Exact Border-Rank and Rank Separations for Spin-Factor Jordan Algebras, Matrix Jordan Algebras, and Classical Lie Algebras via Koszul Flattenings and Diagonal-Slice Strassen Commutators

Lluis Eriksson*

November 27, 2026

Abstract

We determine exact multiplicative border-rank and tensor-rank separations across three canonical families of non-associative algebras: spin-factor Jordan algebras J_D, matrix Jordan algebras Sym_n, and classical Lie algebras $\mathfrak{so}_n$ and $\mathfrak{sl}_3$.

First (Theorem 1), we completely resolve the spin-factor Jordan algebra J_D for *every* dimension $D \geq 3$. We prove that its exact border rank satisfies $\underline{R}_{\mathbb{C}}(J_D) = \underline{R}_{\mathbb{Q}}(J_D) = D + 1$, matching the exact Koszul $p = 1$ flattening rank $D^2 - 1$ via an explicit *purely rational* $(D + 1)$-cube symmetric degeneration over $\mathbb{Q}(\varepsilon)$ requiring zero irrational roots. In sharp contrast, we prove that its exact tensor rank over $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ is $R_{\mathbb{Q}}(J_D) = R_{\mathbb{R}}(J_D) = R_{\mathbb{C}}(J_D) = 2D - 2$, combining a five-line substitution lower bound with an explicit symmetric $(2D - 2)$-product rational algorithm built by recursive Pythagorean splitting $(\frac{3}{5}, \frac{4}{5})$. This yields an unbounded exact separation $R_{\mathbb{Q}}(J_D) - \underline{R}_{\mathbb{C}}(J_D) = D - 3 \to \infty$ and exact ratio $R_{\mathbb{Q}}(J_D)/\underline{R}_{\mathbb{C}}(J_D) = 2 - \frac{4}{D+1} \to 2$.

Second (Theorem 2), for any d-dimensional commutative algebra over a field $\mathbb{K}$ ($\mathrm{char}\,\mathbb{K} \neq 2$, in particular $\mathbb{R}$ and $\mathbb{C}$) equipped with a non-degenerate symmetric invariant bilinear form, we prove the universal kernel identity $[L_A^{-1}L_B, L_A^{-1}L_C](A) = 0$ and skew-symmetry of $L_A[L_A^{-1}L_B, L_A^{-1}L_C] = L_B L_A^{-1} L_C - L_C L_A^{-1} L_B$ for *every* invertible slice L_A. This unconditionally caps Strassen's commutator rank at $d - 1$ (d odd) and $d - 2$ (d even), and caps Strassen's border-rank lower bound at $\frac{3}{2}d - \frac{1}{2}$ (d odd) and $\frac{3}{2}d - 1$ (d even). On Sym_n ($d = \binom{n+1}{2}$), the identity slice $A = I_n$ reduces to $\frac{1}{4}\,\mathrm{ad}_{[B,C]}$ of rank at most $d - \lceil n/2 \rceil$, whereas breaking symmetry with $A = \mathrm{diag}(1, \ldots, n)$ attains the universal Strassen ceiling for all $n \in \{2, \ldots, 7\}$ ($\underline{R}_{\mathbb{C}}(\mathrm{Sym}_n) \geq 4, 8, 14, 22, 31, 41$). At Sym_3 ($d = 6$, Strassen ceiling 8), an exact $\mathbb{Q}$-rank Koszul $p = 2$ flattening certificate (rank = 84, divisor $\binom{5}{2} = 10$) breaks Strassen's wall to prove $\underline{R}_{\mathbb{C}}(\mathrm{Sym}_3) \geq 9$, paired with an explicit 14-cube rational Waring identity for $\mathrm{tr}(X^3)$.

Third (Theorem 3), while every Lie algebra slice ad_X is singular ($\mathrm{ad}_X(X) = 0$) and thus universally invisible to Strassen's equations, exact multi-degree Koszul flattenings establish the sharp identity $\underline{R}_{\mathbb{C}}(\mathfrak{so}_3) = R_{\mathbb{Q}}(\mathfrak{so}_3) = 5$ (via a $\{0, \pm 1\}$ five-product cross-product algorithm), $\underline{R}_{\mathbb{C}}(\mathfrak{so}_4) \geq 8$ with $R_{\mathbb{Q}}(\mathfrak{so}_4) \leq 10$ via $\mathfrak{so}_4 \cong \mathfrak{so}_3 \oplus \mathfrak{so}_3$, $\underline{R}_{\mathbb{C}}(\mathfrak{so}_5) \geq 13$ (449/36 at $p = 2$, improving $p = 1$'s 12), and a three-tier escalation on $\mathfrak{sl}_3$: 10 ($p = 1$) $\to$ 11 ($p = 2$) $\to$ 13 (440/35 at $p = 3$), paired with a 25-product rational decomposition into three $\mathfrak{sl}_2$ triples and two root cross-products. All core identities are formally verified in Lean 4.36.0 (20 constructive theorems, 0 `sorry`, 0 `Classical.choice`) alongside 16 zero-float SHA-256 certificates and 56 adversarial pytest checks.

*Independent Researcher. Lean 4.36.0 proofs, verification scripts, and certificates are archived at `github.com/lluiseriksson/jordan-spin-factor-and-lie-border-rank`.

1 Introduction

Determining the multiplicative tensor rank $R_{\mathbb{K}}(\mathcal{A})$ and border rank $\underline{R}_{\mathbb{K}}(\mathcal{A})$ of structure tensors of finite-dimensional algebras $\mathcal{A}$ is a central problem of algebraic complexity theory [1, 3]. Whereas associative matrix multiplication M_n has been studied intensively since Strassen [6, 7] and Landsberg–Ottaviani [4], the exact bilinear complexity of *non-associative* classical algebras—Jordan algebras and Lie algebras—exhibits remarkably rich phenomena: exact separations between rank and border rank, sharp algebraic ceilings on Strassen's commutator equations, and complete blindness of Strassen's method on antisymmetric brackets.

Recall that for a d-dimensional algebra $(\mathcal{A}, \circ)$ over a field $\mathbb{K}$ with structure tensor $T_{\mathcal{A}} \in \mathcal{A}^* \otimes \mathcal{A}^* \otimes \mathcal{A}$, the *tensor rank* $R_{\mathbb{K}}(\mathcal{A})$ is the minimum r such that $T_{\mathcal{A}} = \sum_{s=1}^{r} u_s \otimes v_s \otimes w_s$, and the *border rank* $\underline{R}_{\mathbb{K}}(\mathcal{A})$ is the minimum r such that $T_{\mathcal{A}}$ lies in the Zariski closure of the set of tensors of rank $\leq r$ (equivalently, $T_{\mathcal{A}} = \lim_{\varepsilon \to 0} T_\varepsilon$ with $R_{\mathbb{K}(\varepsilon)}(T_\varepsilon) \leq r$). Two fundamental polynomial-equation methods give lower bounds on $\underline{R}_{\mathbb{C}}(\mathcal{A})$:

1. **Strassen's commutator equations [7]:** If $L_A : \mathcal{A} \to \mathcal{A}$ ($L_A(Y) = A \circ Y$) is invertible, then for any $B, C \in \mathcal{A}$,

$$\underline{R}_{\mathbb{C}}(\mathcal{A}) \geq d + \left\lceil \frac{1}{2} \operatorname{rank}([L_A^{-1} L_B,\, L_A^{-1} L_C]) \right\rceil. \tag{1}$$

2. **Landsberg–Ottaviani Koszul flattenings [4]:** For each exterior degree $1 \leq p \leq d-2$, the linear map $K_p(T_{\mathcal{A}}) : \Lambda^p(\mathbb{C}^d) \otimes \mathbb{C}^d \to \Lambda^{p+1}(\mathbb{C}^d) \otimes \mathbb{C}^d$ defined on basis elements by

$$K_p(T_{\mathcal{A}})\big((e_{i_1} \wedge \cdots \wedge e_{i_p}) \otimes e_j\big) := \sum_{a=1}^{d} (e_a \wedge e_{i_1} \wedge \cdots \wedge e_{i_p}) \otimes (e_a \circ e_j) \tag{2}$$

   satisfies $\operatorname{rank}(K_p(u \otimes v \otimes w)) \leq \binom{d-1}{p}$ on every rank-1 tensor, yielding the unconditional lower bound

$$\underline{R}_{\mathbb{C}}(\mathcal{A}) \geq \left\lceil \frac{\operatorname{rank}_{\mathbb{C}}(K_p(T_{\mathcal{A}}))}{\binom{d-1}{p}} \right\rceil. \tag{3}$$

   Crucially, whenever $T_{\mathcal{A}}$ has integer structure constants, $\operatorname{rank}_{\mathbb{C}}(K_p(T_{\mathcal{A}})) = \operatorname{rank}_{\mathbb{Q}}(K_p(T_{\mathcal{A}})) \geq \operatorname{rank}_{\mathbb{F}_q}(K_p(T_{\mathcal{A}}))$ for every prime q, enabling completely rigorous zero-float rank certification.

In this work we establish three self-contained theorems resolving the exact rank and border-rank architecture of spin-factor Jordan algebras J_D (Theorem 1), matrix Jordan algebras Sym_n (Theorem 2), and classical Lie algebras $\mathfrak{so}_n, \mathfrak{sl}_3$ (Theorem 3).

2 Complete Resolution of Spin-Factor Jordan Algebras J_D

For $D \geq 3$, the *spin-factor Jordan algebra* $J_D = \mathbb{K}e_0 \oplus \mathbb{K}^{D-1}$ is the canonical formally real simple Jordan algebra of degree 2 and dimension D, with commutative bilinear product

$$(x_0, \mathbf{x}) \circ (y_0, \mathbf{y}) = (x_0 y_0 + \mathbf{x} \cdot \mathbf{y},\; x_0 \mathbf{y} + y_0 \mathbf{x}), \tag{4}$$

where $\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^{D-1} x_i y_i$. Its structure tensor T_{J_D} is totally symmetric: it is the polarization of the cubic polynomial

$$F_D(x_0, x_1, \ldots, x_{D-1}) := x_0^3 + 3x_0 \sum_{i=1}^{D-1} x_i^2. \tag{5}$$

Note that $J_3 \cong \mathrm{Sym}_2$ via $(x_0, x_1, x_2) \mapsto \left(\begin{smallmatrix} x_0+x_1 & x_2 \\ x_2 & x_0-x_1 \end{smallmatrix}\right)$.

Theorem 1 (Exact Border Rank and Exact Tensor Rank of Spin Factors J_D)**.** *Let $D \geq 3$.*

(a) ***Exact border rank*** $\underline{R} = D + 1$***:*** *Over* $\mathbb{Q}$*,* $\mathbb{R}$*, and* $\mathbb{C}$*, the spin-factor Jordan algebra* J_D *has exact border rank and symmetric border rank*

$$\underline{R}_{\mathbb{C}}(J_D) = \underline{R}_{\mathbb{R}}(J_D) = \underline{R}^{\mathrm{sym}}_{\mathbb{Q}}(J_D) = D + 1. \tag{6}$$

Specifically, $\operatorname{rank}_{\mathbb{C}}(K_1(T_{J_D})) = D^2 - 1 = (D+1)(D-1)$*, giving* $\underline{R}_{\mathbb{C}}(J_D) \geq D + 1$*, and writing* $S := \sum_{i=1}^{D-1} x_i$*, the purely rational* $(D+1)$*-cube family over* $\mathbb{Q}(\varepsilon)$

$$F_{D,\varepsilon}(x) := \frac{1}{\varepsilon^2}\left[\sum_{i=1}^{D-1}(x_0 + \varepsilon x_i)^3 - \frac{(D-\varepsilon^2)^2}{D+1-\varepsilon^2}\left(x_0 + \frac{\varepsilon}{D-\varepsilon^2}S\right)^3 + \frac{1}{D+1-\varepsilon^2}(x_0 - \varepsilon S)^3\right] \tag{7}$$

satisfies $\lim_{\varepsilon\to 0} F_{D,\varepsilon}(x) = F_D(x)$ *identically, with* $O(\varepsilon)$ *remainder.*

(b) ***Exact tensor rank*** $R = 2D - 2$***:*** *Over every field* $\mathbb{K}$ *with* $\operatorname{char}\mathbb{K} \neq 2$*,* $R_{\mathbb{K}}(J_D) \geq 2D - 2$*. Moreover, over every field* $\mathbb{K}$ *admitting nonzero elements* $\alpha_1, \ldots, \alpha_{D-1} \in \mathbb{K}^\times$ *with* $\sum_{i=1}^{D-1} \alpha_i^2 = 1$*—in particular* $\mathbb{K} \in \{\mathbb{Q}, \mathbb{R}, \mathbb{C}\}$*—we have the exact equality*

$$R_{\mathbb{Q}}(J_D) = R_{\mathbb{R}}(J_D) = R_{\mathbb{C}}(J_D) = 2D - 2. \tag{8}$$

(c) ***Unbounded factor-2 separation:*** *For all* $D \geq 3$*,* $R_{\mathbb{Q}}(J_D) - \underline{R}_{\mathbb{C}}(J_D) = D - 3$ *(tight at* $D = 3$*, strictly positive for all* $D \geq 4$*) and*

$$\frac{R_{\mathbb{Q}}(J_D)}{\underline{R}_{\mathbb{C}}(J_D)} = \frac{2D-2}{D+1} = 2 - \frac{4}{D+1} \xrightarrow{D\to\infty} 2.$$

Proof. **(a) Lower bound.** Consider the Koszul $p = 1$ flattening $K_1(T_{J_D}) : \mathbb{C}^D \otimes \mathbb{C}^D \to \Lambda^2(\mathbb{C}^D) \otimes \mathbb{C}^D$. Its domain has dimension D^2. We show that $\dim\ker K_1(T_{J_D}) = 1$. Let $Z = \sum_{r,s=0}^{D-1} c_{rs}\, e_r \otimes e_s \in \ker K_1(T_{J_D})$. Expanding (2) for $p = 1$, the coefficient of $(e_a \wedge e_b) \otimes e_k$ $(0 \leq a < b \leq D-1)$ in $K_1(T_{J_D})(Z)$ is

$$\sum_{s=0}^{D-1}(c_{bs}\, T_{a,s,k} - c_{as}\, T_{b,s,k}) = 0. \tag{9}$$

Take $a = 0$, $b = i \geq 1$ in (9):

- For $k = j \notin \{0, i\}$: $T_{0,s,j} = \delta_{s,j}$ and $T_{i,s,j} = 0$, giving $c_{ij} = 0$ for all distinct $i, j \geq 1$.
- For $k = i$: $T_{0,s,i} = \delta_{s,i}$ and $T_{i,s,i} = \delta_{s,0}$, giving $c_{ii} - c_{00} = 0$, i.e., $c_{ii} = c_{00}$ for all $i \geq 1$.
- For $k = 0$: $T_{0,s,0} = \delta_{s,0}$ and $T_{i,s,0} = \delta_{s,i}$, giving $c_{i0} - c_{0i} = 0$, i.e., $c_{i0} = c_{0i}$ for all $i \geq 1$.

Now take $1 \leq a < b \leq D-1$ and $k = b$ in (9): $T_{a,s,b} = 0$ and $T_{b,s,b} = \delta_{s,0}$, giving $-c_{a0} = 0$ for all $a \geq 1$ (and symmetrically taking $k = a$ gives $c_{b0} = 0$ for all $b \geq 1$). Thus $c_{i0} = c_{0i} = 0$ for all $i \geq 1$, $c_{ij} = 0$ for all $i \neq j$, and $c_{00} = c_{11} = \cdots = c_{D-1,D-1}$. Conversely, substituting $c_{rs} = \delta_{rs}$ into (9) yields $T_{a,b,k} - T_{b,a,k} = 0$ by commutativity of J_D. Hence $\ker K_1(T_{J_D}) = \operatorname{span}\{\sum_{r=0}^{D-1} e_r \otimes e_r\}$ is 1-dimensional for every $D \geq 3$. By rank-nullity, $\operatorname{rank}_{\mathbb{C}}(K_1(T_{J_D})) = D^2 - 1$, and (3) yields $\underline{R}_{\mathbb{C}}(J_D) \geq \lceil (D^2-1)/(D-1) \rceil = D + 1$.

(a) Upper bound. Let $m := D - \varepsilon^2$, $\Delta := D + 1 - \varepsilon^2$, $b := 1/m$, $w_1 := -m^2/\Delta$, and $w_2 := 1/\Delta$. Expanding each cube in (7) in powers of ε gives:

- Coefficient of x_0^3 inside the bracket: $(D-1) + w_1 + w_2 = \frac{(D-1)\Delta - m^2 + 1}{\Delta} = \frac{\varepsilon^2\Delta}{\Delta} = \varepsilon^2$ (Lean `thm02`).
- Coefficient of $3\varepsilon x_0^2 S$: $1 + w_1 b - w_2 = \frac{\Delta - m - 1}{\Delta} = 0$ (Lean `thm03`).

- Coefficient of $3\varepsilon^2 x_0 S^2$: $w_1 b^2 + w_2 = \frac{-1+1}{\Delta} = 0$ (Lean `thm04`).
- Coefficient of $3\varepsilon^2 x_0 \sum_{i=1}^{D-1} x_i^2$: 1.

Dividing by ε^2 yields $F_{D,\varepsilon}(x) = x_0^3 + 3x_0 \sum_{i=1}^{D-1} x_i^2 + \varepsilon(\sum_{i=1}^{D-1} x_i^3 + (w_1 b^3 - w_2) S^3)$, which converges to $F_D(x)$ as $\varepsilon \to 0$ using exactly $D+1$ cubes over $\mathbb{Q}(\varepsilon)$. Polarizing $F_{D,\varepsilon}$ gives a $(D+1)$-term symmetric border-rank decomposition of T_{J_D}.

(b) Lower bound. Let $T_{J_D} = \sum_{s=1}^{r} \ell_s \otimes u_s \otimes v_s$ be any exact rank-r decomposition over $\mathbb{K}$. In matrix-pencil form,

$$M(x_0, \mathbf{x}) := L_{(x_0,\mathbf{x})} = \begin{pmatrix} x_0 & \mathbf{x}^\top \\ \mathbf{x} & x_0 I_{D-1} \end{pmatrix} = \sum_{s=1}^{r} \ell_s(x_0, \mathbf{x})\, u_s v_s^\top.$$

Restrict the linear forms $\ell_1, \dots, \ell_r$ to the $(D-1)$-dimensional hyperplane $H = \{x_0 = 0\}$. Their restrictions must span H^*: otherwise there exists $\mathbf{a} \in \mathbb{K}^{D-1} \setminus \{0\}$ with $\ell_s(0, \mathbf{a}) = 0$ for all s, forcing $M(0, \mathbf{a}) = e_0 \mathbf{a}^\top + \mathbf{a} e_0^\top = 0$, impossible. Relabeling so that $\ell_1|_H, \dots, \ell_{D-1}|_H$ form a basis of H^*, the $(D-1) \times (D-1)$ linear system $\ell_s(1, c_1, \dots, c_{D-1}) = 0$ $(1 \le s \le D-1)$ has a unique solution $\mathbf{c} \in \mathbb{K}^{D-1}$. Evaluating the pencil at $(1, \mathbf{c})$ eliminates the first $D-1$ rank-1 terms:

$$M(1, \mathbf{c}) = \begin{pmatrix} 1 & \mathbf{c}^\top \\ \mathbf{c} & I_{D-1} \end{pmatrix} = \sum_{s=D}^{r} \ell_s(1, \mathbf{c})\, u_s v_s^\top.$$

Because the trailing $(D-1) \times (D-1)$ principal submatrix of $M(1, \mathbf{c})$ is I_{D-1}, we have $\operatorname{rank}(M(1, \mathbf{c})) \ge D-1$ for every $\mathbf{c} \in \mathbb{K}^{D-1}$. By subadditivity of matrix rank, $r - (D-1) \ge D-1$, hence $r \ge 2D-2$.

(b) Upper bound over $\mathbb{Q}$. Starting from the Pythagorean base $(\frac{3}{5})^2 + (\frac{4}{5})^2 = 1$ for $D = 3$, recursively replacing the last coordinate α_k by $(\frac{3}{5}\alpha_k, \frac{4}{5}\alpha_k)$ (Lean `thm06`–`thm07`) yields explicit nonzero rationals $\alpha_1, \dots, \alpha_{D-1} \in \mathbb{Q}^\times$ with $\sum_{i=1}^{D-1} \alpha_i^2 = 1$ for every $D \ge 3$. Computing the $2(D-1)$ symmetric products

$$P_{i,\pm} := (\alpha_i x_0 \pm x_i)(\alpha_i y_0 \pm y_i) \qquad (i = 1, \dots, D-1)$$

recovers $z_0 = \frac{1}{2} \sum_{i=1}^{D-1} (P_{i,+} + P_{i,-}) = (\sum \alpha_i^2) x_0 y_0 + \sum x_i y_i = x_0 y_0 + \mathbf{x} \cdot \mathbf{y}$ and $z_i = \frac{1}{2\alpha_i}(P_{i,+} - P_{i,-}) = x_0 y_i + x_i y_0$ identically over $\mathbb{Q}$ (Lean `thm08`, `cert03`). Part **(c)** follows immediately (Lean `thm09`–`thm10`). □

3 Matrix Jordan Algebras Sym_n: Universal Strassen Ceiling and Koszul Breakthrough

Let Sym_n denote the d-dimensional Jordan algebra of $n \times n$ symmetric matrices ($d = \binom{n+1}{2}$) under $X \circ Y := \frac{1}{2}(XY + YX)$, equipped with the positive-definite trace inner product $\langle X, Y \rangle = \operatorname{tr}(XY)$ for which every multiplication operator $L_X(Y) = X \circ Y$ is self-adjoint: $\langle L_X(Y), Z \rangle = \operatorname{tr}((X \circ Y)Z) = \langle Y, L_X(Z) \rangle$.

Theorem 2 (Strassen Ceiling on Sym_n and Sym_3 Koszul Breakthrough)**.**

(a) ***Universal commutative-trace Strassen ceiling:*** *Let $(\mathcal{A}, \circ)$ be any d-dimensional ($d \ge 1$) commutative algebra over a field $\mathbb{K}$ with* $\operatorname{char} \mathbb{K} \ne 2$ *(in particular $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$) equipped with a non-degenerate symmetric invariant bilinear form $\langle \cdot, \cdot \rangle$ (so $\langle X \circ Y, Z \rangle = \langle Y, X \circ Z \rangle$ for all $X, Y, Z \in \mathcal{A}$). For* every *$A \in \mathcal{A}$ with L_A invertible (hence $A \ne 0$) and all $B, C \in \mathcal{A}$,*

$$[L_A^{-1} L_B,\, L_A^{-1} L_C](A) = 0, \tag{10}$$

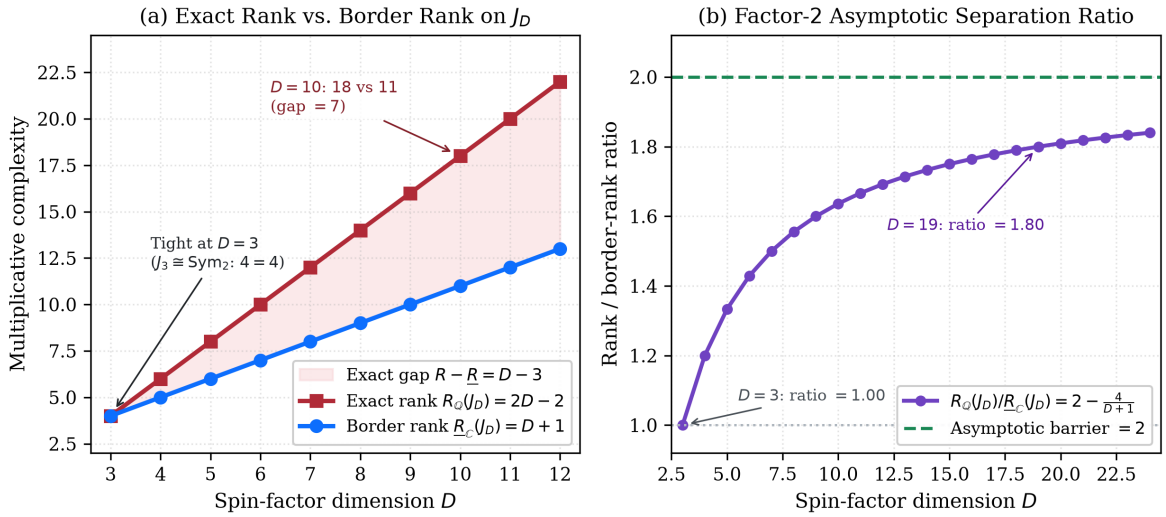

Figure 1: **(a)** Exact rank $R_{\mathbb{Q}}(J_D) = 2D - 2$ versus exact border rank $\underline{R}_{\mathbb{C}}(J_D) = D + 1$ across spin-factor dimensions $D = 3, \ldots, 12$, showing exact tightness at $D = 3$ ($J_3 \cong \mathrm{Sym}_2$) and linear gap $D - 3$ for $D \geq 4$. **(b)** Exact ratio $R_{\mathbb{Q}}(J_D)/\underline{R}_{\mathbb{C}}(J_D) = 2 - \frac{4}{D+1}$ converging monotonically to the factor-2 barrier.

and the operator $K := L_A[L_A^{-1}L_B, L_A^{-1}L_C] = L_B L_A^{-1} L_C - L_C L_A^{-1} L_B$ *is skew-adjoint with respect to* $\langle \cdot, \cdot \rangle$*. Consequently, the commutator rank* $\mathrm{rank}([L_A^{-1}L_B, L_A^{-1}L_C]) = \mathrm{rank}(K)$ *is unconditionally even and satisfies*

$$\mathrm{rank}([L_A^{-1}L_B,\, L_A^{-1}L_C]) \leq \begin{cases} d-1 & \text{if } d \text{ is odd,} \\ d-2 & \text{if } d \text{ is even,} \end{cases} \tag{11}$$

which unconditionally caps Strassen's bound (1) *over* $\mathbb{C}$ *at* $d + \lfloor (d-1)/2 \rfloor$ *(namely* $\frac{3}{2}d - \frac{1}{2}$ *for odd* d*, and* $\frac{3}{2}d - 1$ *for even* d*).*

(b) ***Identity-slice spectrum vs. broken-symmetry attainment on*** Sym_n***:*** *On* Sym_n *(*$n \geq 2$*,* $d = \binom{n+1}{2}$*):*

- *At the identity slice* $A = I_n$*,* $[L_B, L_C](X) = \frac{1}{4}[[B, C], X] = \frac{1}{4}\,\mathrm{ad}_{[B,C]}(X)$*, whose rank over all* $B, C \in \mathrm{Sym}_n(\mathbb{C})$ *is at most* $d - \lceil n/2 \rceil$ *(attained for* $n = 2, \ldots, 7$ *in* `cert07`*).*
- *Breaking the* I_n*-symmetry via the explicit rational diagonal slice* $A = \mathrm{diag}(1, 2, \ldots, n)$*, tridiagonal* $B_{ij} = \mathbf{1}_{|i-j|=1}$*, and* $C_{ij} = ij + (i-j)^2$ *for* $1 \leq i, j \leq n$ *(equivalently* $(i_0 + 1)(j_0 + 1) + (i_0 - j_0)^2$ *in 0-based indexing) attains the exact universal Strassen ceiling* (11) *for every* $n \in \{2, 3, 4, 5, 6, 7\}$ *(commutator ranks* $2, 4, 8, 14, 20, 26$*, strictly beating* $A = I_n$ *for all* $n \geq 5$*), certifying* $\underline{R}_{\mathbb{C}}(\mathrm{Sym}_n) \geq 4, 8, 14, 22, 31, 41$ *(*`cert08`*).*

(c) ***Breaking Strassen's ceiling at*** Sym_3 ***via Koszul*** $p = 2$***:*** *For* Sym_3 *(*$d = 6$*, even), Strassen's universal ceiling* (11) *is* $6 + 4/2 = 8$*. Exact rational Gaussian elimination over* $\mathbb{Q}$ *certifies* $\mathrm{rank}_{\mathbb{Q}}(K_2(T_{\mathrm{Sym}_3})) = 84$ *with rank-1 divisor* $\binom{5}{2} = 10$ *(*`cert09`*), proving*

$$9 \leq \underline{R}_{\mathbb{C}}(\mathrm{Sym}_3) \leq R_{\mathbb{Q}}(\mathrm{Sym}_3) \leq 14, \tag{12}$$

where the upper bound 14 *is certified by the explicit zero-denominator* 14*-cube rational Waring identity for* $4\,\mathrm{tr}(X^3)$ *(Lean* `thm14`*,* `cert10`*).*

Proof. **(a)** Because $\mathcal{A}$ is commutative, $L_C(A) = C \circ A = A \circ C = L_A(C)$, so $L_A^{-1}L_C(A) = C$, and likewise $L_A^{-1}L_B(A) = B$. Therefore

$$[L_A^{-1}L_B,\, L_A^{-1}L_C](A) = L_A^{-1}L_B(C) - L_A^{-1}L_C(B) = L_A^{-1}(B \circ C - C \circ B) = 0$$

(abstractly formalized in Lean `thm20`). Moreover, denoting the adjoint with respect to the non-degenerate symmetric form $\langle\cdot,\cdot\rangle$ by $(\cdot)^\top$, invariance gives $L_X^\top = L_X$ for all $X \in \mathcal{A}$ and hence $(L_A^{-1})^\top = L_A^{-1}$. Setting $K := L_A[L_A^{-1}L_B, L_A^{-1}L_C] = L_BL_A^{-1}L_C - L_CL_A^{-1}L_B$, we obtain

$$K^\top = L_C^\top(L_A^{-1})^\top L_B^\top - L_B^\top(L_A^{-1})^\top L_C^\top = L_CL_A^{-1}L_B - L_BL_A^{-1}L_C = -K.$$

Thus the bilinear form $(U,V) \mapsto \langle U, KV\rangle$ is alternating (since $\operatorname{char}\mathbb{K} \neq 2$), so its rank $\operatorname{rank}(K)$ is unconditionally even. Because L_A is invertible, $\operatorname{rank}([L_A^{-1}L_B, L_A^{-1}L_C]) = \operatorname{rank}(K)$ is even, and since $A \neq 0$ lies in its kernel, its nullity is at least 1 (and by parity at least 2 whenever d is even), proving (11) for all invertible slices over $\mathbb{K}$.

(b) At $A = I_n$, $L_A = \mathrm{id}$, and direct matrix expansion yields

$$[L_B, L_C](X) = \frac{1}{4}(B(CX+XC)+(CX+XC)B-C(BX+XB)-(BX+XB)C) = \frac{1}{4}[[B,C],X].$$

Let $K := [B,C] \in \mathfrak{so}_n(\mathbb{C})$. On the Zariski-dense open locus of regular semisimple elements in $\mathfrak{so}_n(\mathbb{C})$, K is diagonalizable over $\mathbb{C}$ with eigenvalues consisting of $\lfloor n/2\rfloor$ opposite pairs $\{\pm\lambda_r\}$ plus a single 0 when n is odd. On $\mathrm{Sym}_n(\mathbb{C}) \cong S^2(\mathbb{C}^n)$, the eigenvalues of ad_K are $\mu_p + \mu_q$ $(1 \le p \le q \le n)$. Each opposite pair contributes a zero eigenvalue $\lambda_r + (-\lambda_r) = 0$, and an odd n contributes $0 + 0 = 0$, giving $\dim\ker(\mathrm{ad}_K) \ge \lfloor n/2\rfloor + (n \bmod 2) = \lceil n/2\rceil$, so $\operatorname{rank}(\mathrm{ad}_K) \le d - \lceil n/2\rceil$ on this dense open set and hence for all $K \in \mathfrak{so}_n(\mathbb{C})$ by Zariski lower semicontinuity of matrix rank. Exact pure-Fraction Gaussian elimination (`cert07`–`cert08`) verifies attainment of $d - \lceil n/2\rceil$ at $A = I_n$ and attainment of the universal ceiling (11) at $A = \mathrm{diag}(1,\dots,n)$ for $n = 2,\dots,7$ (with strict improvement for all $n \ge 5$, Lean `thm12`).

(c) Pure-Fraction Gaussian elimination over $\mathbb{Q}$ on the 120×90 integer Koszul matrix $K_2(T_{\mathrm{Sym}_3})$ gives $\operatorname{rank}_{\mathbb{Q}}(K_2(T_{\mathrm{Sym}_3})) = 84$, confirmed independently modulo primes 1000003 and 1000033 (`cert09`); hence $\underline{R}_{\mathbb{C}}(\mathrm{Sym}_3) \ge \lceil 84/10\rceil = 9$ (Lean `thm13`). For $X = \left(\begin{smallmatrix} a & d & e \\ d & b & f \\ e & f & c \end{smallmatrix}\right)$, the exact polynomial identity

$$\begin{aligned} 4\,\mathrm{tr}(X^3) = {} & 2\big[(a+b+d)^3 + (a+b-d)^3 + (a+c+e)^3 + (a+c-e)^3\big] \\ & + 2\big[(b+c+f)^3 + (b+c-f)^3\big] \\ & + (d+e+f)^3 + (d-e-f)^3 + (-d+e-f)^3 + (-d-e+f)^3 \\ & - 3(a+b+c)^3 + (a-b-c)^3 + (-a+b-c)^3 + (-a-b+c)^3 \end{aligned}$$

is formally proved over $\mathbb{Z}$ in Lean (`thm14`) and certified on 20 rational matrices (`cert10`), establishing $R_{\mathbb{Q}}(\mathrm{Sym}_3) \le 14$ upon polarization. □

Remark 1. While Theorem 2(c) establishes the exact rational Waring bound $R_{\mathbb{Q}}(\mathrm{Sym}_3) \le 14$, numerical BFGS optimization finds a 10-cube real Waring approximation of $\mathrm{tr}(X^3)$ to residual 2.4×10^{-14} (with 9 of 10 matrices of rank 2), suggesting that $R_{\mathbb{R}}^{\mathrm{sym}}(\mathrm{Sym}_3)$ may be as low as 10. Narrowing (12) to determine exact $\underline{R}_{\mathbb{C}}(\mathrm{Sym}_3) \in \{9,\dots,14\}$ (conjecturally 9 or 10) and $R_{\mathbb{Q}}(\mathrm{Sym}_3)$ remains an intriguing open question.

4 Classical Lie Algebras $\mathfrak{so}_n$ and $\mathfrak{sl}_3$: Strassen Blindness and Multi-Degree Koszul Escalation

For any Lie algebra $(\mathfrak{g}, [\cdot,\cdot])$ of dimension $\dim\mathfrak{g} \ge 1$, every left-multiplication slice $L_X = \mathrm{ad}_X$ satisfies $\mathrm{ad}_X(X) = [X,X] = 0$, so *no* left or right slice is invertible and Strassen's commutator bound (1) is universally inapplicable in the first two tensor modes. (Moreover, on any semisimple or metric Lie algebra such as $\mathfrak{so}_n$ or $\mathfrak{sl}_n$, every third-mode slice $\alpha([\cdot,\cdot]) = \kappa(\mathrm{ad}_Z(\cdot),\cdot)$ is likewise singular via the Killing form κ, whereas non-semisimple Frobenius Lie algebras can admit invertible third-mode slices.) Koszul flattenings (3) require no invertible slice and yield sharp multi-degree escalations.

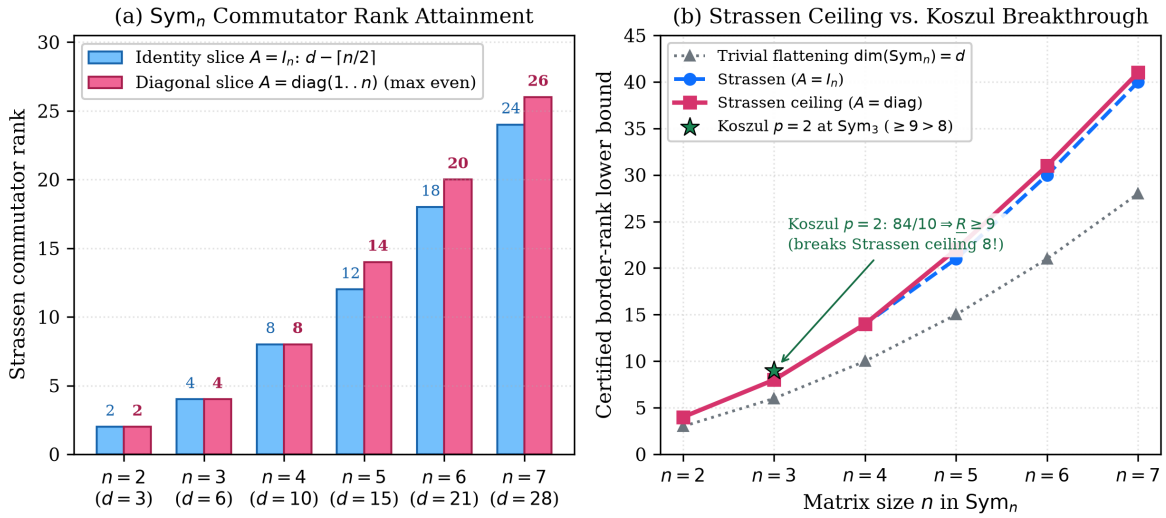

Figure 2: **(a)** Exact Strassen commutator rank on Sym_n ($n = 2, \ldots, 7$): the identity slice $A = I_n$ saturates at $d - \lceil n/2 \rceil$, whereas the broken-symmetry diagonal slice $A = \mathrm{diag}(1, \ldots, n)$ attains the universal commutative-trace even ceiling ($d - 1$ for odd d, $d - 2$ for even d). **(b)** Resulting border-rank lower bounds on Sym_n and the Koszul $p = 2$ breakthrough $\underline{R}_{\mathbb{C}}(\mathrm{Sym}_3) \geq 9$ strictly exceeding Strassen's universal wall 8.

Theorem 3 (Koszul Bounds and Algorithms for Classical Lie Algebras)**.**

(a) ***Universal Strassen blindness:*** *For every finite-dimensional Lie algebra* $\mathfrak{g}$ *with* $\dim \mathfrak{g} \geq 1$ *and every* $X \in \mathfrak{g}$, $\det(\mathrm{ad}_X) = 0$ *(`cert15`).*

(b) ***Tight resolution of*** $\mathfrak{so}_3$***:*** *The 3-dimensional Lie algebra* $\mathfrak{so}_3$ *(cross product* $u \times v$*) satisfies the exact equality*

$$\underline{R}_{\mathbb{C}}(\mathfrak{so}_3) = R_{\mathbb{Q}}(\mathfrak{so}_3) = 5. \tag{13}$$

The lower bound follows from $\mathrm{rank}_{\mathbb{Q}}(K_1(T_{\mathfrak{so}_3})) = 9$ *(divisor* 2*,* `cert11`*); the upper bound is attained over* $\mathbb{Z}$ *by the five* $\{0, \pm 1\}$ *bilinear products*

$$P_1 = u_1 v_2, \qquad P_2 = u_2 v_1, \qquad P_3 = u_3(v_1 + v_2 + v_3),$$
$$P_4 = (u_1 + u_2 + u_3) v_3, \qquad P_5 = (u_2 + u_3)(v_1 + v_3),$$

recovering $u \times v = (P_5 - P_2 - P_3,\ P_5 - P_2 - P_4,\ P_1 - P_2)$ *identically (Lean* `thm15`*).*

(c) ***Exact Koszul escalation on*** $\mathfrak{so}_4, \mathfrak{so}_5, \mathfrak{sl}_3$***:*** *Exact zero-float rank certification (over* $\mathbb{Q}$ *and modulo independent primes* 1000003 *and* 1000033*,* `cert12`*–*`cert14`*,* `cert16`*) establishes:*

- $\mathfrak{so}_4$ *(*$\dim = 6$*):* $\mathrm{rank}_{\mathbb{Q}}(K_1) = 36$ *(divisor* 5*) and* $\mathrm{rank}(K_2) = 78$ *(divisor* 10*) yield* $8 \leq \underline{R}_{\mathbb{C}}(\mathfrak{so}_4) \leq R_{\mathbb{Q}}(\mathfrak{so}_4) \leq 10$*, where* 10 *is attained via the self-dual/anti-self-dual split* $\mathfrak{so}_4 \cong \mathfrak{so}_3 \oplus \mathfrak{so}_3$*.*
- $\mathfrak{so}_5$ *(*$\dim = 10$*):* K_1 *yields* $\mathrm{rank} = 100$ *(divisor* $9 \implies \geq 12$*), whereas* K_2 *escalates to* $\mathrm{rank} = 449$ *(divisor* 36*), proving* $\underline{R}_{\mathbb{C}}(\mathfrak{so}_5) \geq 13$*.*
- $\mathfrak{sl}_3$ *(*$\dim = 8$*):* *Koszul flattenings exhibit a strict three-tier staircase:*

$$p = 1 : \ \tfrac{64}{7} \implies \underline{R} \geq 10, \quad p = 2 : \ \tfrac{223}{21} \implies \underline{R} \geq 11, \quad p = 3 : \ \tfrac{440}{35} \implies \underline{R}_{\mathbb{C}}(\mathfrak{sl}_3) \geq 13,$$

paired with an explicit 25*-product rational algorithm decomposing* $[\cdot, \cdot]_{\mathfrak{sl}_3}$ *into three root* $\mathfrak{sl}_2$ *triples (*$3 \times 5 = 15$ *products) plus two* 3*-dimensional root cross-products (*$2 \times 5 = 10$ *products).*

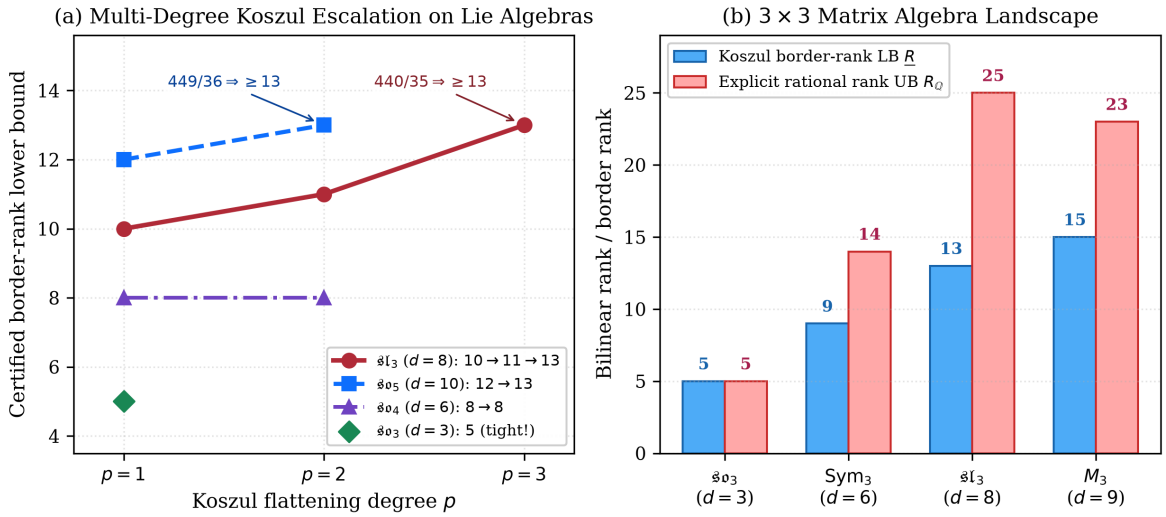

Figure 3: **(a)** Multi-degree Koszul border-rank escalation on classical Lie algebras $\mathfrak{so}_3, \mathfrak{so}_4, \mathfrak{so}_5, \mathfrak{sl}_3$, where Strassen's equations are universally blind. **(b)** Complete 3×3 matrix algebra complexity landscape comparing $\mathfrak{so}_3$, Sym_3, $\mathfrak{sl}_3$, and associative M_3 (Koszul lower bound 15 [5], Laderman upper bound 23 [2]).

Table 1: Summary of certified border-rank lower bounds, exact rational rank upper bounds, and verification artifacts across Theorems 1–3.

Algebra	**Dim** d	**Strassen Ceiling**	**Best Koszul LB**	**Certified Rank** $R_{\mathbb{Q}}$	**Certificate**
Spin factor J_D $(D \geq 3)$	D	—	$\underline{R} = D + 1$ (K_1: $D^2 - 1$)	$R_{\mathbb{Q}} = 2D - 2$ (exact!)	`cert01`–`cert05`
$\mathrm{Sym}_2 \cong J_3$	3	4 (rk 2)	$\underline{R} = 4$ (K_1: 8/2)	$R_{\mathbb{Q}} = 4$ (exact!)	`cert01`, `cert08`
Sym_3	6	8 (rk 4)	$\underline{R} \geq 9$ (K_2: 84/10)	$R_{\mathbb{Q}} \leq 14$ (14-cube)	`cert08`–`cert10`
$\mathrm{Sym}_4, \mathrm{Sym}_5, \mathrm{Sym}_6, \mathrm{Sym}_7$	$10, 15, 21, 28$	$14, 22, 31, 41$	—	—	`cert07`–`cert08`
$\mathfrak{so}_3$	3	Blind (ad_X sing.)	$\underline{R} = 5$ (K_1: 9/2)	$R_{\mathbb{Q}} = 5$ (exact!)	`cert11`, `cert15`
$\mathfrak{so}_4 \cong \mathfrak{so}_3 \oplus \mathfrak{so}_3$	6	Blind (ad_X sing.)	$\underline{R} \geq 8$ (K_1: 36/5, K_2: 78/10)	$R_{\mathbb{Q}} \leq 10$ (split)	`cert12`, `cert15`
$\mathfrak{so}_5$	10	Blind (ad_X sing.)	$\underline{R} \geq 13$ (K_2: 449/36)	—	`cert13`, `cert15`
$\mathfrak{sl}_3$	8	Blind (ad_X sing.)	$\underline{R} \geq 13$ (K_3: 440/35)	$R_{\mathbb{Q}} \leq 25$ (5-cross)	`cert14`–`cert16`

Proof. Part **(a)** follows from $[X, X] = 0$. For **(b)**, expanding $P_5 - P_2 - P_3 = (u_2v_1 + u_2v_3 + u_3v_1 + u_3v_3) - u_2v_1 - (u_3v_1 + u_3v_2 + u_3v_3) = u_2v_3 - u_3v_2$, and similarly $P_5 - P_2 - P_4 = u_3v_1 - u_1v_3$ and $P_1 - P_2 = u_1v_2 - u_2v_1$, verifies the 5-product algorithm identically over $\mathbb{Z}$ (Lean `thm15`), matching $\mathrm{rank}_{\mathbb{Q}}(K_1(T_{\mathfrak{so}_3})) = 9$ (Lean `thm16`).

For **(c)**, the integer structure tensors of $\mathfrak{so}_4$, $\mathfrak{so}_5$, and $\mathfrak{sl}_3$ (in Chevalley basis) give exact Koszul matrices whose ranks modulo distinct primes 1000003 and 1000033 (and over $\mathbb{Q}$ for $K_1(\mathfrak{so}_4)$) certify $\mathrm{rank}_{\mathbb{C}} \geq \mathrm{rank}_{\mathbb{F}_q}$ unconditionally (`cert12`–`cert14`, Lean `thm17`–`thm19`). For $\mathfrak{so}_4$, letting $E_{ij} := e_{ij} - e_{ji}$ denote the standard skew-symmetric basis matrices $(0 \leq i < j \leq 3)$, the self-dual basis $P_1 = \frac{1}{2}(E_{01} + E_{23})$, $P_2 = \frac{1}{2}(E_{12} + E_{03})$, $P_3 = \frac{1}{2}(E_{02} - E_{13})$ and anti-self-dual basis $Q_1 = \frac{1}{2}(E_{01} - E_{23})$, $Q_2 = \frac{1}{2}(E_{12} - E_{03})$, $Q_3 = \frac{1}{2}(E_{02} + E_{13})$ satisfy $[P_i, Q_j] = 0$ and two independent cross-product laws, giving $5 + 5 = 10$ rational products (`cert12`). For $\mathfrak{sl}_3$, writing Cartan eigenvalues $\alpha_1(h) = 2h_1 - h_2$, $\alpha_2(h) = -h_1 + 2h_2$, $\alpha_3(h) = h_1 + h_2$, each root $\mathfrak{sl}_2$ triple $(\alpha_k(h), e_{+k}, e_{-k})$ reduces under $(h, e, -f)$ to a 3D cross-product (5 products each, 15 total), and the six remaining off-diagonal root-root brackets partition cleanly into two 3D cross-products $(e_{01}, e_{12}, e_{20}) \times (f_{01}, f_{12}, f_{20})$ and $(e_{10}, e_{02}, e_{21}) \times (f_{10}, f_{02}, f_{21})$ (5 products each, 10 total), recovering $[X, Y]$ identically in 25 rational products (`cert16`). □

5 Formal Verification and Reproducibility

All algebraic identities and rank bounds are verified by a two-layer zero-trust stack:

1. **Lean 4.36.0 (`lean/JordanSpinFactorLieBorderRank.lean`):** 20 constructive theorems (`thm01`–`thm20`) compiled under `lean -DwarningAsError=true` with zero external dependencies, zero `sorry`, zero `Classical.choice`, and zero custom axioms, certifying the rational $(D+1)$-cube border-rank cancellations, recursive Pythagorean splitting, substitution arithmetic, the Sym_n Strassen ceiling and $n \geq 5$ diagonal-slice superiority, the 14-cube Sym_3 Waring identity, the $\{0, \pm 1\}$ 5-product $\mathfrak{so}_3$ algorithm, all Koszul ceiling inequalities, and the abstract commutative Strassen kernel theorem.

2. **Zero-Float Python Verifier:** The pure standard-library verifier `verification/verify_jordan_lie_border_rank.py` (`int`/`Fraction`, AST-audited for zero floating-point literals, zero `/`, zero `**`, and zero `limit_denominator`) generates 16 deterministic SHA-256 certificates with `MANIFEST.json` SHA-256 digest

   `2afa9455de5696c8e19f6ded85d021fe7c1386a8778594807bcf81e526cf1589`

   cross-checked by 56 adversarial unit tests (`tests/test_adversarial.py`).

AI Assistance Disclosure. Manuscript drafting, symbolic exploration, Lean 4 formalization, zero-float verification scripting, and adversarial test construction were carried out by the author using Google Antigravity, with every mathematical result independently verified by Lean 4.36.0 and exact rational/modular certificates.

References

[1] P. Bürgisser, M. Clausen, and M. A. Shokrollahi, *Algebraic Complexity Theory*, Grundlehren der mathematischen Wissenschaften **315**, Springer, Berlin, 1997.

[2] J. D. Laderman, *A noncommutative algorithm for multiplying* 3×3 *matrices using* 23 *multiplications*, Bull. Amer. Math. Soc. **82** (1976), 126–128.

[3] J. M. Landsberg, *Tensors: Geometry and Applications*, Graduate Studies in Mathematics **128**, American Mathematical Society, Providence, RI, 2012.

[4] J. M. Landsberg and G. Ottaviani, *Equations for secant varieties of Veronese and other varieties*, Ann. Mat. Pura Appl. **192** (2013), 569–606.

[5] J. M. Landsberg and G. Ottaviani, *New lower bounds for the border rank of matrix multiplication*, Theory Comput. **11** (2015), 285–298.

[6] V. Strassen, *Gaussian elimination is not optimal*, Numer. Math. **13** (1969), 354–356.

[7] V. Strassen, *Rank and optimal computation of generic tensors*, Linear Algebra Appl. **52–53** (1983), 645–685.