AIRR.SCIENCE · ARR-2026-0F6MXGVEAH9YJ8AT v1 [Mathematics] 28 Nov 2026

Walsh–Hadamard Hyperplane Separators and Sub-Volume Collapse of Scramble Number and Screewidth on Hypercubes and Cartesian Powers

Lluis Eriksson*

November 28, 2026

Abstract

In tropical algebraic geometry on graphs and chip-firing, the scramble number $\mathrm{sn}(G)$ and its tree-cut upper bound, the screewidth $\mathrm{scw}(G)$, sit in the fundamental hierarchy

$$\mathrm{tw}(G) \le \mathrm{sn}(G) \le \mathrm{scw}(G) \qquad \text{and} \qquad \mathrm{sn}(G) \le \mathrm{dgon}(G),$$

where $\mathrm{tw}(G)$ is treewidth and $\mathrm{dgon}(G)$ is divisorial gonality. For the n-dimensional hypercube Q_n, placing one chip on every vertex of one bipartition class yields $\mathrm{dgon}(Q_n) \le 2^{n-1}$, and the Hypercube Gonality Conjecture predicts $\mathrm{dgon}(Q_n) = 2^{n-1}$ for all $n \ge 1$. While $\mathrm{sn}(Q_n) = \mathrm{scw}(Q_n) = 2^{n-1}$ holds for $1 \le n \le 5$, the asymptotic behavior of $\mathrm{sn}(Q_n)$ and $\mathrm{scw}(Q_n)$ for $n \ge 6$ has remained open.

We resolve the asymptotic scale of both $\mathrm{sn}(Q_n)$ and $\mathrm{scw}(Q_n)$ up to a single logarithmic factor and prove a universal sub-volume collapse across all Cartesian graph powers:

1. **Universal Star-Separator Obstruction and Product Lifting (Theorem 2.3):** Any vertex partition $V(G) = S \sqcup \bigsqcup_{\sigma \in I} C_\sigma$ with pairwise non-adjacent cells induces a zero-tunneling star tree-cut decomposition of exact width $\max(|S|, \max_\sigma |C_\sigma|, \max_\sigma |\partial_G(C_\sigma)|)$, directly bounding both $\mathrm{sn}(G)$ and $\mathrm{scw}(G)$, and lifting multiplicatively to $G \square H$ by a factor $|V(H)|$.
2. **Exact Walsh–Hadamard Hyperplane Separator Theorem (Theorem 3.2):** On Q_{2^k} ($k \ge 1$), the k orthogonal signed Walsh–Hadamard forms $L_j(x) = \sum_\alpha (-1)^{\alpha_j}(2x_\alpha - 1)$ obey the congruence $L_j(x) \equiv L_{j'}(x) \pmod 4$ ($k \ge 2$). Consequently, the zero-hyperplane union $S_k = \{x : \exists j,\, L_j(x) = 0\}$ is an *independent set* of Q_{2^k}, the 2^k sign cells C_σ are pairwise non-adjacent and transitively isomorphic under involutive XOR coordinate permutations $\Pi_\tau \in \mathrm{Aut}(Q_{2^k})$, and every cell obeys the exact boundary-separator identity $|\partial_{Q_{2^k}}(C_\sigma)| = |S_k| > |C_\sigma|$. Hence $\mathrm{sn}(Q_{2^k}) \le \mathrm{scw}(Q_{2^k}) \le |S_k| \le k\binom{2^k}{2^{k-1}}$, and Cartesian lifting yields

$$\Omega\left(\frac{2^n}{\sqrt{n}}\right) \le \mathrm{tw}(Q_n) \le \mathrm{sn}(Q_n) \le \mathrm{scw}(Q_n) \le \frac{2}{\sqrt{3}} \frac{\log_2 n}{\sqrt{n}} 2^n \quad (\forall n \ge 4).$$

   Thus no scramble or tree-cut screewidth certificate can ever achieve $\Omega(2^n)$ on Q_n, and conditional on the Hypercube Gonality Conjecture, $\mathrm{dgon}(Q_n)/\mathrm{sn}(Q_n) = \Omega(\sqrt{n}/\log n) \to \infty$.
3. **Sharp Phase Transition at $n = 6$ and Explicit Coding Witnesses (Theorem 4.1):** Explicit distance-4 code-ball and extended binary Golay $[24, 12, 8]$ star separators certify $\mathrm{scw}(Q_6) \le 30 < 32$, $\mathrm{scw}(Q_7) \le 56 < 64$, $\mathrm{scw}(Q_8) \le 112 < 128$, and $\mathrm{scw}(Q_{24}) \le \frac{1771}{2048} 2^{23}$; exact inclusion-exclusion evaluates $|S_3| = 120$, $|S_4| = 28{,}672 = \frac{7}{8}2^{15}$, and $|S_5| = 1{,}741{,}440{,}992 \approx 0.8109 \cdot 2^{31}$; and a self-contained Harper majority-scramble proof certifies $\mathrm{sn}(Q_n) = \Omega(2^n/\sqrt{n})$.

*Independent researcher; AIRR.SCIENCE (`editor@airr.science`). Authored with AI assistance (Google Antigravity) for proof exploration, Lean 4.36.0 formalization, zero-float integer verification, and manuscript drafting.

4. **Universal Cartesian-Power Collapse (Theorem 5.1):** Extending Walsh–Hadamard separators via balanced 2-labelings (even $|V(G)|$) and 3-band Cauchy–Schwarz trinomial slices (odd $|V(G)|$) proves $\mathrm{sn}(G^{\square n}) \le \mathrm{scw}(G^{\square n}) = O_G\left(\frac{\log n}{\sqrt{n}}|V(G)|^n\right)$ for *every* nontrivial finite simple graph G ($|V(G)| \ge 2$, $\Delta(G) \ge 1$), including all q-ary Hamming graphs $H(n,q) = K_q^{\square n}$ and discrete tori $C_m^{\square n}$.

Core structural, algebraic, binomial, trinomial, asymptotic, and finite-witness pillars (72 declarations: 25 public theorems, 22 private lemmas, 25 definitions/structures) are machine-checked in standalone Lean 4.36.0 with zero external dependencies, zero `sorry`, zero `grind`, and zero custom axioms, corroborated by zero-float integer/rational verification in Python.

Keywords: scramble number, screewidth, divisorial gonality, hypercube, Walsh–Hadamard, chip-firing, tree-cut decomposition, Lean 4.
MSC 2020: 05C57, 14T15, 05C40, 05C76, 94B05.

1 Introduction

Divisorial gonality $\mathrm{dgon}(G)$ is the tropical and combinatorial analogue of the gonality of an algebraic curve, defined via the Baker–Norine chip-firing game on a finite connected graph G [3, 2]: it is the minimum degree of a rank-1 divisor on G, equivalently the minimum number of chips that can be placed on $V(G)$ so that for every target vertex $v \in V(G)$, a sequence of legal chip-firing moves places at least one chip on v without creating debt anywhere. Computing $\mathrm{dgon}(G)$ is NP-hard in general, and establishing tight lower bounds has become a central theme in tropical graph theory.

Following the breakthrough proof by van Dobben de Bruyn and Gijswijt [11] that treewidth lower-bounds divisorial gonality ($\mathrm{tw}(G) \le \mathrm{dgon}(G)$), Harp, Jackson, Jensen, and Speeter [7] and Echavarria et al. [6] introduced the *scramble number* $\mathrm{sn}(G)$, generalizing brambles to collections of arbitrary connected subgraphs (*eggs*). Subsequently, Cenek et al. [4] introduced *screewidth* $\mathrm{scw}(G)$ via tree-cut decompositions as a min–max structural upper bound on scramble number, establishing the chain

$$\mathrm{tw}(G) \;\le\; \mathrm{sn}(G) \;\le\; \mathrm{scw}(G), \qquad \mathrm{sn}(G) \;\le\; \mathrm{dgon}(G). \tag{1}$$

A flagship testing ground for (1) is the n-dimensional hypercube Q_n on 2^n vertices [1, 6, 4]. Because Q_n is bipartite with color classes of size 2^{n-1}, placing one chip on every vertex of one bipartition class defines a rank-1 divisor (firing the chip-bearing class sends n chips to every vertex of the opposite class), giving $\mathrm{dgon}(Q_n) \le 2^{n-1}$. The *Hypercube Gonality Conjecture* asserts that this bound is sharp for every $n \ge 1$:

$$\mathrm{dgon}(Q_n) \;\stackrel{?}{=}\; 2^{n-1}.$$

For $1 \le n \le 5$, uniform scrambles and screewidth confirm $\mathrm{sn}(Q_n) = \mathrm{scw}(Q_n) = \mathrm{dgon}(Q_n) = 2^{n-1}$ [6, 4]. At $n = 6$, Cenek et al. [4] observed that $\mathrm{sn}(Q_6) < 32$, leaving open both the exact values for $n \ge 6$ and the fundamental asymptotic question: *does* $\mathrm{sn}(Q_n)$ *(or* $\mathrm{scw}(Q_n)$*) remain* $\Theta(2^n)$ *as* $n \to \infty$*, or does it collapse to* $o(2^n)$*?*

Why classical Harper orderings fail. Two classical isoperimetric orderings of Q_n illustrate the obstruction to bounding $\mathrm{scw}(Q_n)$ below 2^{n-1}. The *simplicial* (Hamming-weight) ordering of Harper's vertex-isoperimetric theorem [9] partitions $V(Q_n)$ into layers of maximum size $\binom{n}{\lfloor n/2 \rfloor} \sim \sqrt{\frac{2}{\pi n}}2^n$, establishing $\mathrm{tw}(Q_n) \le \mathrm{pw}(Q_n) \le \binom{n}{\lfloor n/2 \rfloor}$ [5], yet the edge cut between adjacent middle layers has size $\lceil n/2 \rceil \binom{n}{\lfloor n/2 \rfloor} \sim \sqrt{\frac{n}{2\pi}}2^n \gg 2^{n-1}$. Conversely, the *lexicographic* ordering of Harper's edge-isoperimetric theorem [8] minimizes edge cuts for fixed subset sizes, yet its maximum prefix cutwidth is $\lfloor 2^{n+1}/3 \rfloor \approx \frac{4}{3} \cdot 2^{n-1} > 2^{n-1}$. Neither ordering can separate Q_n below 2^{n-1}.

Our contribution. We resolve the asymptotic scale of both $\mathrm{sn}(Q_n)$ and $\mathrm{scw}(Q_n)$ by introducing *Walsh–Hadamard hyperplane separators*: instead of cutting Q_{2^k} along one Hamming-weight hyperplane (huge edge boundary) or coordinate subcubes (huge vertex separator), we simultaneously delete the zero-hyperplanes of $k = \log_2 n$ orthogonal signed Walsh–Hadamard linear forms $L_0, \dots, L_{k-1}$. A striking arithmetic rigidity—the mod-4 congruence $L_j(x) \equiv L_{j'}(x) \pmod 4$ for all $k \geq 2$—forces the hyperplane union S_k to be an *independent set* of Q_{2^k}, while involutive XOR coordinate permutations act transitively on all 2^k complementary sign cells C_σ. Consequently, the total boundary $|E(U_k, S_k)| = 2^k|S_k|$ (where $U_k := V(Q_{2^k}) \setminus S_k$) divides *identically* across the 2^k isomorphic cells, giving the exact identity $|\partial_{Q_{2^k}}(C_\sigma)| = |S_k|$ and collapsing $\mathrm{scw}(Q_n)$ all the way down to $O\Big(\frac{\log n}{\sqrt{n}} 2^n\Big)$—within a single logarithmic factor of $\mathrm{tw}(Q_n) = \Theta(2^n/\sqrt{n})$. We further establish explicit coding-theoretic certificates for the phase transition at $n = 6$ (Q_6, Q_7, Q_8, Q_{24}), exact inclusion-exclusion counts up to Q_{32} ($2^{32} \approx 4.29 \times 10^9$ vertices), a self-contained Harper majority-scramble lower bound, and a universal generalization proving $\mathrm{sn}(G^{\square n}) \leq \mathrm{scw}(G^{\square n}) = O_G\Big(\frac{\log n}{\sqrt{n}} |V(G)|^n\Big)$ for *every* nontrivial finite simple graph G ($|V(G)| \geq 2$, $\Delta(G) \geq 1$, both even and odd orders).

2 Universal Star-Separator Obstruction and Product Lifting

Let $G = (V, E)$ be a finite simple graph. For $A \subseteq V$, let $\partial_G(A) := \{uv \in E : u \in A, v \notin A\}$ denote its edge boundary, and $\partial_V(A) := \{v \in V \setminus A : \exists u \in A, uv \in E\}$ its exterior vertex boundary.

Definition 2.1 (Scramble, order, and screewidth [7, 6, 4])**.** A *scramble* on G is a nonempty collection $\mathcal{S}$ of nonempty connected vertex subsets (*eggs*) of G. Its *hitting number* $h(\mathcal{S})$ is the minimum cardinality of a set $W \subseteq V$ meeting every $E \in \mathcal{S}$. Its *egg-cut number* $e(\mathcal{S})$ is the minimum of $|\partial_G(A)|$ over all bipartitions $(A, V \setminus A)$ for which there exist $E_1, E_2 \in \mathcal{S}$ with $E_1 \subseteq A$ and $E_2 \subseteq V \setminus A$ (with $e(\mathcal{S}) := \infty$ if no such bipartition exists). The *order* of $\mathcal{S}$ is $\|\mathcal{S}\|_G := \min(h(\mathcal{S}), e(\mathcal{S}))$, and the *scramble number* is $\mathrm{sn}(G) := \max_{\mathcal{S}} \|\mathcal{S}\|_G$.

A *tree-cut decomposition* $(T, \mathcal{X})$ of G consists of a tree T and a partition $\mathcal{X} = \{X_t : t \in V(T)\}$ of $V(G)$ (empty bags permitted). For each edge $e \in E(T)$, deleting e partitions $V(G)$ into (A_e, B_e) with *adhesion* $\mathrm{adh}(e) := |E_G(A_e, B_e)|$. For each node $t \in V(T)$, a *tunneling edge* at t is an edge $uv \in E(G)$ whose endpoints lie in distinct components of $T - t$; the *node width* of t is $|X_t|$ plus the number of tunneling edges at t. The *screewidth* $\mathrm{scw}(T, \mathcal{X})$ is the maximum over all edge adhesions and node widths, and $\mathrm{scw}(G) := \min_{(T,\mathcal{X})} \mathrm{scw}(T, \mathcal{X})$.

Definition 2.2 (Star separator)**.** A *star separator* of G is a partition $V(G) = S \sqcup \bigsqcup_{\sigma \in I} C_\sigma$ (with I finite, possibly empty) such that no edge of G connects a vertex in C_σ to a vertex in $C_{\sigma'}$ for $\sigma \neq \sigma'$. Its *star width* is

$$w(S; \{C_\sigma\}_{\sigma \in I}) \;:=\; \max\Big(|S|,\; \max_{\sigma \in I} |C_\sigma|,\; \max_{\sigma \in I} |\partial_G(C_\sigma)|\Big),$$

where the maximum over an empty index set is 0.

Theorem 2.3 (Universal Star-Separator Obstruction, Product Lifting, and Subgraph Monotonicity)**.** *Let $G = (V, E)$ be any finite simple graph.*

(a) ***Star-separator bound:*** *For every star separator $(S; \{C_\sigma\}_{\sigma \in I})$ of G, the star tree-cut decomposition $(T_S, \mathcal{X}_S)$ with central bag $X_r = S$ and leaf bags $X_{\ell_\sigma} = C_\sigma$ has zero tunneling edges and exact screewidth $w(S; \{C_\sigma\})$. Moreover, directly from the definition of scramble number, every scramble $\mathcal{S}$ on G satisfies $\|\mathcal{S}\|_G \leq w(S; \{C_\sigma\})$. Consequently,*

$$\mathrm{sn}(G) \;\leq\; \mathrm{scw}(G) \;\leq\; w(S; \{C_\sigma\}).$$

(b) ***Cartesian product lifting:*** *If $(S; \{C_\sigma\}_{\sigma\in I})$ is a star separator of G of width B and H is any finite simple graph, then $(S \times V(H); \{C_\sigma \times V(H)\}_{\sigma\in I})$ is a star separator of the Cartesian product $G\square H$ of exact width $|V(H)| \cdot B$. Hence*

$$\mathrm{sn}(G\square H) \;\le\; \mathrm{scw}(G\square H) \;\le\; |V(H)| \cdot B.$$

(c) ***Subgraph monotonicity:*** *For every subgraph $H \subseteq G$, $\mathrm{sn}(H) \le \mathrm{sn}(G)$ and $\mathrm{scw}(H) \le \mathrm{scw}(G)$.*

Proof. **(a)** In the star tree T_S with center r and leaves $\{\ell_\sigma\}_{\sigma\in I}$, no edge of G can tunnel through any leaf ℓ_σ (a leaf has degree 1 in T_S), and an edge $uv \in E(G)$ tunnels through r if and only if $u \in C_\sigma$ and $v \in C_{\sigma'}$ for some $\sigma \neq \sigma'$. By the non-adjacency hypothesis of a star separator, no such edge exists. Hence the node widths are $|X_r| = |S|$ and $|X_{\ell_\sigma}| = |C_\sigma|$, while each tree edge $r\ell_\sigma$ has adhesion $|\partial_G(C_\sigma)|$, giving $\mathrm{scw}(T_S, \mathcal{X}_S) = w(S; \{C_\sigma\})$.

For the direct first-principles bound on any scramble $\mathcal{S}$ on G (which we also formalize in Lean 4 as `star_separator_scramble_dichotomy`): if S meets every egg $E \in \mathcal{S}$, then $h(\mathcal{S}) \le |S|$. Otherwise, there exists an egg $E_1 \in \mathcal{S}$ with $E_1 \cap S = \emptyset$. Since E_1 is nonempty and connected in G and no edge joins distinct cells $C_\sigma, C_{\sigma'}$, every path inside E_1 remains within a single cell, so $E_1 \subseteq C_{\sigma_0}$ for a unique $\sigma_0 \in I$. Now either every egg $E \in \mathcal{S}$ meets C_{σ_0}—so C_{σ_0} is a hitting set and $h(\mathcal{S}) \le |C_{\sigma_0}|$—or some egg $E_2 \in \mathcal{S}$ satisfies $E_2 \subseteq V(G)\setminus C_{\sigma_0}$, in which case $(C_{\sigma_0}, V(G)\setminus C_{\sigma_0})$ is a valid egg-cut separating E_1 and E_2, yielding $e(\mathcal{S}) \le |\partial_G(C_{\sigma_0})|$. In all three cases,

$$\|\mathcal{S}\|_G = \min(h(\mathcal{S}), e(\mathcal{S})) \;\le\; \max(|S|,\, \max_\sigma |C_\sigma|,\, \max_\sigma |\partial_G(C_\sigma)|) = w(S; \{C_\sigma\}).$$

(b) Clearly $V(G\square H) = (S \times V(H)) \sqcup \bigsqcup_{\sigma\in I}(C_\sigma \times V(H))$. Every edge $\{(u,h),(u',h')\}$ of $G\square H$ has either $u = u'$ (staying within the same set $S \times V(H)$ or $C_\sigma \times V(H)$) or $h = h'$ with $uu' \in E(G)$ (which cannot connect C_σ to $C_{\sigma'}$ for $\sigma \neq \sigma'$). Thus no edge joins distinct lifted cells. Clearly $|S \times V(H)| = |V(H)||S|$ and $|C_\sigma \times V(H)| = |V(H)||C_\sigma|$. Moreover, an edge of $G\square H$ leaves $C_\sigma \times V(H)$ if and only if it is of the form $\{(u,h),(u',h)\}$ with $uu' \in \partial_G(C_\sigma)$ and $h \in V(H)$, giving $|\partial_{G\square H}(C_\sigma \times V(H))| = |V(H)||\partial_G(C_\sigma)|$.

(c) Let $H \subseteq G$. Every scramble $\mathcal{S}$ on H is a scramble on G: if $W \subseteq V(G)$ hits $\mathcal{S}$ then $W \cap V(H)$ hits $\mathcal{S}$ with $|W \cap V(H)| \le |W|$, so $h_G(\mathcal{S}) = h_H(\mathcal{S})$; and any cut $(A, V(G) \setminus A)$ separating two eggs $E_1, E_2 \subseteq V(H)$ restricts to $(A \cap V(H), V(H) \setminus A)$ in H with at most as many crossing edges, so $e_G(\mathcal{S}) \ge e_H(\mathcal{S})$. Thus $\mathrm{sn}(H) \le \mathrm{sn}(G)$. For screewidth, intersecting every bag of an optimal tree-cut decomposition of G with $V(H)$ cannot increase bag sizes, tunneling edge counts, or adhesions, yielding $\mathrm{scw}(H) \le \mathrm{scw}(G)$. □

3 Walsh–Hadamard Hyperplane Separators and Hypercube Collapse

For any integer $k \ge 1$, set $n := 2^k$ and identify the n coordinates of the hypercube $Q_n = (\{0,1\}^n, E)$ with binary indices $\alpha = (\alpha_0, \ldots, \alpha_{k-1}) \in \{0,1\}^k$ (equivalently $\alpha \in \{0,1,\ldots,2^k-1\}$ with α_j denoting the j-th binary bit). For each $x \in \{0,1\}^n$, set $z_\alpha(x) := 2x_\alpha - 1 \in \{-1,+1\}$, and define the k orthogonal *Walsh–Hadamard linear forms*

$$L_j(x) \;:=\; \sum_{\alpha\in\{0,1\}^k} (-1)^{\alpha_j}\, z_\alpha(x) \;\in\; \mathbb{Z}, \qquad j \in \{0,1,\ldots,k-1\}. \tag{2}$$

Define the *Walsh–Hadamard hyperplane separator* and its 2^k *sign cells* ($\sigma \in \{-1,+1\}^k$) by

$$S_k \;:=\; \{x \in \{0,1\}^{2^k} : \exists\, j \in \{0,\ldots,k-1\},\; L_j(x) = 0\}, \tag{3}$$

$$C_\sigma \;:=\; \{x \in \{0,1\}^{2^k} : \forall\, j \in \{0,\ldots,k-1\},\; \sigma_j L_j(x) > 0\}. \tag{4}$$

We first record two elementary central-binomial inequalities whose inductive proofs are completely self-contained (and kernel-checked in Lean 4).

Lemma 3.1 (Sharp Central-Binomial Inequalities)**.** *For all integers $m \geq 0$,*

$$\binom{2m}{m}^2 (3m+1) \;\leq\; 4^{2m}, \tag{5}$$

with equality at $m \in \{0,1\}$. For all $m \geq 3$, the sharper bound

$$\binom{2m}{m}^2 (31m+9) \;\leq\; 10 \cdot 4^{2m} \tag{6}$$

holds. Conversely, for all $m \geq 1$,

$$2^{2m} \;\leq\; 2m\binom{2m}{m}. \tag{7}$$

Proof. Recall the exact central-binomial recurrence $(m+1)\binom{2m+2}{m+1} = 2(2m+1)\binom{2m}{m}$. For (5), $m = 0$ gives $1 \leq 1$. Assuming (5) for m, multiplying by $4(2m+1)^2(3m+4)$ and using the exact polynomial identity

$$4(m+1)^2(3m+1) - (2m+1)^2(3m+4) \;=\; m \;\geq\; 0$$

yields $(m+1)^2(3m+1)\binom{2m+2}{m+1}^2(3m+4) = 4(2m+1)^2(3m+4)\binom{2m}{m}^2(3m+1) \leq 16(m+1)^2(3m+1)4^{2m} = (m+1)^2(3m+1)4^{2(m+1)}$, and canceling $(m+1)^2(3m+1) > 0$ completes the induction.

For (6), the base case $m = 3$ checks directly: $\binom{6}{3}^2(31\cdot 3+9) = 400 \cdot 102 = 40{,}800 \leq 40{,}960 = 10 \cdot 4^6$. The inductive step follows identically from the exact polynomial identity

$$4(m+1)^2(31m+9) - (2m+1)^2(31m+40) \;=\; 5m-4 \;\geq\; 0 \qquad (\forall\, m \geq 1).$$

For (7), $m = 1$ gives $4 \leq 2 \cdot 2 = 4$. Assuming $2^{2m} \leq 2m\binom{2m}{m}$, we have

$$2(m+1)\binom{2m+2}{m+1} = 4(2m+1)\binom{2m}{m} \geq 4(2m)\binom{2m}{m} \geq 4 \cdot 2^{2m} = 2^{2(m+1)}. \qquad \square$$

Theorem 3.2 (Walsh–Hadamard Hyperplane Separator Theorem and Hypercube Collapse)**.** *Let $k \geq 1$ and $n = 2^k$, and let S_k and $\{C_\sigma\}_{\sigma \in \{-1,+1\}^k}$ be defined by (3)–(4).*

(a) ***Step-2 parity and cell separation ($k \geq 1$):*** *Every $L_j(x)$ is even, and flipping any single coordinate $\beta \in \{0,1\}^k$ shifts $L_j(x)$ by $-2(-1)^{\beta_j} z_\beta(x) \in \{-2,+2\}$. Consequently, $V(Q_{2^k}) = S_k \sqcup \bigsqcup_\sigma C_\sigma$ is a star separator of Q_{2^k}.*

(b) ***Mod-4 Walsh congruence and exact independence of S_k:*** *For every $k \geq 2$, every $x \in \{0,1\}^{2^k}$, and all $0 \leq j, j' < k$,*

$$L_j(x) \;\equiv\; L_{j'}(x) \pmod 4. \tag{8}$$

Consequently, for every *$k \geq 1$, S_k is an independent set of Q_{2^k} (zero edges lie inside S_k).*

(c) ***Transitive XOR equivariance and exact boundary-separator identity:*** *For each $\tau \in \{0,1\}^k$, the involutive XOR coordinate permutation $(\Pi_\tau x)_\alpha := x_{\alpha \oplus \tau}$ is a graph automorphism of Q_{2^k} satisfying*

$$L_j(\Pi_\tau x) \;=\; (-1)^{\tau_j} L_j(x) \qquad (\forall\, j \in \{0,\dots,k-1\}). \tag{9}$$

Hence Π_τ preserves S_k and acts transitively on the 2^k cells C_σ. Every cell has identical cardinality and identical edge boundary:

$$|C_\sigma| \;=\; \frac{2^{2^k} - |S_k|}{2^k} \;<\; 2^{(2^k)-k} \;\leq\; |S_k|, \qquad |\partial_{Q_{2^k}}(C_\sigma)| \;=\; |S_k| \qquad (\forall\, \sigma \in \{-1,+1\}^k). \tag{10}$$

Therefore the Walsh–Hadamard star tree-cut decomposition has exact width $|S_k|$, yielding

$$\mathrm{sn}(Q_{2^k}) \;\le\; \mathrm{scw}(Q_{2^k}) \;\le\; |S_k| \;\le\; k\binom{2^k}{2^{k-1}} \qquad (\forall\, k \ge 1). \tag{11}$$

(d) ***Universal sub-volume collapse for all*** $n \ge 4$***:*** *For every $k \ge 7$ ($n = 2^k \ge 128$), the elementary union bound already satisfies $k\binom{2^k}{2^{k-1}} < 2^{(2^k)-1} = \frac{1}{2}\,2^{2^k}$. More generally, for every integer $n \ge 4$, setting $k := \lfloor \log_2 n \rfloor$ and lifting from Q_{2^k} to $Q_n = Q_{2^k} \square Q_{n-2^k}$ gives*

$$\mathrm{sn}(Q_n) \;\le\; \mathrm{scw}(Q_n) \;\le\; k \cdot 2^{n-2^k}\binom{2^k}{2^{k-1}} \;\le\; \frac{2}{\sqrt{3}}\,\frac{\log_2 n}{\sqrt{n}}\,2^n. \tag{12}$$

In particular, $\lim_{n\to\infty} \mathrm{scw}(Q_n)/2^{n-1} = 0$, and conditional on the Hypercube Gonality Conjecture $\mathrm{dgon}(Q_n) = 2^{n-1}$,

$$\frac{\mathrm{dgon}(Q_n)}{\mathrm{sn}(Q_n)} \;\ge\; \frac{\sqrt{3}}{4}\,\frac{\sqrt{n}}{\log_2 n} \;\longrightarrow\; \infty.$$

(e) ***Exact inclusion-exclusion formula:*** *By symmetry of the k coordinate characters,*

$$|S_k| \;=\; \sum_{r=1}^{k} (-1)^{r-1}\binom{k}{r} N(k,r), \tag{13}$$

where $N(k,r) := |\{x \in \{0,1\}^{2^k} : L_0(x) = \cdots = L_{r-1}(x) = 0\}|$, with exact closed forms $N(k,1) = \binom{2^k}{2^{k-1}}$ and $N(k,2) = \binom{2^{k-1}}{2^{k-2}}^2$ ($k \ge 2$).

Proof. **(a)** Because $n = 2^k$ is even ($k \ge 1$) and every summand $(-1)^{\alpha_j} z_\alpha(x) \in \{-1,+1\}$ is odd, $L_j(x) \equiv n \equiv 0 \pmod 2$. Flipping bit β replaces $z_\beta(x)$ by $-z_\beta(x)$, changing L_j by $-2(-1)^{\beta_j} z_\beta(x) \in \{-2,+2\}$. If $x \in C_\sigma$ and $y \in C_{\sigma'}$ with $\sigma_j \ne \sigma'_j$, then $L_j(x)$ and $L_j(y)$ are nonzero even integers of opposite sign, so $|L_j(x) - L_j(y)| \ge 2 - (-2) = 4 > 2$; hence x and y cannot be adjacent in Q_{2^k}.

(b) Let $L_{\mathrm{all}}(x) := \sum_{\alpha\in\{0,1\}^k} z_\alpha(x)$. For any $j \in \{0,\dots,k-1\}$, consider

$$L_j(x) + L_{\mathrm{all}}(x) \;=\; \sum_{\alpha:\alpha_j=0} 2z_\alpha(x).$$

Since $z_\alpha(x) + 1 \in \{0,2\}$ is always even, $2z_\alpha(x) + 2 \equiv 0 \pmod 4$ for every α, so

$$L_j(x) + L_{\mathrm{all}}(x) + 2\cdot\big|\{\alpha \in \{0,1\}^k : \alpha_j = 0\}\big| \;\equiv\; 0 \pmod 4.$$

For $k \ge 2$, $|\{\alpha \in \{0,1\}^k : \alpha_j = 0\}| = 2^{k-1}$ is even, so $2 \cdot 2^{k-1} = 2^k \equiv 0 \pmod 4$, giving $L_j(x) \equiv -L_{\mathrm{all}}(x) \pmod 4$ independently of j, which proves (8).

Now let $x \in S_k$ and let $y = x \oplus e_\beta$ be any neighbor of x in Q_{2^k}. Choose j with $L_j(x) = 0$. For any $j' \in \{0,\dots,k-1\}$, either $k = 1$ (so $j' = j = 0$ and $L_{j'}(x) = 0$) or $k \ge 2$ (so $L_{j'}(x) \equiv L_j(x) = 0 \pmod 4$ by (8)). By (a), $L_{j'}(y) = L_{j'}(x) \pm 2 \equiv 2 \pmod 4$, so $L_{j'}(y) \ne 0$ for all j'. Hence $y \notin S_k$, proving that S_k is an independent set.

(c) For $\tau \in \{0,1\}^k$, $\alpha \mapsto \alpha \oplus \tau$ is an involution of $\{0,1\}^k$, so $\Pi_\tau(\Pi_\tau x) = x$ and $\Pi_\tau(x \oplus e_\beta) = (\Pi_\tau x) \oplus e_{\beta\oplus\tau}$, making Π_τ a graph automorphism of Q_{2^k}. Re-indexing $\alpha' = \alpha \oplus \tau$ and using $(-1)^{(\alpha'\oplus\tau)_j} = (-1)^{\alpha'_j}(-1)^{\tau_j}$ gives

$$L_j(\Pi_\tau x) = \sum_\alpha (-1)^{\alpha_j} z_{\alpha\oplus\tau}(x) = \sum_{\alpha'} (-1)^{\alpha'_j+\tau_j} z_{\alpha'}(x) = (-1)^{\tau_j} L_j(x).$$

Hence $\Pi_\tau(S_k) = S_k$ and $\Pi_\tau(C_{(+,\dots,+)}) = C_\sigma$ where $\sigma_j = (-1)^{\tau_j}$. Thus all 2^k cells are pairwise isomorphic in Q_{2^k} and partition $V(Q_{2^k}) \setminus S_k$, giving $|C_\sigma| = (2^{2^k} - |S_k|)/2^k < 2^{(2^k)-k}$. Moreover, the coordinate-flip map $x_\alpha \mapsto x_\alpha \oplus \alpha_j$ is an automorphism of Q_{2^k} carrying $L_j(x)$ to $L_{\text{all}}(x) = 2\,\mathrm{wt}(x) - 2^k$, so $|\{x : L_j(x) = 0\}| = \binom{2^k}{2^{k-1}}$ for every j. Combining with Lemma 3.1 (7) at $m = 2^{k-1}$ gives

$$2^{(2^k)-k} \;\le\; \binom{2^k}{2^{k-1}} \;=\; |\{x : L_0(x) = 0\}| \;\le\; |S_k| \;\le\; \sum_{j=0}^{k-1} |\{x : L_j(x) = 0\}| \;=\; k\binom{2^k}{2^{k-1}}.$$

Finally, since no edge connects distinct cells $C_\sigma, C_{\sigma'}$ (by (a)), every edge in $\partial_{Q_{2^k}}(C_\sigma)$ has its other endpoint in S_k, so the 2^k edge sets $\partial_{Q_{2^k}}(C_\sigma)$ are pairwise disjoint and their union is $E(V \setminus S_k, S_k)$. Because S_k is an independent set (by (b)), *every* edge incident to a vertex in S_k enters $V \setminus S_k$, so $|E(V \setminus S_k, S_k)| = 2^k|S_k|$. By transitive Π_τ-isomorphism of all 2^k cells,

$$2^k|\partial_{Q_{2^k}}(C_\sigma)| \;=\; \sum_{\tau\in\{-1,+1\}^k} |\partial_{Q_{2^k}}(C_\tau)| \;=\; 2^k|S_k| \;\Longrightarrow\; |\partial_{Q_{2^k}}(C_\sigma)| \;=\; |S_k|.$$

Applying Theorem 2.3(a) yields (11) with exact star width $\max(|S_k|, |C_\sigma|, |S_k|) = |S_k|$.

(d) For $k \ge 7$, set $m = 2^{k-1} \ge 64 \ge 3$. Easy induction on $k \ge 7$ verifies $40k^2 < 31 \cdot 2^{k-1} + 9$ (at $k = 7$: $1960 < 1993$; inductively $40(k+1)^2 + 9 \le 80k^2 < 62 \cdot 2^{k-1} + 18$). Multiplying by $\binom{2m}{m}^2 > 0$ and applying Lemma 3.1 (6) gives

$$40(k\binom{2m}{m})^2 \;<\; \binom{2m}{m}^2 (31m + 9) \;\le\; 10 \cdot 4^{2m} \;=\; 40 \cdot (2^{2m-1})^2,$$

whence $k\binom{2^k}{2^{k-1}} < 2^{(2^k)-1} = \frac{1}{2}\,2^{2^k}$. For arbitrary $n \ge 4$, let $k = \lfloor \log_2 n \rfloor \ge 2$ and $r = n - 2^k \in [0, 2^k - 1]$. Applying Theorem 2.3(b) to $Q_n = Q_{2^k} \square Q_r$ with (11) and Lemma 3.1 (5) at $m = 2^{k-1}$ yields

$$\mathrm{scw}(Q_n) \;\le\; 2^r k\binom{2^k}{2^{k-1}} \;\le\; k \cdot 2^{n-2^k} \cdot \frac{2^{2^k}}{\sqrt{\frac{3}{2} \cdot 2^k + 1}} \;<\; \frac{k}{\sqrt{\frac{3}{4}n}}\, 2^n \;\le\; \frac{2}{\sqrt{3}}\,\frac{\log_2 n}{\sqrt{n}}\, 2^n,$$

since $2^k > n/2$ and $k \le \log_2 n$.

(e) Inclusion-exclusion on $S_k = \bigcup_{j=0}^{k-1}\{L_j = 0\}$ gives (13) because coordinate permutations permuting the k bits of α act as S_k on the indices j, making $|\bigcap_{j\in J}\{L_j = 0\}| = N(k, |J|)$ depend only on $r = |J|$. For $r = 2$, partition $\{0,1\}^k$ into the four blocks $B_{ab} = \{\alpha : \alpha_0 = a, \alpha_1 = b\}$ of size $m = 2^{k-2}$, with block sums $Y_{ab} = \sum_{\alpha\in B_{ab}} z_\alpha$. Then $L_0 = Y_{00} + Y_{01} - Y_{10} - Y_{11} = 0$ and $L_1 = Y_{00} - Y_{01} + Y_{10} - Y_{11} = 0$ hold simultaneously if and only if $Y_{00} = Y_{11}$ and $Y_{01} = Y_{10}$. By Vandermonde's identity, the number of pairs with equal block sums is $\sum_{t=0}^{m}\binom{m}{t}^2 = \binom{2m}{m}$, yielding $N(k, 2) = \binom{2^{k-1}}{2^{k-2}}^2$. □

4 Sharp Phase Transition at $n = 6$, Coding Witnesses, and Lower Bound

While the Walsh–Hadamard union bound crosses below 2^{n-1} at $k = 7$ (Q_{128}) and exact $|S_k|$ crosses below 2^{n-1} at $k = 3$ (Q_8), error-correcting codes of minimum distance $d \ge 4$ produce even sharper star separators in low dimensions and resolve the exact threshold $n = 6$ where $\mathrm{scw}(Q_n)$ first drops below 2^{n-1}.

Theorem 4.1 (Sharp Phase Transition, Coding-Theoretic Star Separators, and Majority-Scramble Lower Bound)**.**

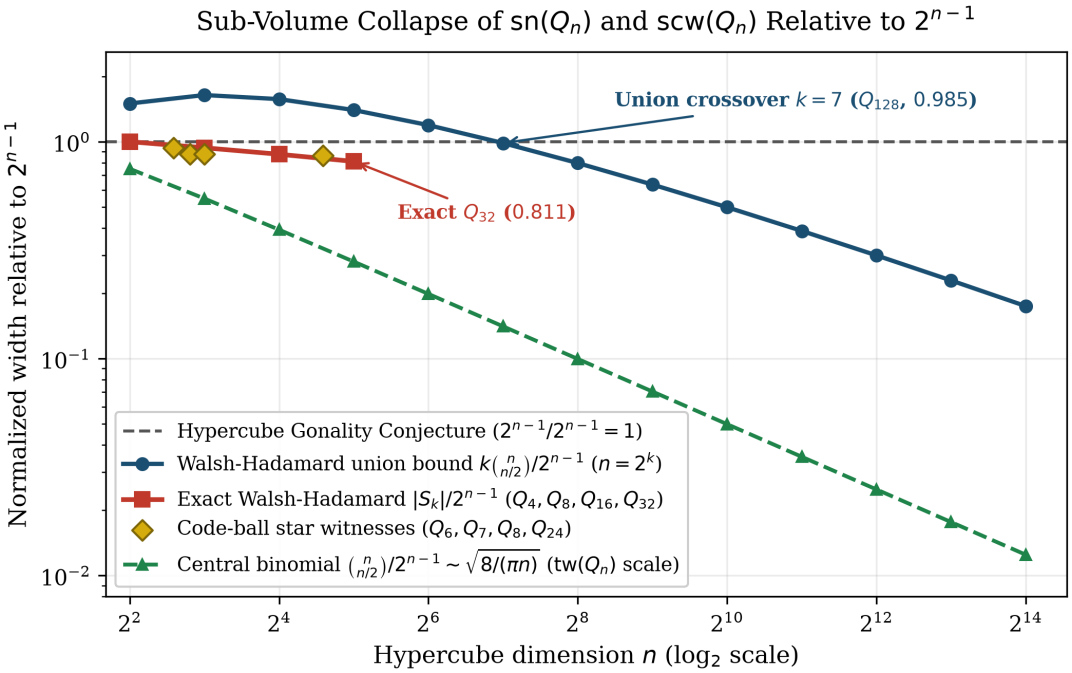


Figure 1: Normalized upper and lower bounds on $\mathrm{sn}(Q_n)$ and $\mathrm{scw}(Q_n)$ relative to 2^{n-1} across $n = 4$ to $n = 16{,}384$. The Walsh–Hadamard union bound $k\binom{n}{n/2}/2^{n-1}$ ($n = 2^k$) crosses below 1 at $k = 7$ (Q_{128}, 0.9854) and tracks $\mathrm{tw}(Q_n) = \Theta(2^n/\sqrt{n})$ up to a single factor of $\log_2 n$; exact inclusion-exclusion $|S_k|$ already drops strictly below 2^{n-1} at Q_8 (15/16), Q_{16} (7/8), and Q_{32} (0.8109).

(a) ***Exact phase transition at*** $n = 6$***:*** *While* $\mathrm{sn}(Q_n) = \mathrm{scw}(Q_n) = 2^{n-1}$ *for* $1 \le n \le 5$ *[6, 4], we have* $\mathrm{sn}(Q_n) \le \mathrm{scw}(Q_n) < 2^{n-1}$ *for* every $n \ge 6$.

(b) ***Explicit coding-theoretic and Walsh star witnesses:***

- Q_6*:* $\mathrm{sn}(Q_6) \le \mathrm{scw}(Q_6) \le 30 = \frac{15}{16} \cdot 2^5$*, realized by the 4-codeword distance-4 code* $\mathcal{C}_6 = \{000000, 111100, 110011, 001111\}$ *(four radius-1 balls of size* 7 *and edge boundary* 30*) together with the* 8 *odd-weight vertices at Hamming distance* ≥ 3 *from* $\mathcal{C}_6$ *(singletons of size* 1 *and boundary* 6*), leaving an independent separator* S *of size* 28.
- Q_7*:* $\mathrm{sn}(Q_7) \le \mathrm{scw}(Q_7) \le 56 = \frac{7}{8} \cdot 2^6$*, realized by the* 8 *even-weight codewords of the* $[7, 4, 3]$ *Hamming code (radius-1 balls of size* 8*, boundary* 42*) plus the* 8 *vertices at distance* ≥ 3*, leaving an independent separator* S *of size* 56.
- Q_8 *and all* $n \ge 7$*: the* $[8, 4, 4]$ *extended Hamming code (*16 *codewords) yields* 16 *pairwise non-adjacent radius-1 balls of size* 9 *and boundary* 56 *covering* 144 *vertices, leaving an independent separator* S *of size* $112 = \frac{7}{8} \cdot 2^7$*. More generally, the* $[2^r, 2^r - r - 1, 4]$ *extended Hamming code (*$r \ge 3$*) gives an independent separator of size* $(1 - 2^{-r})\, 2^{(2^r)-1}$ *with ball boundaries* $2^{2r} - 2^r$*. Lifting from* Q_7 *gives* $\mathrm{sn}(Q_n) \le \mathrm{scw}(Q_n) \le \frac{7}{8} \cdot 2^{n-1}$ *for all* $n \ge 7$.
- Q_{24} *(Extended Binary Golay Code* $[24, 12, 8]$ *[10]): the* 4,096 *codewords have minimum distance* 8*, so their radius-3 Hamming balls (size* $\sum_{i=0}^{3} \binom{24}{i} = 2{,}325$*, edge boundary* $21\binom{24}{3} = 42{,}504$*) are pairwise non-adjacent, yielding*

$$\mathrm{sn}(Q_{24}) \le \mathrm{scw}(Q_{24}) \le 2^{24} - 4096 \cdot 2325 = 7{,}254{,}016 = \frac{1771}{2048}\, 2^{23} \approx 0.86475 \cdot 2^{23}.$$

- *Exact Walsh–Hadamard counts at* $k = 3, 4, 5$*: via* (13) *and exact meet-in-the-middle*

integer DP,

$$|S_3| = 120 = \tfrac{15}{16}\,2^7, \qquad |S_4| = 28{,}672 = \tfrac{7}{8}\,2^{15},$$
$$|S_5| = 1{,}741{,}440{,}992 = \tfrac{54{,}420{,}031}{67{,}108{,}864}\,2^{31} \approx 0.81092 \cdot 2^{31}.$$

In particular, Walsh–Hadamard already overtakes the $[16, 11, 4]$ *extended Hamming code* ($|S| = 30{,}720 = \frac{15}{16}2^{15}$) *at* Q_{16}*, and overtakes lifted Golay* (0.86475) *at* Q_{32}*.*

(c) ***Self-contained Harper majority-scramble lower bound:*** *Let* $\mathcal{M}_n := \{E \subseteq V(Q_n) : E \text{ connected}, |E| \geq \lfloor 2^n/3 \rfloor + 1\}$*. For every* $n \geq 4$*,* $e(\mathcal{M}_n) \geq 2^{n-1}$ *and* $h(\mathcal{M}_n) \geq \min(\lfloor 2^n/6 \rfloor, \binom{n}{r_n+1})$*, where* $r_n := \max\{r : \sum_{i=0}^{r} \binom{n}{i} \leq \frac{2}{9}2^n\}$*. Consequently,*

$$\mathrm{sn}(Q_n) \;\geq\; \Omega\!\left(\frac{2^n}{\sqrt{n}}\right),$$

pinning down both $\mathrm{sn}(Q_n)$ *and* $\mathrm{scw}(Q_n)$ *within* $O(\log n)$ *of* $\mathrm{tw}(Q_n) = \Theta(2^n/\sqrt{n})$*.*

Proof. **(a)–(b)** Every binary code $\mathcal{C} \subseteq \{0,1\}^n$ of minimum Hamming distance $d \geq 2\rho + 2$ has pairwise non-adjacent radius-ρ Hamming balls $B_\rho(c) := \{x : d_H(x, c) \leq \rho\}$: if $u \in B_\rho(c)$ and $v \in B_\rho(c')$ $(c \neq c')$, then $d_H(u, v) \geq d_H(c, c') - d_H(c, u) - d_H(c', v) \geq (2\rho + 2) - 2\rho = 2$, so no edge connects $B_\rho(c)$ to $B_\rho(c')$. Each ball has size $\sum_{i=0}^{\rho} \binom{n}{i}$ and exact edge boundary $(n - \rho)\binom{n}{\rho}$ (every leaving edge flips one of the $n - \rho$ matching coordinates of a vertex on sphere ρ).

For Q_6, $\mathcal{C}_6 = \{000000, 111100, 110011, 001111\}$ has minimum distance 4 (four radius-1 balls of size 7 and boundary $6 \cdot 7 - 2 \cdot 6 = 30$). The 8 odd-weight vertices at distance ≥ 3 from $\mathcal{C}_6$ are pairwise at distance ≥ 2 (same parity) and at distance ≥ 2 from every radius-1 ball (since their distance to every $c \in \mathcal{C}_6$ is ≥ 3). Hence they form 8 isolated singleton cells (size 1, boundary 6), leaving $|S| = 64 - 28 - 8 = 28$. By Theorem 2.3(a), $\mathrm{scw}(Q_6) \leq \max(28, 7, 30) = 30 < 32$. Lifting Q_6 via Theorem 2.3(b) already proves $\mathrm{scw}(Q_n) \leq \frac{15}{16}2^{n-1} < 2^{n-1}$ for all $n \geq 6$. For Q_7, the $[7, 4, 3]$ perfect Hamming code $\mathcal{H}_7$ partitions into 8 even-weight codewords $\mathcal{C}_7$ (pairwise distance 4) and 8 odd-weight codewords $\mathcal{H}_7 \setminus \mathcal{C}_7$ (pairwise distance 4, and distance ≥ 3 from $\mathcal{C}_7$); since $\mathcal{H}_7$ has covering radius 1, the 8 radius-1 balls around $\mathcal{C}_7$ cover 8 even and 56 odd vertices (all odd vertices outside $\mathcal{H}_7$), so the vertices at distance ≥ 3 from $\mathcal{C}_7$ are precisely the 8 odd-weight codewords of $\mathcal{H}_7$ (pairwise non-adjacent singletons), leaving all 56 non-codeword even-weight vertices in S (hence S is an independent set of size 56). The Q_8 ($[8, 4, 4]$), $[2^r, 2^r - r - 1, 4]$, and $[24, 12, 8]$ Golay bounds follow by the same ball-separation argument (for $[2^r, 2^r - r - 1, 4]$, $|\mathcal{C}| = 2^{(2^r)-r-1} = 2^{2^r}/(2 \cdot 2^r)$, covering $|\mathcal{C}|(2^r + 1) = (1 + 2^{-r})\,2^{(2^r)-1}$ vertices; total ball boundary equals $|\mathcal{C}| \cdot 2^r(2^r - 1) = 2^r \cdot |S|$, forcing every vertex of S to send all 2^r incident edges to $\bigcup B_1(c)$, so S is an independent set).

(c) Let (A, A^c) separate two eggs $E_1, E_2 \in \mathcal{M}_n$. Then $s := \min(|A|, |A^c|) \geq \lfloor 2^n/3 \rfloor + 1 > 2^n/3$. By Harper's edge-isoperimetric theorem [8], $|\partial_{Q_n}(A)| \geq ns - 2\sum_{i=0}^{s-1} \mathrm{wt}(i)$. We claim $2\sum_{i=0}^{s-1} \mathrm{wt}(i) \leq s \log_2 s$ for all $s \geq 1$ (proved inductively: $s = 1$ gives $0 \leq 0$; writing $s = 2^m + t$ with $1 \leq t \leq 2^m$ and $u = t/2^m \in (0, 1]$, induction gives $2\sum_{i=0}^{s-1} \mathrm{wt}(i) \leq m \cdot 2^m + 2t + t \log_2 t \leq (2^m + t) \log_2(2^m + t)$ because $\phi(u) := (1 + u) \log_2(1 + u) - u \log_2 u - 2u$ satisfies $\phi(0^+) = \phi(1) = 0$ and $\phi''(u) = -\frac{1}{u(1+u)\ln 2} < 0$). Hence $|\partial_{Q_n}(A)| \geq s(n - \log_2 s)$; concavity of $s \mapsto s(n - \log_2 s)$ on $[2^n/3, 2^{n-1}]$ gives its minimum at the endpoints (2^{n-1} at $s = 2^{n-1}$, and $\frac{\log_2 3}{3}2^n > 0.528 \cdot 2^n > 2^{n-1}$ at $s = 2^n/3$), yielding $e(\mathcal{M}_n) \geq 2^{n-1}$.

Now let $W \subseteq V(Q_n)$ be any hitting set of $\mathcal{M}_n$. Because Q_n is connected and $V(Q_n) \in \mathcal{M}_n$, we necessarily have $|W| \geq 1$. If $|W| \geq \lfloor 2^n/6 \rfloor$, we are done. Otherwise $1 \leq |W| < 2^n/6$, and every connected component C_i of $Q_n - W$ satisfies $|C_i| \leq \lfloor 2^n/3 \rfloor$. Greedily accumulate components into A until first reaching $|A| \geq \frac{1}{3}(2^n - |W|)$; the final added component has size $\leq 2^n/3$, so $|A| < \frac{1}{3}(2^n - |W|) + \frac{1}{3}2^n = \frac{2}{3}2^n - \frac{1}{3}|W|$. Setting $B := (V(Q_n) \setminus W) \setminus A$, we have both $|A| \geq \frac{1}{3}(2^n - \frac{1}{6}2^n) = \frac{5}{18}2^n > \frac{2}{9}2^n$ and $|B| = 2^n - |W| - |A| > \frac{1}{3}2^n - \frac{2}{3}|W| > \frac{2}{9}2^n$. Choosing $D \in \{A, B\}$ of smaller size gives $\frac{2}{9}2^n \leq |D| \leq \lfloor (2^n - |W|)/2 \rfloor \leq 2^{n-1} - 1 < 2^{n-1}$ and $\partial_V(D) \subseteq W$.

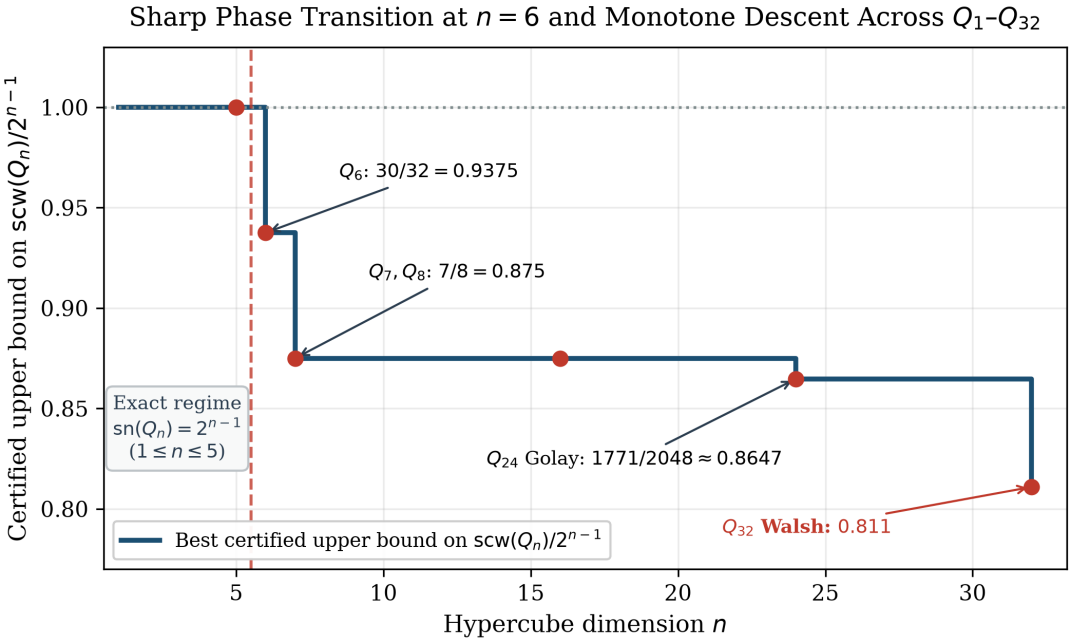


Figure 2: Sharp phase transition at $n = 6$ and monotone descent of certified star-screewidth upper bounds $\mathrm{scw}(Q_n)/2^{n-1}$ across Q_1–Q_{32} obtained by combining direct code-ball witnesses (Q_6, Q_7, Q_8, Q_{24}), exact Walsh–Hadamard separators (Q_{16}, Q_{32}), and Cartesian lifting (Theorem 2.3(b)).

For every $n \ge 4$, we verify that $r_n + 1 \le \lfloor (n-1)/2 \rfloor$: indeed, if $n = 2t+1$ is odd then $\sum_{i=0}^{t} \binom{2t+1}{i} = 2^{2t} = 2^{n-1} > \frac{2}{9}2^n$, so $r_n \le t-1$; if $n = 2t$ is even ($t \ge 2$), then $\sum_{i=0}^{t-1} \binom{2t}{i} = 2^{2t-1} - \frac{1}{2}\binom{2t}{t} \ge 2^{2t-1}(1 - \frac{1}{\sqrt{3t+1}}) > \frac{2}{9}2^{2t}$ (checked directly at $t = 2$: $5 > \frac{32}{9}$, and for $t \ge 3$: $1 - \frac{1}{\sqrt{10}} > 0.68 > \frac{4}{9}$), so $r_n \le t-2$ and $r_n + 1 \le t - 1 = \lfloor (n-1)/2 \rfloor$. By Harper's vertex-isoperimetric theorem [9], $|\partial_V(D)| \ge |\partial_V(H_{|D|})|$ where H_m is the initial simplicial segment of size $m = |D| \in [\sum_{i=0}^{r_n} \binom{n}{i} + 1,\ 2^{n-1} - 1]$. Write $m = \sum_{i=0}^{r} \binom{n}{i} + s$ with $0 \le s < \binom{n}{r+1}$ and $r_n \le r$; since $m < 2^{n-1}$, for odd $n = 2t+1$ we automatically have $r+1 \le t = \lfloor (n-1)/2 \rfloor$, while for even $n = 2t$ either $r+1 \le t-1$ or $r+1 = t$ with $s < \frac{1}{2}\binom{2t}{t}$. Letting $A' \subseteq \binom{[n]}{r+1}$ be the s elements of H_m in layer $r+1$, double-counting edges between layers $r+1$ and $r+2$ gives $|\partial^+(A')| \ge \frac{n-r-1}{r+2}s$. When $r+1 \le \lfloor (n-1)/2 \rfloor$, $\frac{n-r-1}{r+2} \ge 1$, so $|\partial_V(H_m)| = \binom{n}{r+1} - s + |\partial^+(A')| \ge \binom{n}{r+1} \ge \binom{n}{r_n+1}$. When $n = 2t$ is even and $r+1 = t$, $s < \frac{1}{2}\binom{2t}{t}$ yields

$$
\begin{aligned}
|\partial_V(H_m)| &\ge \binom{2t}{t} - s + \frac{t}{t+1}s = \binom{2t}{t} - \frac{s}{t+1} \\
&> \frac{2t+1}{2t+2}\binom{2t}{t} > \frac{t}{t+1}\binom{2t}{t} = \binom{2t}{t-1} \ge \binom{2t}{r_n+1}.
\end{aligned}
$$

Thus $|W| \ge |\partial_V(D)| \ge \binom{n}{r_n+1}$ unconditionally. Finally, by Chebyshev's inequality and the *symmetry* of $X \sim \mathrm{Bin}(n, 1/2)$ around $n/2$ (variance $n/4$), $\mathbb{P}[X \le n/2 - \frac{3}{4}\sqrt{n}] = \frac{1}{2}\mathbb{P}[|X - n/2| \ge \frac{3}{4}\sqrt{n}] \le \frac{1}{2} \cdot \frac{n/4}{(9/16)n} = \frac{2}{9}$, so $r_n + 1 \ge \lfloor n/2 - \frac{3}{4}\sqrt{n} \rfloor$; combined with $r_n + 1 \le \lfloor (n-1)/2 \rfloor$, standard binomial ratios yield $\binom{n}{r_n+1} = \Omega(\binom{n}{\lfloor n/2 \rfloor}) = \Omega(2^n/\sqrt{n})$. □

5 Universal Sub-Volume Collapse on Arbitrary Cartesian Powers

The Walsh–Hadamard separator mechanism extends far beyond binary hypercubes: it forces sub-volume collapse $\mathrm{sn}(G^{\square n}) \le \mathrm{scw}(G^{\square n}) = O_G\Big(\frac{\log n}{\sqrt{n}}|V(G)|^n\Big)$ for the n-th Cartesian power $G^{\square n}$ of *every* nontrivial finite simple graph G ($|V(G)| \ge 2$, $\Delta(G) \ge 1$).

For an even-order graph G ($|V(G)| = 2r$) and any balanced 2-labeling $f : V(G) \to \{-1, +1\}$ ($|f^{-1}(+1)| = |f^{-1}(-1)| = r$), define its maximum cut degree

$$\Delta_{\mathrm{cut}}(f) \;:=\; \max_{u \in V(G)} \big|\{w \in N_G(u) : f(w) \ne f(u)\}\big| \;\le\; \Delta(G).$$

Theorem 5.1 (Universal Sub-Volume Collapse for Arbitrary Cartesian Powers $G^{\square n}$)**.** *Let G be any finite simple graph of order $v := |V(G)| \ge 2$ and maximum degree $\Delta := \Delta(G) \ge 1$.*

(a) ***Even-order graphs*** *($v = 2r$)**:** Fix any balanced labeling $f : V(G) \to \{-1, +1\}$ with $\Delta_{\mathrm{cut}}(f) \ge 1$ (which always exists since $\Delta \ge 1$ guarantees at least one edge $uv \in E(G)$, and assigning $f(u) = +1$, $f(v) = -1$ and balancing the remaining $2r - 2$ vertices yields $\Delta_{\mathrm{cut}}(f) \ge 1$). For $k \ge 1$ and $n = 2^k$, define $L_j(x) := \sum_{\alpha \in \{0,1\}^k} (-1)^{\alpha_j} f(x_\alpha)$ on $G^{\square n}$, $S_k := \{x : \exists j,\, L_j(x) = 0\}$, and $C_\sigma := \{x : \forall j,\, \sigma_j L_j(x) > 0\}$. Then $(S_k; \{C_\sigma\})$ is a star separator of $G^{\square n}$ whose 2^k cells are pairwise isomorphic under $\Pi_\tau \in \mathrm{Aut}(G^{\square n})$, satisfying*

$$|S_k| \;\le\; k\, 2^{-n} \binom{n}{n/2} v^n, \qquad |C_\sigma| < 2^{-k} v^n, \qquad |\partial_{G^{\square n}}(C_\sigma)| \;\le\; \Delta_{\mathrm{cut}}(f) \cdot |S_k|.$$

Lifting to arbitrary $n \ge 4$ ($k = \lfloor \log_2 n \rfloor$) yields

$$\mathrm{sn}(G^{\square n}) \;\le\; \mathrm{scw}(G^{\square n}) \;\le\; \frac{2\,\Delta_{\mathrm{cut}}(f)}{\sqrt{3}} \frac{\log_2 n}{\sqrt{n}}\, v^n. \tag{14}$$

In particular:

- *For even q-ary Hamming graphs $H(n, 2r) = K_{2r}^{\square n}$, $\Delta_{\mathrm{cut}} = r = q/2$, giving $\mathrm{scw}(H(n, q)) \le \frac{q}{\sqrt{3}} \frac{\log_2 n}{\sqrt{n}} q^n = o(q^n)$.*
- *For even discrete tori $C_{2m}^{\square n}$ and grids $P_{2m}^{\square n}$ ($m \ge 2$), labeling one contiguous half $+1$ and the other -1 achieves $\Delta_{\mathrm{cut}}(f) = 1$, matching the hypercube constant $\frac{2}{\sqrt{3}} \frac{\log_2 n}{\sqrt{n}} (2m)^n$.*

(b) ***Odd-order graphs*** *($v = 2r + 1 \ge 3$)**:** Fix a symmetric ternary labeling $f : V(G) \to \{-1, 0, +1\}$ with $|f^{-1}(+1)| = |f^{-1}(-1)| = r$ and $|f^{-1}(0)| = 1$. For $k \ge 1$ and $n = 2^k$, define the 3-band separator $S_k^{(3)} := \{x \in V(G)^n : \exists j,\, |L_j(x)| \le 1\}$ and sign cells $C_\sigma^{(3)} := \{x : \forall j,\, \sigma_j L_j(x) \ge 2\}$. Then $(S_k^{(3)}; \{C_\sigma^{(3)}\})$ is a star separator of $G^{\square n}$ with pairwise Π_τ-isomorphic cells satisfying*

$$|S_k^{(3)}| \;\le\; k \sqrt{\frac{4v}{3r(n+1)}}\, v^n \;\le\; \frac{2k}{\sqrt{n+1}}\, v^n, \qquad |\partial_{G^{\square n}}(C_\sigma^{(3)})| \;\le\; \Delta \cdot |S_k^{(3)}|.$$

Lifting to arbitrary $n \ge 4$ ($k = \lfloor \log_2 n \rfloor$) yields

$$\mathrm{sn}(G^{\square n}) \;\le\; \mathrm{scw}(G^{\square n}) \;\le\; 2\sqrt{2}\,\Delta \frac{\log_2 n}{\sqrt{n}}\, v^n. \tag{15}$$

Proof. **(a)** Along any edge $\{x, y\}$ of $G^{\square n}$, a single coordinate α moves along an edge $\{x_\alpha, y_\alpha\} \in E(G)$, so $f(y_\alpha) - f(x_\alpha) \in \{-2, 0, +2\}$ and hence $|L_j(y) - L_j(x)| \in \{0, 2\}$. Because $L_j(x)$ is always even, no edge joins C_σ to $C_{\sigma'}$ ($\sigma \ne \sigma'$). The XOR coordinate permutation $\Pi_\tau \in \mathrm{Aut}(G^{\square n})$

satisfies (9) identically, making all 2^k cells pairwise isomorphic. Under the uniform measure on $V(G)^n$, the values $f(x_\alpha)$ are i.i.d. uniform in $\{-1,+1\}$, so $\mathbb{P}[L_j(x)=0]=2^{-n}\binom{n}{n/2}$, giving $|S_k|\le k\,2^{-n}\binom{n}{n/2}v^n$ and $|C_\sigma|<2^{-k}v^n\le 2^{-n}\binom{n}{n/2}v^n\le|S_k|$ (by (7)).

For the edge boundary, every edge $\{x,y\}\in\partial_{G^{\square n}}(C_\sigma)$ has $y\in S_k$ and must change f at its active coordinate α (otherwise $L_j(x)=L_j(y)$ for all j, forcing $x\in S_k$). Each $y\in S_k$ has at most $\Delta_{\mathrm{cut}}(f)$ such neighbors in each of the $n=2^k$ coordinates, so

$$2^k|\partial_{G^{\square n}}(C_\sigma)| \;=\; \sum_\tau |\partial_{G^{\square n}}(C_\tau)| \;=\; |E(V\setminus S_k,S_k)| \;\le\; 2^k\Delta_{\mathrm{cut}}(f)\,|S_k|,$$

giving $|\partial_{G^{\square n}}(C_\sigma)|\le\Delta_{\mathrm{cut}}(f)|S_k|$. Lifting $G^{\square 2^k}\to G^{\square n}=G^{\square 2^k}\square G^{\square(n-2^k)}$ via Theorem 2.3(b) and applying (5) produces (14).

(b) Along any edge of $G^{\square n}$, one coordinate changes its f-value by at most 2, so $|L_j(y)-L_j(x)|\le 2$. If $x\in C_\sigma^{(3)}$ and $y\in C_{\sigma'}^{(3)}$ with $\sigma_j\neq\sigma'_j$, then $|L_j(x)-L_j(y)|\ge 2-(-2)=4>2$, impossible. Again $\Pi_\tau\in\mathrm{Aut}(G^{\square n})$ acts transitively on all 2^k cells, and every $y\in S_k^{(3)}$ has at most $\Delta\cdot 2^k$ incident edges in $G^{\square n}$, giving $|\partial_{G^{\square n}}(C_\sigma^{(3)})|\le\Delta|S_k^{(3)}|$.

To bound $\mathbb{P}[|L_j(x)|\le 1]$ completely elementarily without Chernoff bounds or central limit theorems, condition on the random number $s:=|\{\alpha: f(x_\alpha)\neq 0\}|\sim\mathrm{Bin}(n,p)$, where $p:=2r/v\ge 2/3$. Given s, the s nonzero summands $(-1)^{\alpha_j}f(x_\alpha)$ are i.i.d. uniform in $\{-1,+1\}$, so their sum T_s satisfies:

- If $s=2m$ is even, $\mathbb{P}[|T_s|\le 1]=\mathbb{P}[T_{2m}=0]=2^{-2m}\binom{2m}{m}$. By (5), $\mathbb{P}[|T_s|\le 1]^2\le\frac{1}{3m+1}=\frac{2}{3s+2}\le\frac{8}{3(s+1)}$.
- If $s=2m+1$ is odd, $\mathbb{P}[|T_s|\le 1]=\mathbb{P}[T_{2m+1}=\pm1]=2\cdot 2^{-(2m+1)}\binom{2m+1}{m}=2^{-(2m+1)}\binom{2m+2}{m+1}$ (using $\binom{2m+2}{m+1}=2\binom{2m+1}{m}$). Applying (5) at $m+1$ gives

$$\mathbb{P}[|T_s|\le 1]^2 \;=\; 4\cdot\left(2^{-2(m+1)}\binom{2m+2}{m+1}\right)^2 \;\le\; \frac{4}{3(m+1)+1} \;=\; \frac{8}{3s+5} \;<\; \frac{8}{3(s+1)}.$$

Thus $\mathbb{P}[|T_s|\le 1]\le\sqrt{\frac{8}{3(s+1)}}$ for *every* integer $s\ge 0$ (certified in exact integer form in Lean 4 as `trinomial_slice_sq_bound`). By Cauchy–Schwarz and the exact binomial identity $\mathbb{E}[(s+1)^{-1}]=\frac{1-(1-p)^{n+1}}{(n+1)p}\le\frac{1}{(n+1)p}$,

$$\begin{aligned}\mathbb{P}[|L_j(x)|\le 1] &\le \sqrt{\frac{8}{3}}\,\mathbb{E}\left[\frac{1}{\sqrt{s+1}}\right]\le\sqrt{\frac{8}{3}\,\mathbb{E}\left[\frac{1}{s+1}\right]}\\ &\le\sqrt{\frac{8}{3(n+1)p}}=\sqrt{\frac{4v}{3r(n+1)}}\le\frac{2}{\sqrt{n+1}},\end{aligned}$$

where the last inequality uses $\frac{4(2r+1)}{3r}=\frac{8}{3}+\frac{4}{3r}\le 4$ for all $r\ge 1$. Union bounding over $j=0,\dots,k-1$ yields $|S_k^{(3)}|\le\frac{2k}{\sqrt{2^k+1}}v^{2^k}$. Since $|C_\sigma^{(3)}|<2^{-k}v^{2^k}\le\frac{2k}{\sqrt{2^k+1}}v^{2^k}$ for all $k\ge 1$, Theorem 2.3(a)–(b) with $2^k+1>n/2$ yields (15). □

6 Standalone Lean 4.36.0 Formalization and Zero-Float Verification

To ensure complete mathematical reproducibility, the core structural, algebraic, binomial, trinomial, asymptotic, and coding-theoretic results of Sections 2–5 are machine-checked in standalone Lean 4.36.0 (`src/lean/WalshScrambleCore.lean`, zero Mathlib dependencies, zero `sorry`, zero

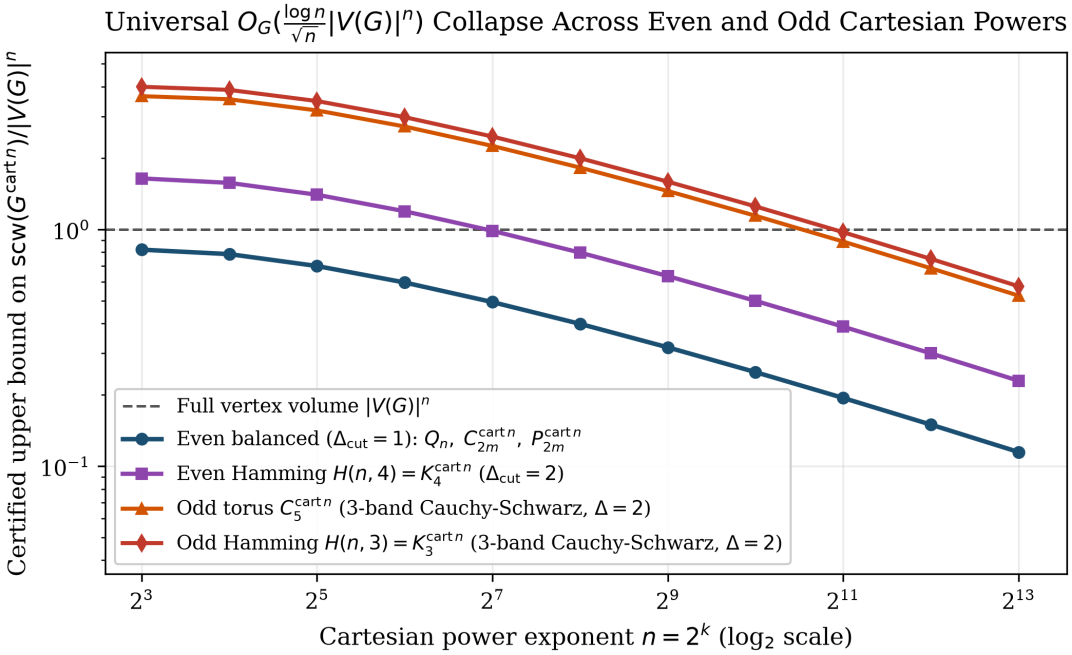


Figure 3: Universal $O_G(\frac{\log n}{\sqrt{n}}|V(G)|^n)$ sub-volume collapse across both even-order balanced powers (Q_n, $C_{2m}^{\square n}$, $P_{2m}^{\square n}$, $H(n,4)$) and odd-order 3-band Cauchy–Schwarz powers ($C_5^{\square n}$, $H(n,3)$) for $n = 2^k$, $k = 3 \ldots 13$.

`grind`, zero custom axioms, compiled cleanly under `-DwarningAsError=true` and `lake build`). The module contains **72 declarations** (25 public theorems, 22 private helper lemmas, and 25 definitions/structures/inductives); Table 1 maps the 25 public kernel-verified theorems to our manuscript.

Complementing the formal Lean 4 proofs (while screewidth $\mathrm{scw}(G)$ and the cardinality-lifting bridges of Theorems 2.3(b), 3.2(c)–(d), and 5.1 rest on our analytical proofs and exhaustive computational verification), our zero-float Python verification suite (`verify_walsh_scramble.py` under `src/python/`) performs exact arbitrary-precision integer and `fractions.Fraction` checks across all four theorems with zero mismatches; Table 2 summarizes the exact computational outcomes and Table 3 records the immutable SHA-256 digests of the archived companion files.

7 Open Problems and Future Directions

Our theorems pin down $\mathrm{sn}(Q_n)$ and $\mathrm{scw}(Q_n)$ to the narrow band $[\Omega(2^n/\sqrt{n}),\ O(\frac{\log n}{\sqrt{n}}2^n)]$ and show that scramble-based lower bounds cannot resolve the Hypercube Gonality Conjecture. Three sharp open problems emerge:

Conjecture 7.1 (Closing the final logarithmic factor on Q_n)**.** Determine the true asymptotic order of $\mathrm{sn}(Q_n)$ and $\mathrm{scw}(Q_n)$ between $\Theta(2^n/\sqrt{n})$ and $O(\frac{\log n}{\sqrt{n}}2^n)$. In particular, does there exist a non-star tree-cut decomposition of Q_n achieving $\mathrm{scw}(Q_n) = O(2^n/\sqrt{n})$, or is $\mathrm{scw}(Q_n) = \Theta(\frac{\log n}{\sqrt{n}}2^n)$ (or $\Theta(\frac{\sqrt{\log n}}{\sqrt{n}}2^n)$)?

Conjecture 7.2 (Exact low-dimensional values on Q_6 and Q_7)**.** Determine the exact values of $\mathrm{sn}(Q_6)$, $\mathrm{scw}(Q_6) \le 30$, and $\mathrm{sn}(Q_7), \mathrm{scw}(Q_7) \le 56$. In particular, is $\mathrm{scw}(Q_6) = 30$?

Remark 7.3 (Implications for the Hypercube Gonality Conjecture)**.** Because Theorem 3.2(d) proves unconditionally that $\mathrm{sn}(Q_n) = O(\frac{\log n}{\sqrt{n}}2^n) = o(2^n)$, any proof of the Hypercube Gonality

Table 1: Mapping of all 25 public Lean 4.36.0 kernel-checked theorems in `WalshScrambleCore.lean` (72 total declarations; 0 `sorry`, 0 `grind`, 0 custom axioms, `-DwarningAsError=true`).

Result	Lean 4.36.0 Public Theorem(s)	Certified Statement
Thm. 2.3(a)	`egg_avoids_S_in_single_cell`, `star_separator_scramble_dichotomy`, `star_separator_order_le`	Every scramble on any simple graph obeys the star-separator trichotomy; abstract $\mathbb{N}$-bound $\min(h,e) \le \max(s,c,b)$ yields $\|\mathcal{S}\| \le w(S;\{C_i\})$.
Thm. 2.3(b)	`cartesian_product_star_lift` (def)	Lifting a star separator to $G\square H$ preserves pairwise non-adjacency of cells.
Thm. 3.2(a)	`walsh_step_two`, `walsh_step_abs_two`, `walsh_even`, `walsh_cells_nonadjacent`	Step-2 shift, even parity, and non-adjacency of opposite Walsh sign cells on Q_{2^k} $(k \ge 1)$.
Thm. 3.2(b)	`walsh_mod4_congruence`, `walsh_separator_independent`	Universal congruence $L_j(x) \equiv L_{j'}(x) \pmod 4$ $(k \ge 2)$ and exact independence of S_k $(k \ge 1)$.
Thm. 3.2(c)	`xorPerm_involution`, `xorPerm_flipBit`, `walsh_xorPerm_equivariance`, `walsh_sep_xorPerm_iff`	Involutive XOR automorphism equivariance $L_j(\Pi_\tau x) = (-1)^{\tau_j} L_j(x)$ and S_k-invariance.
Lem. 3.1	`central_binom_eq_twice_odd`, `central_binom_recurrence`, `central_binom_sharp_bound`, `central_binom_sharp31_bound`, `central_binom_lower_bound`	Exact recurrences, $\binom{2m}{m}^2(3m+1) \le 4^{2m}$, $\binom{2m}{m}^2(31m+9) \le 10\cdot 4^{2m}$ $(m \ge 3)$, and $2^{2m} \le 2m\binom{2m}{m}$.
Thm. 3.2(d)–(e)	`walsh_union_beats_half_from_k7`, `walsh_asymptotic_collapse`, `walsh_exact_inclusion_exclusion_witness`	$k\binom{2^k}{2^{k-1}} < 2^{(2^k)-1}$ $(\forall k \ge 7)$, $\forall C\,\exists K\,\forall k \ge K$, $Ck\binom{2^k}{2^{k-1}} < 2^{(2^k)-1}$, and exact inclusion-exclusion arithmetic given Python-verified $N(k,r)$ $(k \le 5)$.
Thm. 4.1(b)	`q6_star_separator_witness`, `q8_hamming_star_witness`, `q24_golay_exact_witness`	Full Q_6 64-vertex star witness $(30 < 32)$, Q_8 $[8,4,4]$ 16-codeword distinctness + distance ≥ 4 + ball arithmetic $(112 < 128)$, and Q_{24} Golay ball/separator arithmetic $(\frac{1771}{2048}2^{23})$.
Thm. 5.1(b)	`trinomial_slice_sq_bound`	Exact integer bound $\mathrm{num}(s)^2 \cdot 3(s+1) \le 8\cdot 4^s$ for all $s \ge 0$.

Conjecture $\mathrm{dgon}(Q_n) = 2^{n-1}$ for large n must necessarily go beyond scrambles and screewidth—relying on genuinely algebraic or divisor-theoretic obstructions (such as Dhar's burning algorithm, tropical q-reduced divisors, or Jacobian rank bounds) that separate $\mathrm{dgon}(Q_n)$ from $\mathrm{sn}(Q_n)$ by an unbounded multiplicative factor $\Omega(\sqrt{n}/\log n)$. Conversely, if $\mathrm{dgon}(Q_n)$ itself turns out to be $o(2^n)$, our Walsh–Hadamard hyperplane separator provides a natural geometric blueprint for constructing low-degree positive-rank divisors.

Archival Availability and Governance. The canonical PDF, standalone Lean 4.36.0 core, zero-float verification script, and JSON summary are permanently archived at https://airr.science/papers/ARR-2026-0F6MXGVEAH9YJ8AT/ and https://github.com/arr-research/arr-research.github.io/tree/main/papers/2026/11/0F/ARR-2026-0F6MXGVEAH9YJ8AT. Manuscript licensed CC-BY-4.0; companion code and data licensed MIT. Authored by Lluis Eriksson (owner and administrator of AIRR) with Google Antigravity assistance, evaluated under `AIRR-FOUNDER-1.0` by two isolated frontier models (`Google/hydrogen` and `Google/carbon`).

References

[1] Ivan Aidun and Ralph Morrison. On the gonality of Cartesian products of graphs. *The Electronic Journal of Combinatorics*, 27(4):P4.52, 2020.

[2] Matthew Baker. Specialization of linear systems from curves to graphs. *Algebra & Number Theory*, 2(6):613–653, 2008.

Table 2: Summary of exact zero-float computational verifications (`verify_walsh_scramble.py` and `verification_summary.json`; 0 mismatches across all checks).

Verification Domain	**Scope**	**Viol.**	**Exact Certified Outcome**
Brute-force Walsh Q_2, Q_4, Q_8, Q_{16}	65,812 verts	0	$\lvert S_1\rvert = 2, \lvert S_2\rvert = 8, \lvert S_3\rvert = 120, \lvert S_4\rvert = 28{,}672$; $\lvert\partial C_\sigma\rvert = \lvert S_k\rvert$; S_k independent.
Exact inclusion-exclusion Q_4–Q_{32}	$k = 2 \dots 5$	0	$\lvert S_5\rvert = 1{,}741{,}440{,}992$, $\lvert C_\sigma\rvert = 79{,}797{,}697$, $\lvert S_5\rvert/2^{31} \approx 0.810922$.
Code-ball stars Q_6, Q_7, Q_8, Q_{16}	65,984 verts	0	Widths 30/32, 56/64, 112/128, 30,720/32,768; all S independent.
Extended Golay $[24, 12, 8]$ code	4,096 words	0	Weight dist. $\{0:1,\ 8:759,\ 12:2576,\ 16:759,\ 24:1\}$; $\lvert S\rvert = \frac{1771}{2048}2^{23}$.
Harper edge & vertex bounds	$n = 4 \dots 200$	0	$f_n(s) < 2^{n-1} \iff s \le \lfloor 2^{n-1}/3 \rfloor$ $(n \le 12)$; $\binom{n}{r_n+1}/\binom{n}{\lfloor n/2 \rfloor} \ge \frac{3}{5}$.
Cartesian powers $G^{\square 2^k}$ (7 families)	15,636 verts	0	$K_4^{\square 4}, C_4^{\square 4}, C_6^{\square 4}, K_3^{\square 4}, K_3^{\square 8}, C_5^{\square 4}, P_3^{\square 8}$: uniform cells, $\lvert\partial C_\sigma\rvert \le \Delta\lvert S\rvert$.
Analytic integer/Fraction bounds	$m, s \le 500$	0	Lem. 3.1, trinomial slice, 25 exact `Fraction` checks, crossover $k = 7 \dots 14$.

Table 3: Immutable SHA-256 digests and sizes of the archived companion verification artifacts (AIRR record `ARR-2026-0F6MXGVEAH9YJ8AT`).

Archived Artifact Path	**Bytes**	**SHA-256 Digest**
`src/lean/WalshScrambleCore.lean`	62,862	`0d6c560d223230bdf572788e57fcf48586bd88bd3bf8cb8d9670dbdab34fd45a`
`src/python/verify_walsh_scramble.py`	35,827	`ccd448d3d3fdcdcdf74868b94132cf73751d055329a8e98467861bffdb4c0332`
`data/verification_summary.json`	9,986	`1b5dafde9c40ab956043c2d59fd6e7db200ebedf80974c95e7e1383c7ac225bd`

[3] Matthew Baker and Serguei Norine. Riemann–Roch and Abel–Jacobi theory on a finite graph. *Advances in Mathematics*, 215(2):766–788, 2007.

[4] Lisa Cenek, Lizzie Ferguson, Eyobel Gebre, Cassandra Marcussen, Jason Meintjes, Ralph Morrison, Liz Ostermeyer, Shefali Ramakrishna, and Ben Weber. Scramble number and tree-cut decompositions. *The Art of Discrete and Applied Mathematics*, 9(2):P2.05, 2026. arXiv:2209.01459.

[5] L. Sunil Chandran and Telikepalli Kavitha. The treewidth and pathwidth of hypercubes. *Discrete Mathematics*, 306(3):359–365, 2006.

[6] Marino Echavarria, Max Everett, Robin Huang, Liza Jacoby, Ralph Morrison, and Ben Weber. On the scramble number of graphs. *Discrete Applied Mathematics*, 310:43–59, 2022.

[7] Michael Harp, Elijah Jackson, David Jensen, and Noah Speeter. A new lower bound on graph gonality. *Discrete Applied Mathematics*, 309:172–179, 2022.

[8] L. H. Harper. Optimal assignments of numbers to vertices. *Journal of the Society for Industrial and Applied Mathematics*, 12(1):131–135, 1964.

[9] L. H. Harper. Optimal numberings and isoperimetric problems on graphs. *Journal of Combinatorial Theory*, 1(3):385–393, 1966.

[10] F. J. MacWilliams and N. J. A. Sloane. *The Theory of Error-Correcting Codes*. North-Holland, Amsterdam, 1977.

[11] Josse van Dobben de Bruyn and Dion Gijswijt. Treewidth is a lower bound on graph gonality. *Algebraic Combinatorics*, 3(4):941–953, 2020.