mu_m)\tilde s_m}{p_t''(\mu_m)} - \frac{1}{2}\kappa_m = -\frac{3}{2}(\tilde s_m^2 - \tilde w_m) + 2\tilde s_m^2 - \frac{1}{2}\kappa_m = -\frac{1}{2}\kappa_m + \frac{1}{2}\tilde s_m^2 + \frac{3}{2}\tilde w_m.$$

Part (c). First, differentiate $\Phi_d(p_t) = \sum_{i\neq j}(\lambda_i - \lambda_j)^{-2}$ (from Part (a)) along $\dot\lambda_i = s_i$:

$$-2S_2(p) = \dot\Phi_d(p) = -2\sum_{i\neq j}\frac{s_i - s_j}{(\lambda_i - \lambda_j)^3} \implies S_2(p) = \sum_{i\neq j}\frac{s_i - s_j}{(\lambda_i - \lambda_j)^3}.$$

Expanding $s_i - s_j$ gives $S_2 = 2Q + 2\sum_{i,j,k\ \mathrm{dist.}}(\lambda_i - \lambda_j)^{-3}(\lambda_i - \lambda_k)^{-1}$. For any unordered triple of distinct indices $\{i,j,k\}$, set $a = \lambda_i - \lambda_j$, $b = \lambda_j - \lambda_k$, $c = \lambda_k - \lambda_i$, so $a + b + c = 0$. Summing over the 6 ordered permutations of $\{i,j,k\}$ yields the cyclic triple-sum identity:

$$\begin{aligned}\sum_{6\ \mathrm{perms}}\frac{1}{(\lambda_i - \lambda_j)^3(\lambda_i - \lambda_k)} &= -\frac{b+c}{a^3bc} - \frac{c+a}{b^3ca} - \frac{a+b}{c^3ab} = \frac{1}{a^2bc} + \frac{1}{b^2ca} + \frac{1}{c^2ab} \\ &= \frac{ab+bc+ca}{a^2b^2c^2} = -\frac{1}{2}\frac{a^2+b^2+c^2}{a^2b^2c^2} = -\frac{1}{2}\left(\frac{1}{a^2b^2} + \frac{1}{b^2c^2} + \frac{1}{c^2a^2}\right),\end{aligned}$$

where we used $a^2+b^2+c^2 = -2(ab+bc+ca)$. Since the 6 permutations contribute $2(\frac{1}{a^2b^2} + \frac{1}{b^2c^2} + \frac{1}{c^2a^2})$ to T_1, summing over all unordered triples gives $\sum_{i,j,k\ \mathrm{dist.}}(\lambda_i - \lambda_j)^{-3}(\lambda_i - \lambda_k)^{-1} = -\frac{1}{4}T_1$. Consequently,

$$S_2(p) = 2Q - \frac{1}{2}T_1 = \frac{5}{2}Q - \frac{1}{2}(Q + T_1) = \frac{5}{2}Q - \frac{1}{2}\sum_{i=1}^{d} w_i^2. \tag{19}$$

Next, compute the entries of $B = -A^2$: $B_{ii} = \sum_{k\neq i}(\lambda_i - \lambda_k)^{-2} = w_i = L_{ii}$, and for $i \neq j$, partial fractions give

$$B_{ij} = -\sum_{k\notin\{i,j\}}\frac{1}{(\lambda_i - \lambda_k)(\lambda_k - \lambda_j)} = -\frac{1}{\lambda_i - \lambda_j}\sum_{k\notin\{i,j\}}\left(\frac{1}{\lambda_i - \lambda_k} - \frac{1}{\lambda_j - \lambda_k}\right) = \frac{2}{(\lambda_i - \lambda_j)^2} - x_{ij}.$$

Therefore $\mathrm{Tr}(L^2) = \sum_i w_i^2 + Q = 2Q + T_1$, and

$$\begin{aligned}\mathrm{Tr}(BL) &= \sum_{i=1}^{d} w_i^2 + \sum_{i\neq j}\left(\frac{2}{(\lambda_i - \lambda_j)^2} - x_{ij}\right)\left(-\frac{1}{(\lambda_i - \lambda_j)^2}\right) \\ &= (Q + T_1) - 2Q + S_2 = Q + \frac{1}{2}T_1 = \frac{1}{2}\mathrm{Tr}(L^2),\end{aligned}$$

whence $\|L-B\|_F^2 = \operatorname{Tr}(L^2) - 2\operatorname{Tr}(BL) + \operatorname{Tr}(B^2) = \|B\|_F^2$. Similarly,

$$\begin{aligned}\operatorname{Tr}(B^2) &= (Q+T_1) + \sum_{i\neq j}\left(\frac{4}{(\lambda_i-\lambda_j)^4} - \frac{4x_{ij}}{(\lambda_i-\lambda_j)^2} + x_{ij}^2\right)\\ &= 5Q + T_1 - 4S_2 + 2S_2 \;=\; 5Q + T_1 - 2\Big(2Q - \frac{1}{2}T_1\Big) \;=\; Q + 2T_1.\end{aligned}$$

Hence $\frac{3}{2}\operatorname{Tr}(L^2) - \operatorname{Tr}(B^2) = \frac{3}{2}(2Q+T_1) - (Q+2T_1) = 2Q - \frac{1}{2}T_1 = S_2(p)$, establishing (16)–(17).

Part (d). Since $\dot{s}_i = -\sum_{j\neq i}\frac{s_i-s_j}{(\lambda_i-\lambda_j)^2} = -(Ls)_i$, differentiating $S_2 = \frac{1}{2}\sum_{i\neq j}\frac{(s_i-s_j)^2}{(\lambda_i-\lambda_j)^2}$ yields

$$\dot{S}_2 \;=\; \sum_{i\neq j}\frac{(s_i-s_j)(-(Ls)_i+(Ls)_j)}{(\lambda_i-\lambda_j)^2} - \sum_{i\neq j}\frac{(s_i-s_j)^3}{(\lambda_i-\lambda_j)^3} \;=\; -2s^\top L(Ls) - 2S_3 \;=\; -2(\|Ls\|_2^2 + S_3).$$

Thus $\ddot{\Phi}_d = -2\dot{S}_2 = 4(\|Ls\|_2^2 + S_3)$. Differentiating $\frac{d}{dt}(1/S_1) = 2S_2/S_1^2$ gives

$$\frac{d^2}{dt^2}\left(\frac{1}{S_1}\right) \;=\; \frac{2\dot{S}_2S_1 - 4S_2\dot{S}_1}{S_1^3} \;=\; \frac{-4(\|Ls\|_2^2+S_3)S_1 + 8S_2^2}{S_1^3} \;=\; -\frac{4}{S_1^3}\Big[(S_1\|Ls\|_2^2 - S_2^2) + (S_1S_3 - S_2^2)\Big].$$

By Cauchy–Schwarz, $S_2^2 = \langle s, Ls\rangle^2 \le \|s\|_2^2\|Ls\|_2^2 = S_1\|Ls\|_2^2$. If $s_1 \le \cdots \le s_d$, then $x_{ij} = \frac{s_i-s_j}{\lambda_i-\lambda_j} \ge 0$ for all $i \neq j$, so Cauchy–Schwarz on $\sqrt{x_{ij}}$ and $x_{ij}^{3/2}$ yields $\left(\sum x_{ij}^2\right)^2 \le \left(\sum x_{ij}\right)\left(\sum x_{ij}^3\right)$, i.e., $S_1S_3 - S_2^2 \ge 0$. For every Jacobi polynomial $p = P_d^{(\alpha,\beta)}$ $(\alpha,\beta > -1)$, the Jacobi ODE $(1-x^2)p'' + [\beta-\alpha-(\alpha+\beta+2)x]p' + d(d+\alpha+\beta+1)p = 0$ gives at each root $\lambda_i \in (-1,1)$:

$$\begin{aligned} s_i &= \frac{(\alpha+\beta+2)\lambda_i + (\alpha-\beta)}{2(1-\lambda_i^2)},\\ \frac{d}{dx}\left(\frac{(\alpha+\beta+2)x + (\alpha-\beta)}{2(1-x^2)}\right) &= \frac{(\alpha+1)(1+x)^2 + (\beta+1)(1-x)^2}{2(1-x^2)^2} \;>\; 0,\end{aligned}$$

so $s_1 < \cdots < s_d$ strictly. □

4 Exact Cubic Law, Symmetric Quartic Hermite Phase Transition, and Toscani Concavity

In classical probability, Toscani [19] proved that $t \mapsto 1/I(X+\sqrt{t}Z)$ is concave for log-concave densities, and Cheng–Geng [3] showed this fails for general multimodal densities. In stark contrast, we now prove that in finite free probability, $t \mapsto 1/\Phi_d(p_t)$ is globally concave for *every* cubic $(d=3)$ and *every* symmetric quartic $(d=4)$, including strongly bimodal symmetric quartics $(1 < b/a < \sqrt{5})$ whose score vectors are non-monotone and whose pairwise slopes x_{ij} take negative values. At the same time, we determine the exact collapse law and uncover a sharp Hermite phase transition in $d=4$.

Theorem 4.1 (Exact Cubic Law, Symmetric Quartic Phase Transition, and Concavity of $1/\Phi_d$)**.**

(a) ***Exact cubic law*** *$(d=3)$. For every centered cubic $p(x) = x^3 + ax + b \in \mathcal{R}_3^+$ $(a<0$, $\Delta = -4a^3 - 27b^2 > 0)$,*

$$\begin{aligned}\tau_*(p) &= -\frac{a}{3} - \left(\frac{|b|}{2}\right)^{2/3}, \qquad \Phi_3(p_t) = \frac{18(3t-a)^2}{4(3t-a)^3 - 27b^2},\\ \frac{d^2}{dt^2}\left(\frac{1}{\Phi_3(p_t)}\right) &= -\frac{81b^2}{(3t-a)^4} \;\le\; 0\end{aligned} \tag{20}$$

for all $t > -\tau_*(p)$. Moreover, p satisfies the exact score-product identities

$$S_1S_3 - S_2^2 \;=\; 3S_1(s_1s_2s_3)^2, \qquad \|Ls\|_2^2 - S_3 \;=\; 3(s_1s_2s_3)^2, \qquad S_1\|Ls\|_2^2 - S_2^2 \;=\; 6S_1(s_1s_2s_3)^2, \tag{21}$$

and global concavity of $1/\Phi_3(p_t)$ yields the sharp tangent upper bound $\tau_*(p) \le \frac{S_1(p)}{2S_2(p)}$, with equality if and only if $b = 0$ (p is affine Hermite).

(b) ***Symmetric quartic Hermite phase transition*** *($d = 4$). Let $p(x) = (x^2 - a^2)(x^2 - b^2)$ with $0 < a < b$, set $r := b/a > 1$ and $\eta := \frac{12a^2b^2}{(a^2+b^2)^2} = \frac{12r^2}{(1+r^2)^2} \in (0, 3)$. Then $\eta = 1$ precisely at the Hermite root ratio $r_* := \sqrt{3} + \sqrt{2}$, and $\tau_*(p)$ undergoes a sharp collision-channel phase transition:*

$$\frac{\tau_*(p)}{\sigma^2(p)/3} \;=\; \begin{cases} 1 - \sqrt{\dfrac{\eta - 1}{2}}, & 1 < r \le \sqrt{3} + \sqrt{2} \quad (\textit{outer-pair collision at } \pm\sqrt{y_*}), \\ 1 - \sqrt{1 - \eta}, & r \ge \sqrt{3} + \sqrt{2} \quad (\textit{inner-pair collision at } 0), \end{cases} \tag{22}$$

where $\sigma^2(p)/3 = (a^2 + b^2)/6$, with simultaneous 4-fold collapse into x^4 at $r = r_*$ ($p = He_4^{(\sigma^2/3)}$) and universal critical amplitude ratio $\sqrt{2}$ between the two sides of the Hermite peak. Furthermore, $a^8(S_1S_3 - S_2^2)$ factors identically as

$$a^8(S_1S_3 - S_2^2) \;=\; \frac{(r^4 - 10r^2 + 1)^2(r^4 + 2r^2 + 13)(13r^4 + 2r^2 + 1)}{16\, r^6(r^2 - 1)^6} \;\ge\; 0, \tag{23}$$

vanishing if and only if $r = \sqrt{3} + \sqrt{2}$. Consequently, $\frac{d^2}{dt^2}(1/\Phi_4(p_t)) \le 0$ for all $t > -\tau_*(p)$ and every symmetric quartic obeys the global tangent upper bound $\tau_*(p) \le \frac{S_1(p)}{2S_2(p)}$, with equality if and only if $r = \sqrt{3} + \sqrt{2}$.

Proof. Part (a). For $p(x) = x^3 + ax + b$, we have $p_t(x) = x^3 + (a - 3t)x + b$ with discriminant $\Delta(t) = 4(3t - a)^3 - 27b^2$. The condition $\Delta(-\tau) \ge 0$ gives $\tau_*(p) = -\frac{a}{3} - (|b|/2)^{2/3}$. By Theorem 2.2(c),

$$\Phi_3(p_t) \;=\; \frac{1}{2}\frac{\Delta'(t)}{\Delta(t)} \;=\; \frac{18(3t - a)^2}{4(3t - a)^3 - 27b^2} \;\Longrightarrow\; \frac{1}{\Phi_3(p_t)} \;=\; \frac{2}{9}(3t - a) - \frac{3b^2}{2(3t - a)^2}.$$

Differentiating twice gives $\frac{d^2}{dt^2}(1/\Phi_3(p_t)) = -\frac{81b^2}{(3t-a)^4} \le 0$. Because $1/\Phi_3(p_t)$ is concave on $(-\tau_*(p), \infty)$ and vanishes as $t \downarrow -\tau_*(p)$ (directly from the formula when $b \neq 0$, and $1/\Phi_3(p_t) = \frac{2}{3}(t + \tau_*)$ when $b = 0$), its tangent line at $t = 0$ lies above the graph, yielding $0 \le \frac{1}{S_1} + \frac{2S_2}{S_1^2}(-\tau_*(p))$, i.e., $\tau_*(p) \le \frac{S_1}{2S_2}$.

To prove (21), set $a_0 = \lambda_1 - \lambda_2$, $b_0 = \lambda_2 - \lambda_3$, $c_0 = \lambda_3 - \lambda_1$ ($a_0 + b_0 + c_0 = 0$) and let $\alpha = 1/a_0$, $\beta = 1/b_0$, $\gamma = 1/c_0$. Then $e_2 := \alpha\beta + \beta\gamma + \gamma\alpha = \frac{a_0+b_0+c_0}{a_0b_0c_0} = 0$. Setting $e_1 := \alpha + \beta + \gamma$ and $e_3 := \alpha\beta\gamma$, the three real numbers α, β, γ are the roots of $z^3 - e_1z^2 - e_3 = 0$, whose discriminant is $-4e_1^3e_3 - 27e_3^2 = (\alpha - \beta)^2(\beta - \gamma)^2(\gamma - \alpha)^2 = (s_1s_2s_3)^2$ (since $s_1 = \alpha - \gamma$, $s_2 = \beta - \alpha$, $s_3 = \gamma - \beta$). Using $\alpha^3 = e_1\alpha^2 + e_3$, the pairwise slope is $x_{12} = \frac{s_1 - s_2}{a_0} = \frac{2}{a_0^2} - \frac{b_0 + c_0}{a_0b_0c_0} = 2\alpha^2 + \beta\gamma = 2\alpha^2 + e_3/\alpha = 3\alpha^2 - e_1\alpha$, and cyclically for x_{23}, x_{31}. From the power sums $p_1 = e_1$, $p_2 = e_1^2$, $p_3 = e_1^3 + 3e_3$, $p_4 = e_1^4 + 4e_1e_3$, $p_5 = e_1^5 + 5e_1^2e_3$, $p_6 = e_1^6 + 6e_1^3e_3 + 3e_3^2$, direct substitution gives

$$S_1 = 2e_1^2, \qquad S_2 = 4e_1^4 + 18e_1e_3, \qquad S_3 = 8e_1^6 + 60e_1^3e_3 + 81e_3^2.$$

Hence $S_1S_3 - S_2^2 = -24e_1^5e_3 - 162e_1^2e_3^2 = 6e_1^2(-4e_1^3e_3 - 27e_3^2) = 3S_1(s_1s_2s_3)^2$. On the other hand, since $s_i = \frac{3\lambda_i}{p'(\lambda_i)}$ for $p(x) = x^3 + ax + b$ and $\prod_i p'(\lambda_i) = -\Delta$, we have $s_1s_2s_3 = \frac{27b}{\Delta}$, so comparing (18) with (20) gives

$$(S_1\|Ls\|_2^2 - S_2^2) + (S_1S_3 - S_2^2) \;=\; \frac{81b^2}{4a^4}S_1^3 \;=\; \frac{81b^2}{4a^4} \cdot \frac{324a^4}{\Delta^2}S_1 \;=\; 9S_1(s_1s_2s_3)^2.$$

Subtracting $S_1S_3 - S_2^2 = 3S_1(s_1s_2s_3)^2$ yields $S_1\|Ls\|_2^2 - S_2^2 = 6S_1(s_1s_2s_3)^2$ and $\|Ls\|_2^2 - S_3 = 3(s_1s_2s_3)^2$.

Part (b). For $p(x) = x^4 - (a^2+b^2)x^2 + a^2b^2$, applying $e^{+\frac{\tau}{2}\partial_x^2}$ gives $p_{-\tau}(x) = q_\tau(x^2)$ where

$$q_\tau(y) \;=\; y^2 + B(\tau)\,y + C(\tau), \qquad B(\tau) := 6\tau - (a^2+b^2), \qquad C(\tau) := 3\tau^2 - (a^2+b^2)\tau + a^2b^2.$$

Thus $p_{-\tau} \in \mathcal{R}_4$ iff q_τ has two real nonnegative roots, i.e., $B(\tau) \le 0$, $C(\tau) \ge 0$, and $\Delta_q(\tau) := B(\tau)^2 - 4C(\tau) = 24\tau^2 - 8(a^2+b^2)\tau + (b^2-a^2)^2 \ge 0$. Remarkably, the τ-discriminants of C and Δ_q are exact opposite multiples of one another:

$$\begin{aligned}\mathrm{Disc}_\tau(C) = a^4 - 10a^2b^2 + b^4 \;&=\; (a^2+b^2)^2(1-\eta),\\ \mathrm{Disc}_\tau(\Delta_q) = -32(a^4 - 10a^2b^2 + b^4) \;&=\; 32(a^2+b^2)^2(\eta-1).\end{aligned}$$

Since $r^4 - 10r^2 + 1 = 0$ $(r > 1)$ has the unique root $r_* = \sqrt{5+2\sqrt{6}} = \sqrt{3}+\sqrt{2}$, we obtain two disjoint regimes:

- If $1 < r \le r_*$ $(\eta \ge 1)$, then $C(\tau) \ge 0$ for all τ (strictly for $r < r_*$), while $\Delta_q(\tau)$ first vanishes at $\tau_{\text{out}} = \frac{a^2+b^2}{6}(1 - \sqrt{(\eta-1)/2}) \le \frac{a^2+b^2}{6}$, where the outer pairs collapse at $\pm\sqrt{-B(\tau_{\text{out}})/2}$.
- If $r \ge r_*$ $(\eta \le 1)$, then $\Delta_q(\tau) \ge 0$ for all τ (strictly for $r > r_*$), while $C(\tau)$ first vanishes at $\tau_{\text{in}} = \frac{a^2+b^2}{6}(1 - \sqrt{1-\eta}) \le \frac{a^2+b^2}{6}$, where the inner pair collides at 0.

At $r = r_*$, both vanish simultaneously at $\tau = (a^2+b^2)/6 = \sigma^2/3$ and $p_{-\tau}(x) = x^4$. This proves (22).

For (23), by scale invariance set $a = 1$ and $b = r > 1$, so roots are $(-r, -1, 1, r)$. By symmetry $s(-1) = -s(1)$ and $s(-r) = -s(r)$, with $s(1) = \frac{r^2-5}{2(r^2-1)}$ (notice $s(1) < 0$ for $1 < r < \sqrt{5}$, giving non-monotone scores!) and $s(r) = \frac{5r^2-1}{2r(r^2-1)}$. The six unordered pairs give four distinct slopes:

$$\begin{aligned} x_{-1,1} &= \frac{r^2-5}{2(r^2-1)}, & x_{-r,r} &= \frac{5r^2-1}{2r^2(r^2-1)},\\ x_{1,r} = x_{-r,-1} &= \frac{-r^2+6r-1}{2r(r-1)^2}, & x_{-1,r} = x_{-r,1} &= \frac{r^2+6r+1}{2r(r+1)^2}.\end{aligned}$$

Summing $S_k = x_{-1,1}^k + x_{-r,r}^k + 2x_{1,r}^k + 2x_{-1,r}^k$ $(k = 1, 2, 3)$ over the common denominator gives the exact rational expressions

$$\begin{aligned} S_1 &= \frac{(r^2+1)(r^4+14r^2+1)}{2r^2(r^2-1)^2}, \qquad S_2 = \frac{r^{12} - 8r^{10} + 55r^8 + 416r^6 + 55r^4 - 8r^2 + 1}{4r^4(r^2-1)^4},\\ S_3 &= \frac{(r^2+1)(r^{16} - 19r^{14} + 190r^{12} - 1117r^{10} + 5986r^8 - 1117r^6 + 190r^4 - 19r^2 + 1)}{8r^6(r^2-1)^6},\end{aligned}$$

from which direct polynomial multiplication produces (23). Because heat flow preserves evenness of p_t, p_t remains a symmetric quartic for every $t > -\tau_*(p)$, so (18) and (23) imply $\frac{d^2}{dt^2}(1/\Phi_4(p_t)) \le 0$ on $(-\tau_*(p), \infty)$, strictly unless $r = \sqrt{3}+\sqrt{2}$. Since $\lim_{t\downarrow -\tau_*(p)} 1/\Phi_4(p_t) = 0$ (as $\Phi_4(p_t) \ge 2/\Delta_{\min}(p_t)^2 \to \infty$ or directly from $\int_{-\tau_*}^0 \Phi_4 = \infty$ and monotonicity of Φ_4), concavity implies $\tau_*(p) \le \frac{S_1(p)}{2S_2(p)}$. □

Theorem 3.1(d) (unconditional local concavity for all monotone-score polynomials), Theorem 4.1 (exact global concavity for all $d = 3$ and all symmetric $d = 4$ including strongly bimodal non-monotone scores), and exhaustive numerical testing across $d \in \{3, 4, 5, 6, 8\}$ (Section 6) motivate the following conjecture:

Conjecture 4.2 (Finite Free Toscani Concavity)**.** *For every $d \geq 4$ and every $p \in \mathcal{R}_d^+$, the reciprocal finite free Fisher information $t \mapsto 1/\Phi_d(p_t)$ is concave on $(-\tau_*(p), \infty)$—equivalently, $S_1(\|Ls\|_2^2 + S_3) \geq 2S_2^2$ on all of $\mathcal{R}_d^+$, with equality if and only if p is an affine Hermite polynomial. In particular, every $p \in \mathcal{R}_d^+$ satisfies the local-curvature collapse bound $\tau_*(p) \leq \frac{S_1(p)}{2S_2(p)}$.*

Remark 4.3. We emphasize that Conjecture 4.2 is strictly separated from our unconditional results (Theorems 2.2, 3.1, 4.1, and 5.1). As Figure 1(b) illustrates, if Conjecture 4.2 holds, the bound $\frac{S_1}{2S_2}$ captures the sharp $\Theta(d^{-4})$ hard-edge collapse scaling on Chebyshev and Legendre polynomials—closing the quadratic gap left by the $\Theta(d^{-2})$ Stam bound.

5 Ultraspherical Jacobi Laws and Hard-Edge Scaling

We now determine the exact Fisher information, variance, and microscopic hard-edge collapse scaling of the classical ultraspherical (symmetric Jacobi) polynomials $P_d^{(\alpha,\alpha)}$ ($\alpha > -1$), which unify Chebyshev polynomials of the first kind (T_d, $\alpha = -\frac{1}{2}$), Legendre polynomials (P_d, $\alpha = 0$), and Chebyshev polynomials of the second kind (U_d, $\alpha = +\frac{1}{2}$). Because multiplying a polynomial by a nonzero constant leaves its roots (and hence τ_*, Φ_d, and σ^2) invariant, we identify $P_d^{(\alpha,\alpha)}$ with its monic normalization $\widehat{P}_d^{(\alpha,\alpha)} \in \mathcal{R}_d^+$.

Theorem 5.1 (Ultraspherical Exact Laws, Hard-Edge Entire-Function Limit, and Three-Scale Separation)**.** *Let $\alpha > -1$ and $d \geq 2$.*

(a) ***Exact Fisher information and variance.*** *Every ultraspherical Jacobi polynomial satisfies*

$$\Phi_d(P_d^{(\alpha,\alpha)}) \;=\; \frac{(d-1)\,d\,(d+2\alpha+1)(d+2\alpha+2)}{8(\alpha+2)}, \qquad \frac{\sigma^2(P_d^{(\alpha,\alpha)})}{d-1} \;=\; \frac{1}{2d+2\alpha-1}. \tag{24}$$

Consequently, the Stam and variance upper bounds of Theorem 2.2(a) scale as

$$\frac{d(d-1)}{4\Phi_d(P_d^{(\alpha,\alpha)})} \;=\; \frac{2(\alpha+2)}{(d+2\alpha+1)(d+2\alpha+2)} \;=\; \Theta_\alpha(d^{-2}), \qquad \frac{\sigma^2(P_d^{(\alpha,\alpha)})}{d-1} \;=\; \Theta_\alpha(d^{-1}).$$

(b) ***Uniform hard-edge entire-function convergence.*** *Define the hard-edge rescaled backward heat-flow polynomial*

$$Q_d(y;t) \;:=\; \frac{\big(e^{+\frac{t}{2d^4}\partial_x^2} P_d^{(\alpha,\alpha)}\big)\big(1 - \frac{y}{2d^2}\big)}{P_d^{(\alpha,\alpha)}(1)}, \qquad (y,t) \in \mathbb{C}^2. \tag{25}$$

As $d \to \infty$, $Q_d(y;t)$ converges uniformly on every compact subset of $(y,t) \in \mathbb{C}^2$ to the entire function

$$F_\alpha(y;t) \;:=\; \sum_{m=0}^{\infty} \frac{(-1)^m}{4^m\, m!\, (\alpha+1)_m}\, e^{+2t\partial_y^2}(y^m), \qquad F_\alpha(y;0) \;=\; \Gamma(\alpha+1)\left(\frac{y}{4}\right)^{-\alpha/2} J_\alpha(\sqrt{y}), \tag{26}$$

where $(\alpha+1)_m = \prod_{j=0}^{m-1}(\alpha+1+j)$ is the Pochhammer symbol and J_α is the Bessel function of the first kind.

(c) ***Asymptotic hard-edge bounds.*** *Define the hard-edge entire-function threshold $t^*_{\mathrm{edge}}(\alpha) := \inf\{t > 0 : F_\alpha(\cdot;t)$ has a non-real zero$\}$ and the Laguerre functional $L_\alpha(y;t) := F'_\alpha(y;t)^2 - F_\alpha(y;t)F''_\alpha(y;t)$. Then*

$$\limsup_{d\to\infty} d^4\, \tau_*(P_d^{(\alpha,\alpha)}) \;\leq\; t^*_{\mathrm{edge}}(\alpha), \tag{27}$$

and if $L_\alpha(y_0;t_0) < 0$ for some $y_0 \in \mathbb{R}$ and $t_0 > 0$, then $t^*_{\mathrm{edge}}(\alpha) \le t_0$. Furthermore, for every $\alpha \in [-\frac{1}{2}, \frac{1}{2}]$, the minimal root gap of $P_d^{(\alpha,\alpha)}$ is attained at the extremal edge pair $x_{d,1} - x_{d,2}$, yielding

$$\liminf_{d\to\infty} d^4\, \tau_*(P_d^{(\alpha,\alpha)}) \;\ge\; \frac{\left(j_{\alpha,2}^2 - j_{\alpha,1}^2\right)^2}{16}, \tag{28}$$

where $0 < j_{\alpha,1} < j_{\alpha,2} < \dots$ are the positive zeros of J_α.

(d) ***Certified two-sided $\Theta(d^{-4})$ scaling and three-scale separation.*** *For Chebyshev T_d ($\alpha = -\frac{1}{2}$), Legendre P_d ($\alpha = 0$), and Chebyshev U_d ($\alpha = +\frac{1}{2}$), 40-digit interval-arithmetic evaluation of L_α with explicit truncation remainder bounds ($< 10^{-25}$) certifies*

$$L_{-1/2}(18.5;\, 26.0) < -3.15 \times 10^{-5}, \qquad L_0(26.67;\, 42.0) < -1.64 \times 10^{-5},$$
$$L_{1/2}(36.67;\, 61.0) < -2.12 \times 10^{-6}.$$

Consequently, all three families satisfy the rigorous two-sided $\Theta(d^{-4})$ law

$$24.352 < \liminf_{d\to\infty} d^4\tau_*(T_d) \le \limsup_{d\to\infty} d^4\tau_*(T_d) \le 26.0,$$
$$38.093 < \liminf_{d\to\infty} d^4\tau_*(P_d) \le \limsup_{d\to\infty} d^4\tau_*(P_d) \le 42.0,$$
$$54.792 < \liminf_{d\to\infty} d^4\tau_*(U_d) \le \limsup_{d\to\infty} d^4\tau_*(U_d) \le 61.0,$$

establishing the exact three-scale separation $\tau_* = \Theta(d^{-4}) \ll \frac{d(d-1)}{4\Phi_d} = \Theta(d^{-2}) \ll \frac{\sigma^2}{d-1} = \Theta(d^{-1})$.

Proof. Part (a). Let $p = \widehat{P}_d^{(\alpha,\alpha)}$, which obeys the Jacobi ODE $(1-x^2)p''(x) - 2(\alpha+1)xp'(x) + \Lambda p(x) = 0$ with $\Lambda := d(d+2\alpha+1)$. At each root $\lambda_i \in (-1,1)$, $s_i = \frac{p''(\lambda_i)}{2p'(\lambda_i)} = \frac{(\alpha+1)\lambda_i}{1-\lambda_i^2}$. Using the partial-fraction identity $\frac{4x^2}{(1-x^2)^2} = \frac{1}{(1-x)^2} + \frac{1}{(1+x)^2} - \frac{2}{1-x^2}$ and symmetry $\{\lambda_i\} = \{-\lambda_i\}$, we obtain

$$\begin{aligned}\Phi_d(p) &= \frac{(\alpha+1)^2}{4}\sum_{i=1}^{d}\left[\frac{1}{(1-\lambda_i)^2} + \frac{1}{(1+\lambda_i)^2} - \frac{1}{1-\lambda_i} - \frac{1}{1+\lambda_i}\right] \\ &= \frac{(\alpha+1)^2}{2}\left[\left(\frac{p'(1)}{p(1)}\right)^2 - \frac{p''(1)}{p(1)} - \frac{p'(1)}{p(1)}\right],\end{aligned}$$

where we used $\sum_{i=1}^d (1-\lambda_i)^{-1} = p'(1)/p(1)$ and $\sum_{i=1}^d (1-\lambda_i)^{-2} = -\frac{d}{dx}(p'/p)\big|_{x=1} = (p'(1)/p(1))^2 - p''(1)/p(1)$. Setting $x = 1$ in the Jacobi ODE gives $\frac{p'(1)}{p(1)} = \frac{\Lambda}{2(\alpha+1)}$. Differentiating the ODE once yields $(1-x^2)p''' - 2(\alpha+2)xp'' + (\Lambda - 2\alpha - 2)p' = 0$, and setting $x = 1$ gives $\frac{p''(1)}{p(1)} = \frac{\Lambda(\Lambda-2\alpha-2)}{4(\alpha+1)(\alpha+2)}$. Substituting both ratios into $\Phi_d(p)$ yields

$$\Phi_d(p) \;=\; \frac{\Lambda}{8(\alpha+2)}\Big[\Lambda(\alpha+2) - (\alpha+1)(\Lambda-2\alpha-2) - 2(\alpha+1)(\alpha+2)\Big] \;=\; \frac{\Lambda(\Lambda-2\alpha-2)}{8(\alpha+2)}.$$

Factoring $\Lambda - 2\alpha - 2 = d^2 + (2\alpha+1)d - (2\alpha+2) = (d-1)(d+2\alpha+2)$ gives the first identity of (24). For the variance, write $p(x) = x^d + c_2 x^{d-2} + \dots$ (with $c_1 = 0$ by symmetry and $-2c_2 = \sum \lambda_i^2 = d\sigma^2$). Equating the coefficient of x^{d-2} in the Jacobi ODE gives

$$d(d-1) + c_2\Big[\Lambda - (d-2)(d-3+2\alpha+2)\Big] \;=\; d(d-1) + 2(2d+2\alpha-1)c_2 \;=\; 0,$$

whence $\frac{\sigma^2}{d-1} = \frac{-2c_2}{d(d-1)} = \frac{1}{2d+2\alpha-1}$.

Part (b). By the classical hypergeometric representation of Jacobi polynomials [17, Eq. (4.21.2)],

$$\frac{P_d^{(\alpha,\alpha)}(1-\frac{y}{2d^2})}{P_d^{(\alpha,\alpha)}(1)} = {}_2F_1\Big(-d,\, d+2\alpha+1;\, \alpha+1;\, \frac{y}{4d^2}\Big) = \sum_{m=0}^{d} c_{d,m}\, y^m,$$
$$c_{d,m} := \frac{(-1)^m}{4^m m!(\alpha+1)_m}\,\rho_{d,m},$$

where $\rho_{d,m} := d^{-2m}\prod_{j=0}^{m-1}(d-j)(d+2\alpha+1+j)$. Under the substitution $x = 1-\frac{y}{2d^2}$, we have $\partial_x = -2d^2\partial_y$, so $\frac{t}{2d^4}\partial_x^2 = 2t\partial_y^2$. Applying $e^{+2t\partial_y^2}$ term-by-term to the degree-d polynomial yields

$$Q_d(y;t) \;=\; \sum_{m=0}^{d} c_{d,m}\, H_m(y;t), \qquad H_m(y;t) := e^{+2t\partial_y^2}(y^m) = \sum_{k=0}^{\lfloor m/2\rfloor}\binom{m}{2k}\frac{(2k)!}{k!}(2t)^k y^{m-2k}.$$

Fix compact radii $|y|\le R$ and $|t|\le T$. Set $A_\alpha := \max(0,\alpha+\frac12)$. By AM–GM, for every $0\le j\le m-1\le d-1$,

$$0 \;<\; \frac{(d-j)(d+2\alpha+1+j)}{d^2} \;\le\; \left(\frac{2d+2\alpha+1}{2d}\right)^2 \;\le\; \left(1+\frac{A_\alpha}{d}\right)^2,$$

so $0<\rho_{d,m}\le(1+A_\alpha/d)^{2m}\le(1+A_\alpha/d)^{2d}\le e^{2A_\alpha}$ uniformly for all $0\le m\le d$. For $2k\le m$, we have $\frac{(2k)!}{k!}=\prod_{r=1}^{k}(k+r)\le m^k$, so

$$|H_m(y;t)| \;\le\; \sum_{k=0}^{\lfloor m/2\rfloor}\binom{m}{2k}(\sqrt{2Tm})^{2k}R^{m-2k} \;\le\; \big(R+\sqrt{2Tm}\big)^m \;\le\; 2^m\Big(R^m+(2T)^{m/2}m^{m/2}\Big).$$

Since $(m!)^2=\prod_{r=1}^{m} r(m+1-r)\ge m^m$, we have $m^{m/2}\le m!$ for all $m\ge1$, giving the d-independent summable majorant

$$|c_{d,m}H_m(y;t)| \;\le\; \frac{e^{2A_\alpha}}{(\alpha+1)_m}\left(\frac{(R/2)^m}{m!}+\left(\frac{T}{2}\right)^{m/2}\right) =:\; M_m, \qquad \sum_{m=0}^{\infty}M_m<\infty. \tag{29}$$

Since $\lim_{d\to\infty}\rho_{d,m}=1$ for every fixed $m\ge0$, the Weierstrass M-test implies $Q_d(y;t)\to F_\alpha(y;t)$ uniformly on $|y|\le R$, $|t|\le T$. At $t=0$, $H_m(y;0)=y^m$ and $\sum_{m=0}^{\infty}\frac{(-y/4)^m}{m!(\alpha+1)_m}=\Gamma(\alpha+1)(y/4)^{-\alpha/2}J_\alpha(\sqrt{y})$.

Part (c). Suppose $F_\alpha(\cdot;t_1)$ has a non-real zero $y_*\in\mathbb{C}\setminus\mathbb{R}$. Pick $r>0$ so that $\overline{D(y_*,r)}\subset\mathbb{C}\setminus\mathbb{R}$ and $F_\alpha(\cdot;t_1)$ has no zeros on $\partial D(y_*,r)$. By uniform convergence $Q_d(\cdot;t_1)\to F_\alpha(\cdot;t_1)$ and Hurwitz's theorem, for all sufficiently large d, $Q_d(\cdot;t_1)$ has a non-real zero in $D(y_*,r)$, so $e^{+\frac{t_1}{2d^4}\partial_x^2}P_d^{(\alpha,\alpha)}\notin\mathcal{R}_d$ and hence $d^4\tau_*(P_d^{(\alpha,\alpha)})<t_1$. Taking the infimum over such t_1 proves (27).

To prove that $L_\alpha(y_0;t_0)<0$ implies $t^*_{\mathrm{edge}}(\alpha)\le t_0$ (and hence $\limsup_{d\to\infty}d^4\tau_*(P_d^{(\alpha,\alpha)})\le t_0$), note first from (29) that for any fixed $t_0\ge0$ and $|y|\le R$,

$$|F_\alpha(y;t_0)| \;\le\; e^{2A_\alpha}\sum_{m=0}^{\infty}\frac{(R/2)^m}{m!\,(\alpha+1)_m}+e^{2A_\alpha}\sum_{m=0}^{\infty}\frac{(t_0/2)^{m/2}}{(\alpha+1)_m}.$$

Because $(\alpha+1)_m\ge m!$ for $\alpha\ge0$, $(\alpha+1)_m\ge\frac{\alpha+1}{m+1}m!$ for $-1<\alpha<0$, and $(2m)!\le4^m(m!)^2$, we have $\sum_{m=0}^{\infty}\frac{(R/2)^m}{m!(\alpha+1)_m}\le C_\alpha(1+R)\cosh(\sqrt{2R})$, showing that $F_\alpha(\cdot;t_0)$ is an entire function of growth order at most $\frac12<1$ (hence genus 0). Moreover, $F_\alpha(0;t_0)=\sum_{k=0}^{\infty}\frac{(2t_0)^k}{16^k k!(\alpha+1)_{2k}}\ge1>0$. By Hadamard's factorization theorem, if all zeros $\{z_k\}_{k\ge1}$ of $F_\alpha(\cdot;t_0)$ were real, then $F_\alpha(y;t_0)=F_\alpha(0;t_0)\prod_{k\ge1}(1-y/z_k)$ with $\sum_{k\ge1}|z_k|^{-1}<\infty$, whence term-by-term logarithmic

differentiation would yield $L_\alpha(y;t_0) = F_\alpha(y;t_0)^2\sum_{k\ge1}(y-z_k)^{-2} \ge 0$ for all $y \in \mathbb{R}$ (with $L_\alpha(z_j;t_0) = F'_\alpha(z_j;t_0)^2 \ge 0$ at any real zero z_j). Contrapositively, $L_\alpha(y_0;t_0) < 0$ implies that $F_\alpha(\cdot;t_0)$ has at least one non-real zero, so $t^*_{\mathrm{edge}}(\alpha) \le t_0$. (Independently, $Q'_d(\cdot;t_0)^2 - Q_d(\cdot;t_0)Q''_d(\cdot;t_0) \to L_\alpha(\cdot;t_0) < 0$ at y_0 also yields $Q_d(\cdot;t_0) \notin \mathcal{R}_d$ for all large d and hence $\limsup_{d\to\infty} d^4\tau_*(P_d^{(\alpha,\alpha)}) \le t_0$ directly.)

For the lower bound (28) when $\alpha \in [-\frac12,\frac12]$, let $0<\theta_1<\theta_2<\cdots<\theta_d<\pi$ be the zeros of $P_d^{(\alpha,\alpha)}(\cos\theta)$ (so $x_{d,k}=\cos\theta_k$ and $\theta_{d+1-k}=\pi-\theta_k$ by symmetry). For $d\in\{2,3\}$ all adjacent chord gaps coincide by symmetry, so assume $d\ge4$. Put $u(\theta) := (\sin\theta)^{\alpha+1/2}P_d^{(\alpha,\alpha)}(\cos\theta)$, which satisfies the Sturm–Liouville equation [17, Theorem 4.24.2]

$$u''(\theta)+q(\theta)\,u(\theta) \;=\; 0, \qquad q(\theta) := \left(d+\alpha+\frac12\right)^2 + \frac{\frac14-\alpha^2}{\sin^2\theta}.$$

Because $\frac14-\alpha^2\ge0$ on $[-\frac12,\frac12]$, $q(\theta)$ is symmetric about $\frac\pi2$ and nondecreasing as a function of $|\theta-\frac\pi2|$ (nonincreasing on $(0,\frac\pi2]$). The zeros in $(0,\frac\pi2]$ are $0<\theta_1<\theta_2<\cdots<\theta_{\lceil d/2\rceil}\le\frac\pi2$ (with $\theta_{\lceil d/2\rceil}=\frac\pi2$ when d is odd). By the Sturm comparison theorem [17, §6.3] on $(0,\frac\pi2]$, the angular gaps satisfy $\theta_2-\theta_1\le\theta_3-\theta_2\le\cdots\le\theta_{\lceil d/2\rceil}-\theta_{\lceil d/2\rceil-1}$. Since $\sin\theta$ is strictly increasing on $(0,\frac\pi2]$, the chord gaps

$$x_{d,k}-x_{d,k+1} \;=\; \cos\theta_k-\cos\theta_{k+1} \;=\; 2\sin\left(\frac{\theta_k+\theta_{k+1}}{2}\right)\sin\left(\frac{\theta_{k+1}-\theta_k}{2}\right)$$

are strictly increasing for $1\le k\le\lceil d/2\rceil-1$, and symmetrically on $[\frac\pi2,\pi)$. When $d=2m\ge4$ is even, it remains to compare the central gap $[\theta_m,\theta_{m+1}]=[\theta_m,\pi-\theta_m]$ straddling $\frac\pi2$: let $h:=\theta_{m+1}-\theta_m$. If $h<\theta_m-\theta_{m-1}$, then $I_1:=[\theta_m-h,\theta_m]\subset(\theta_{m-1},\theta_m]$, and for every $s\in(0,h)$ we have $|\frac\pi2-(\theta_m-h+s)| = \frac{3h}{2}-s > \frac h2 > |\frac h2-s| = |\frac\pi2-(\theta_m+s)|$, whence $q(\theta_m-h+s)\ge q(\theta_m+s)$ on $[0,h]$. Since $u(\theta_m+s)$ vanishes at $s=0$ and $s=h$, Sturm comparison forces $u(\theta_m-h+s)$ to vanish on $[\theta_m-h,\theta_m)\subset(\theta_{m-1},\theta_m)$, contradicting that θ_{m-1} and θ_m are consecutive zeros. Thus $\theta_{m+1}-\theta_m\ge\theta_m-\theta_{m-1}\ge\theta_2-\theta_1$, and since $\sin(\frac{\theta_m+\theta_{m+1}}{2})=\sin(\frac\pi2)=1>\sin(\frac{\theta_1+\theta_2}{2})$, we conclude $x_{d,m}-x_{d,m+1}>x_{d,1}-x_{d,2}$. Hence $\Delta_{\min}(P_d^{(\alpha,\alpha)})=x_{d,1}-x_{d,2}$ for every $d\ge2$ and every $\alpha\in[-\frac12,\frac12]$. By Part (b) at $t=0$ and Hurwitz's theorem, $y_{d,k}=2d^2(1-x_{d,k})\to j_{\alpha,k}^2$ for each fixed $k\ge1$, so $d^2\Delta_{\min}(P_d^{(\alpha,\alpha)})=\frac12(y_{d,2}-y_{d,1})\to\frac12(j_{\alpha,2}^2-j_{\alpha,1}^2)$. Applying Theorem 2.2(a) gives (28).

Part (d). Using $|H_m^{(r)}(y_0;t_0)| = \frac{m!}{(m-r)!}|H_{m-r}(y_0;t_0)| \le m^r(y_0+\sqrt{2t_0m})^m$ for $r\in\{0,1,2\}$ and $y_0\ge1$, truncating (26) at $N=75$ yields rigorous remainder bounds $<10^{-100}$ for $F_\alpha,F'_\alpha,F''_\alpha$ at the three test points $(y_0,t_0)=(18.5,26.0)$ ($\alpha=-1/2$), $(26.67,42.0)$ ($\alpha=0$), and $(36.67,61.0)$ ($\alpha=1/2$). Evaluating the partial sum in 40-digit interval arithmetic (`mpmath.iv`) in `scripts/verify_polynomial_heat_collapse.py` certifies the strict negativity of $L_\alpha(y_0;t_0)$ reported in the theorem statement. Substituting the known Bessel zeros $j_{-1/2,1}=\frac\pi2$, $j_{-1/2,2}=\frac{3\pi}{2}$ (giving $\frac{\pi^4}{4}\approx24.35227$), $j_{0,1}\approx2.404826$, $j_{0,2}\approx5.520078$ (giving ≈38.09382), and $j_{1/2,1}=\pi$, $j_{1/2,2}=2\pi$ (giving $\frac{9\pi^4}{16}\approx54.79261$) into (28) completes the proof. □

6 Computational Verification and Quantitative Benchmarks

All theorems, exact identities, phase-transition curves, and interval-arithmetic certificates in this paper are independently verified by the open-source Python suite in https://github.com/lluiseriksson/polynomial-heat-collapse (tag `v1.0.1`, scripts `scripts/verify_polynomial_heat_collapse.py`, `scripts/make_figures.py`, and automated test suite `pytest` on `tests/test_polynomial_heat_collapse.py`, passing 3/3 test modules).

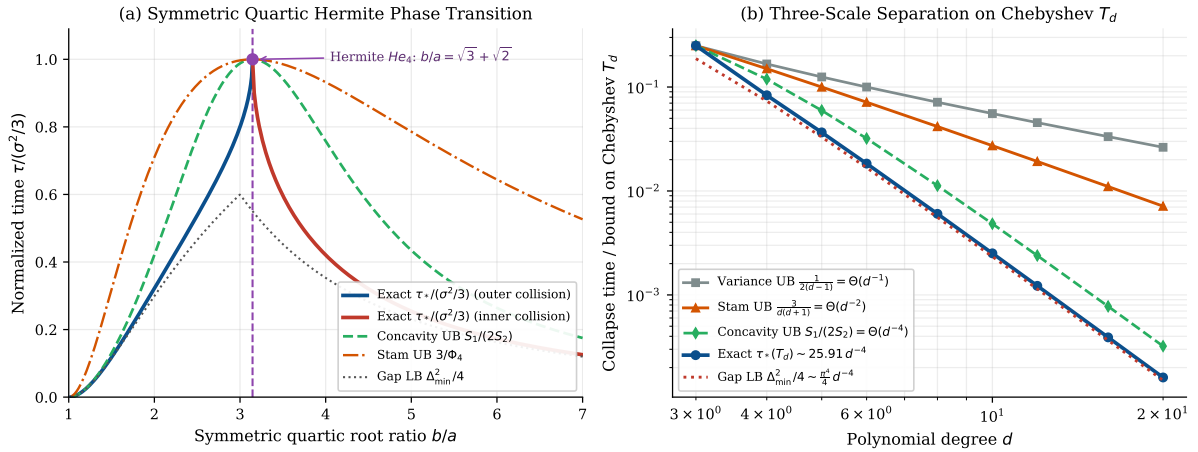


Figure 1: **(a)** Normalized collapse time $\tau_*/(\sigma^2/3)$ for the symmetric quartic $(x^2-1)(x^2-b^2)$ across the exact Hermite phase transition $b/a = \sqrt{3}+\sqrt{2}$ (Theorem 4.1(b)), comparing the outer-collision branch (blue), inner-collision branch (red), concavity tangent bound $S_1/(2S_2)$ (green dashed), Stam bound $3/\Phi_4$ (orange dash-dot), and gap lower bound $\Delta^2_{\min}/4$ (gray dotted). **(b)** Log-log scaling of $\tau_*(T_d)$ versus degree $d \in [3, 20]$ on Chebyshev T_d, illustrating the exact three-scale separation $\Theta(d^{-4}) \ll \Theta(d^{-2}) \ll \Theta(d^{-1})$ (Theorem 5.1) and the $\Theta(d^{-4})$ sharpness of $S_1/(2S_2)$.

6.1 Universal Operator Identities, Sandwich Ratios, and Concavity Tests

Across random real-rooted ensembles of degrees $d \in \{3, 4, 5, 6, 8\}$ (30 samples per degree, minimum initial separation 0.08, fixed RNG seed 2026), every algebraic identity of Theorem 3.1 holds to machine precision:

- Commutator moment collapse $\mathbf{1}_d^\top B^k \mathbf{1}_d = \mathrm{Tr}(B^k)$ $(k = 1, 2, 3, 4)$: max relative error 5.01×10^{-16}.
- Critical-point orthogonality $C^\top C = \mathrm{diag}(\kappa_m)$: max relative error 4.80×10^{-12}.
- Trace and Frobenius laws $\mathrm{Tr}(L^2) = 2\,\mathrm{Tr}(BL)$ and $\|L - B\|_F = \|B\|_F$: max relative errors 3.60×10^{-16} and 3.75×10^{-16}.
- Exact dissipation formulas $S_2 = \frac{3}{2}\mathrm{Tr}(L^2) - \mathrm{Tr}(B^2) = \frac{5}{2}Q - \frac{1}{2}\sum w_i^2$: max relative errors 1.27×10^{-15} and 8.92×10^{-16}.

Across the same ensemble, every inequality of Theorem 2.2 held with zero violations: minimum gap ratio $\tau_*/(\Delta^2_{\min}/4) = 1.002456$, maximum Stam ratio 0.881715, maximum span ratio 0.880367, maximum skewness ratio 1.000000 (attained on cubics), minimum finite-free superadditivity margin $\tau_*(p \boxplus_d q) - \tau_*(p) - \tau_*(q) = +0.012115$, maximum Costa factor $\dot{\Phi}_d + \frac{4}{d(d-1)}\Phi_d^2 = -9.21 \times 10^{-3}$, and maximum reciprocal-Fisher second derivative $(1/\Phi_d)'' = -9.17 \times 10^{-3}$ (providing numerical support for Conjecture 4.2).

6.2 Cubic, Symmetric Quartic, and Ultraspherical Hard-Edge Benchmarks

On centered cubics $x^3 + ax + b$, the closed-form formulas (20) matched numerical bisection and Φ_3 with maximum errors 1.22×10^{-15} and 2.66×10^{-15}. Table 1 reports the symmetric quartic phase-transition benchmarks across both collision channels and the exact 4-fold Hermite transition $b/a = \sqrt{3} + \sqrt{2} \approx 3.14626437$.

For ultraspherical Jacobi polynomials across $\alpha \in \{-\frac{1}{2}, 0, \frac{1}{2}, 1, 2\}$ and $d \in \{3, 4, 6, 8, 12\}$, the closed forms (24) matched direct root computation to relative errors 5.41×10^{-16} (Φ_d) and

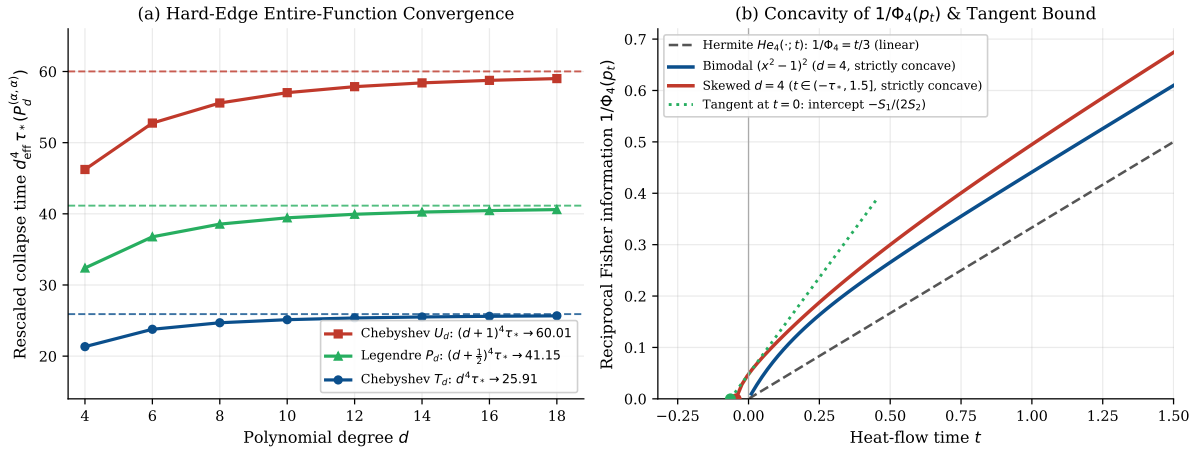


Figure 2: **(a)** Convergence of the rescaled collapse times $d_{\text{eff}}^4\tau_*(P_d^{(\alpha,\alpha)})$ ($d_{\text{eff}} = d+\alpha+\frac{1}{2}$) for T_d, P_d, and U_d ($d \in [4,18]$) toward the 35-digit hard-edge entire-function critical constants $t^*_{\text{edge}}(-1/2) \approx 25.91$, $t^*_{\text{edge}}(0) \approx 41.15$, and $t^*_{\text{edge}}(1/2) \approx 60.01$ (dashed horizontals). **(b)** Global concavity of $t \mapsto 1/\Phi_4(p_t)$ for bimodal $(x^2-1)^2$ and skewed $d=4$ roots $(-1.2, -0.8, -0.2, 2.2)$ relative to the linear Hermite trajectory $1/\Phi_4(He_4(\cdot;t)) = t/3$, illustrating the tangent intercept bound $-\tau_* \geq -S_1/(2S_2)$.

3.75×10^{-16} ($\sigma^2/(d-1)$). Table 2 displays the finite-d hierarchy and hard-edge convergence $d_{\text{eff}}^4\tau_*(P_d^{(\alpha,\alpha)}) \to t^*_{\text{edge}}(\alpha)$ ($d_{\text{eff}} = d+\alpha+\frac{1}{2}$), approaching the 35-digit entire-function thresholds $t^*_{\text{edge}}(-1/2) = 25.906417$, $t^*_{\text{edge}}(0) = 41.150192$, and $t^*_{\text{edge}}(1/2) = 60.013098$.

7 Concluding Remarks and Open Problems

We have established a unified quantitative theory for backward heat-flow collapse times of real-rooted polynomials, bridging finite free information inequalities, Dyson–Stieltjes operator algebra, low-degree collision-channel phase transitions, and hard-edge entire-function limits. Three natural open problems emerge directly from this work:

1. **Finite Free Toscani Concavity (Conjecture 4.2).** Prove (or find a high-degree counterexample to) $S_1(\|Ls\|_2^2 + S_3) \geq 2S_2^2$ on $\mathcal{R}_d^+$ for $d \geq 4$. Note that by Theorem 3.1(d), $S_1\|Ls\|_2^2 \geq S_2^2$ always holds, and Theorem 4.1 proves the stronger inequality $S_1S_3 \geq S_2^2$ in $d=3$ and symmetric $d=4$; whether $S_1S_3 \geq S_2^2$ itself persists for general asymmetric $d \geq 4$ or only the combined inequality $S_1(\|Ls\|_2^2 + S_3) \geq 2S_2^2$ holds is an intriguing algebraic question.

2. **Exact Hard-Edge Limit Equality.** Theorem 5.1(c)–(d) establishes the two-sided hard-edge bounds $(j_{\alpha,2}^2 - j_{\alpha,1}^2)^2/16 \leq \liminf_{d\to\infty} d^4\tau_*(P_d^{(\alpha,\alpha)}) \leq \limsup_{d\to\infty} d^4\tau_*(P_d^{(\alpha,\alpha)}) \leq t^*_{\text{edge}}(\alpha)$. Proving that bulk roots never collide prior to the extremal edge pair would upgrade this to the exact limit $\lim_{d\to\infty} d^4\tau_*(P_d^{(\alpha,\alpha)}) = t^*_{\text{edge}}(\alpha)$ for all $\alpha > -1$.

3. **Soft-Edge and Laguerre Ensembles.** Extending the hard-edge entire-function analysis to Laguerre polynomials $L_d^{(\alpha)}$ (which exhibit a hard edge at 0 and an Airy soft edge at $4d$) and to unitary/circular real-trigonometric polynomials under the heat flow on S^1 should reveal new universal Airy and Bessel collapse constants.

Table 1: Exact symmetric quartic phase transition benchmarks for $p(x) = (x^2-1)(x^2-b^2)$ (Theorem 4.1(b)), comparing closed-form τ_{exact} against 70-step bisection τ_{num} and the three upper bounds $S_1/(2S_2) \le 3/\Phi_4 \le \sigma^2/3$. At the 4-fold Hermite degeneracy $b = \sqrt{3}+\sqrt{2}$, both $C(\tau)$ and $\Delta_q(\tau)$ have double roots in τ (giving $\varepsilon^{1/2}$ threshold sensitivity combined with $\varepsilon^{1/4}$ quartic-root conditioning under IEEE-754 arithmetic, yielding $|\tau_{\text{exact}} - \tau_{\text{num}}| = 3.76 \times 10^{-9}$ at tolerance 10^{-10}); away from the degeneracy, $|\tau_{\text{exact}} - \tau_{\text{num}}| \le 8.88 \times 10^{-16}$.

b	Channel	τ_{exact}	$\lvert\tau_{\text{exact}} - \tau_{\text{num}}\rvert$	$S_1/(2S_2)$	Stam $3/\Phi_4$	Var $\sigma^2/3$
1.2000	outer	0.0100420190	2.48×10^{-16}	0.010252	0.029506	0.406667
1.5000	outer	0.0638606275	2.78×10^{-17}	0.070613	0.172789	0.541667
2.0000	outer	0.2681391681	1.67×10^{-16}	0.350802	0.591781	0.833333
$1+\sqrt{2}$	outer	0.5690355937	3.33×10^{-16}	0.812908	1.024264	1.138071
2.8000	outer	1.0028961844	8.88×10^{-16}	1.372643	1.445538	1.473333
$\sqrt{3}+\sqrt{2}$	Hermite (4-fold)	1.8164965809	3.76×10^{-9}	1.816497	1.816497	1.816497
3.5000	inner	1.3176017364	0.00×10^{0}	2.081261	2.176520	2.208333
5.0000	inner	1.1015467617	2.22×10^{-16}	1.883248	3.404792	4.333333
10.0000	inner	1.0210666464	4.44×10^{-16}	1.218885	5.106900	16.833333

Table 2: Three-scale collapse hierarchy and hard-edge convergence for Chebyshev T_d ($\alpha = -1/2$), Legendre P_d ($\alpha = 0$), and Chebyshev U_d ($\alpha = +1/2$) (Theorem 5.1). The effective-degree rescaling $d_{\text{eff}} = d + \alpha + \frac{1}{2}$ accelerates convergence to $t^*_{\text{edge}}(-1/2) = 25.906417$, $t^*_{\text{edge}}(0) = 41.150192$, and $t^*_{\text{edge}}(1/2) = 60.013098$.

Family	d	$\Delta^2_{\min}/4$	τ_*	$S_1/(2S_2)$	Stam UB	Var UB	$d^4_{\text{eff}}\tau_*$
T_d ($\alpha = -\frac{1}{2}$)	4	0.073223	0.083333	0.119048	0.150000	0.166667	21.3333
T_d ($\alpha = -\frac{1}{2}$)	8	0.005574	0.006028	0.011254	0.041667	0.071429	24.6910
T_d ($\alpha = -\frac{1}{2}$)	12	0.001141	0.001223	0.002392	0.019231	0.045455	25.3614
T_d ($\alpha = -\frac{1}{2}$)	18	0.000229	0.000244	0.000488	0.008772	0.029412	25.6633
P_d ($\alpha = 0$)	4	0.067901	0.078969	0.113208	0.133333	0.142857	32.3824
P_d ($\alpha = 0$)	8	0.006693	0.007386	0.015094	0.044444	0.066667	38.5529
P_d ($\alpha = 0$)	12	0.001499	0.001636	0.003597	0.021978	0.043478	39.9380
P_d ($\alpha = 0$)	18	0.000319	0.000347	0.000787	0.010526	0.028571	40.5945
U_d ($\alpha = +\frac{1}{2}$)	4	0.062500	0.073969	0.105422	0.119048	0.125000	46.2306
U_d ($\alpha = +\frac{1}{2}$)	8	0.007538	0.008471	0.018421	0.045455	0.062500	55.5759
U_d ($\alpha = +\frac{1}{2}$)	12	0.001827	0.002026	0.004875	0.023810	0.041667	57.8704
U_d ($\alpha = +\frac{1}{2}$)	18	0.000411	0.000453	0.001142	0.011905	0.027778	59.0066

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