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Anisotropic Hubbard–Stratonovich Sub-Gaussian Concentration, Spectral-Span Glauber Gaps, Free-Energy Sandwiches, and Projective Stability for Frustrated Ising Spin Systems

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Abstract

We establish four self-contained, non-asymptotic structural theorems governing concentration, spectral mixing, variational free energy, and measure perturbation stability for finite Ising spin systems $\mu_{J,h}(\sigma) \propto \exp(\frac{1}{2}\sigma^\top J\sigma + \langle h, \sigma\rangle)$ on $\{-1,+1\}^n$ ($n \geq 2$) with arbitrary symmetric (possibly frustrated) coupling matrix $J = J^\top$ and external field $h \in \mathbb{R}^n$. Exploiting diagonal shift invariance $\mu_{J+D,h} = \mu_{J,h}$ across diagonal matrices D satisfying $0 \prec A := J + D \prec I_n$ (nonempty whenever the spectral span $\Delta_\lambda(J) := \lambda_{\max}(J) - \lambda_{\min}(J) < 1$), we first combine the Hubbard–Stratonovich Gaussian decomposition $Y = A\sigma + \xi$ ($\xi \sim \mathcal{N}(0, A)$) with the Brascamp–Lieb inequality on exponentially tilted log-concave marginals to prove the exact anisotropic sub-Gaussian cumulant bound $\ln \mathbb{E}[\exp\langle w, \sigma - \mathbb{E}\sigma\rangle] \leq \frac{1}{2} w^\top (I_n - A)^{-1} w$ for all $w \in \mathbb{R}^n$. This yields the anisotropic covariance ceiling $\mathrm{Cov}(\sigma) \preceq (I_n - A)^{-1}$, the spectral-span bound $\|\mathrm{Cov}(\sigma)\|_{\mathrm{op}} \leq (1 - \Delta_\lambda(J))^{-1}$, and exact external-field KL and Total Variation perturbation bounds. Second, via an elementary eight-line variance-chain proof combining product Efron–Stein, Brascamp–Lieb, and conditional variance monotonicity, we prove the universal continuous-time heat-bath Glauber spectral-gap floor $\lambda_1(-L_G) \geq 1 - \lambda_{\max}(A)$ for every admissible shift $0 \prec A = J + D \prec I_n$, yielding $\lambda_1(-L_G) \geq 1 - \Delta_\lambda(J) > 0$ whenever $\Delta_\lambda(J) < 1$, for arbitrary frustrated couplings and external fields. Third, assuming $\mathrm{diag}(J) = 0$ and centering Hubbard–Stratonovich at the unique shifted mean-field fixed point $m^* = \tanh(Am^* + h)$, we derive a closed-form two-sided free-energy sandwich $-\mathcal{F}_{J,h}(m^*) \leq \ln Z(J,h) \leq -\mathcal{F}_{J,h}(m^*) - \frac{1}{2}\sum_{i=1}^n d_i(1 - (m_i^*)^2) - \frac{1}{2}\ln\det(I_n - A)$ whose zero-field gap cancels all first-order terms in A and expands as $\frac{1}{4}\mathrm{tr}(J^2) + \frac{1}{4}\mathrm{tr}(D^2) + O(n\|A\|_{\mathrm{op}}^3)$, explicitly isolating the positive second-order shift overhead $\frac{1}{4}\mathrm{tr}(D^2) > \frac{1}{4}\lambda_{\min}(J)^2$ for $J \neq 0$. Fourth, for arbitrary temperatures we prove exact Hilbert projective stability bounds for joint coupling-and-field perturbations $(J_1, h_1) \to (J_2, h_2)$, yielding sharp nonlinear hyperbolic Total Variation $\tanh(d_H/4)$ and Jeffreys-divergence $d_H \tanh(d_H/4)$ bounds together with a high-temperature magnetization Lipschitz law. All four theorems are verified against exact 2^n partition sums and $2^n \times 2^n$ Glauber generators across 240 spin-glass benchmarks with zero violations.

Keywords: Ising model, frustrated spin glasses, Hubbard–Stratonovich transform, Brascamp–Lieb inequality, sub-Gaussian concentration, Glauber spectral gap, mean-field free energy, Hilbert projective metric.
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1 Introduction

Finite Ising spin systems on the hypercube $\{-1,+1\}^n$ ($n \geq 2$), defined for any real symmetric coupling matrix $J = J^\top \in \mathbb{R}^{n\times n}$ and external field $h \in \mathbb{R}^n$ by

$$\mu_{J,h}(\sigma) := \frac{1}{Z(J,h)} \exp\left(\frac{1}{2}\sigma^\top J\sigma + \langle h,\sigma\rangle\right), \qquad Z(J,h) := \sum_{\sigma\in\{-1,+1\}^n} \exp\left(\frac{1}{2}\sigma^\top J\sigma + \langle h,\sigma\rangle\right), \tag{1}$$

sit at the intersection of statistical physics, Markov chain mixing, high-dimensional probability, and graphical models [6, 7, 8]. A classical high-temperature uniqueness and mixing criterion is Dobrushin's ℓ_∞ condition $\max_i \sum_{j\neq i} |J_{ij}| < 1$ [5]. Recent advances via Polchinski renormalization, Hubbard–Stratonovich decompositions, and spectral independence [2, 4, 3] revealed that the sharper spectral condition $\Delta_\lambda(J) := \lambda_{\max}(J) - \lambda_{\min}(J) < 1$ suffices for covariance bounds and rapid Glauber mixing even on dense frustrated spin glasses where $\max_i \sum_j |J_{ij}| \gg 1$.

The unifying algebraic mechanism is *diagonal shift invariance*: because $\sigma_i^2 = 1$ for every $\sigma \in \{-1,+1\}^n$, replacing J by $A := J + D$ for any diagonal matrix $D = \mathrm{diag}(d_1,\dots,d_n)$ shifts the energy by the constant $\frac{1}{2}\operatorname{tr}(D)$ and leaves the Gibbs measure invariant ($\mu_{J+D,h} = \mu_{J,h}$), while multiplying the partition function by $e^{\frac{1}{2}\operatorname{tr}(D)}$. Whenever D can be chosen so that $0 \prec A = J + D \prec I_n$, the auxiliary Gaussian variable $Y = A\sigma + \xi$ ($\xi \sim \mathcal{N}(0,A)$ independent of σ) renders $\sigma \mid Y$ an exact product Rademacher measure while giving the marginal density $p(y)$ a uniformly strictly log-concave potential $U(y)$ with $\nabla^2 U(y) \succeq A^{-1} - I_n \succ 0$.

In this work we exploit this single Hubbard–Stratonovich mechanism to deliver four concise, self-contained, sharp non-asymptotic theorems:

1. *Anisotropic sub-Gaussian concentration and external-field KL/TV stability:* By applying the Brascamp–Lieb inequality [1] to exponentially tilted log-concave densities $p_v(y) \propto e^{\langle v,y\rangle - U(y)}$, we obtain the exact anisotropic CGF bound $\ln \mathbb{E}[e^{\langle w,\sigma-\mathbb{E}\sigma\rangle}] \leq \frac{1}{2} w^\top (I_n - A)^{-1} w$ for all $w \in \mathbb{R}^n$ without requiring Herbst's log-Sobolev machinery, simultaneously yielding $\mathrm{Cov}(\sigma) \preceq (I_n - A)^{-1}$ and sharp KL/TV field perturbation bounds.

2. *Elementary Glauber spectral-gap floor:* Using only product Efron–Stein, Brascamp–Lieb, Cauchy–Schwarz, and conditional variance monotonicity, we prove in eight lines that the continuous-time heat-bath Glauber generator L_G satisfies $\lambda_1(-L_G) \geq 1 - \lambda_{\max}(A)$ for every admissible shift $0 \prec A = J + D \prec I_n$, and hence $\lambda_1(-L_G) \geq 1 - \Delta_\lambda(J) > 0$ whenever $\Delta_\lambda(J) < 1$, for every frustrated J and every external field h.

3. *Two-sided free-energy sandwich:* Centering the Gaussian integral at the unique shifted mean-field fixed point $m^* = \tanh(Am^* + h)$, we prove a closed-form two-sided bound trapping $\ln Z(J,h)$ between $-\mathcal{F}_{J,h}(m^*)$ and an explicit log-determinant Hubbard–Stratonovich ceiling whose zero-field ($h = 0$) gap cancels identically at first order in A and expands at second order as $\frac{1}{4}\operatorname{tr}(J^2) + \frac{1}{4}\operatorname{tr}(D^2) + O(n\|A\|_{\mathrm{op}}^3)$.

4. *Projective stability across arbitrary temperatures:* Without high-temperature restrictions, we bound Hilbert's projective distance $d_H(\mu_1,\mu_2)$ under joint coupling-and-field shifts $(J_1,h_1) \to (J_2,h_2)$, yielding sharp nonlinear Total Variation $\tanh(d_H/4)$ and Jeffreys-divergence $d_H \tanh(d_H/4)$ bounds alongside a high-temperature magnetization Lipschitz law.

Notation and Preliminaries

For any symmetric matrix $M \in \mathbb{R}^{n\times n}$, $\lambda_{\min}(M) \leq \cdots \leq \lambda_{\max}(M)$ denote its eigenvalues, $\Delta_\lambda(M) := \lambda_{\max}(M) - \lambda_{\min}(M) \geq 0$ its spectral span, and $\|M\|_{\mathrm{op}} := \max(|\lambda_{\max}(M)|, |\lambda_{\min}(M)|)$ its operator norm. We write $M \succ 0$ (resp. $M \succeq 0$) for positive (semi)definiteness. The

binary entropy is $H_2(p) := -p\ln p - (1-p)\ln(1-p)$ for $p \in (0,1)$, and for probability distributions μ, ν on $\{-1,+1\}^n$ with full support, Hilbert's projective distance [9] is $d_H(\mu,\nu) := \max_\sigma \ln(\mu(\sigma)/\nu(\sigma)) - \min_\tau \ln(\mu(\tau)/\nu(\tau))$, Total Variation is $\|\mu-\nu\|_{\mathrm{TV}} := \frac{1}{2}\sum_\sigma |\mu(\sigma)-\nu(\sigma)|$, Kullback–Leibler divergence is $D_{\mathrm{KL}}(\mu\|\nu) := \sum_\sigma \mu(\sigma)\ln(\mu(\sigma)/\nu(\sigma))$, and Jeffreys divergence is $D_J(\mu,\nu) := D_{\mathrm{KL}}(\mu\|\nu) + D_{\mathrm{KL}}(\nu\|\mu)$.

We recall the finite-dimensional Brascamp–Lieb inequality [1]: if $p(y) \propto e^{-U(y)}$ is a probability density on $\mathbb{R}^n$ with $U \in C^2(\mathbb{R}^n)$ and $\nabla^2 U(y) \succeq H \succ 0$ uniformly for all $y \in \mathbb{R}^n$, then every smooth $g \in L^2(p)$ satisfies

$$\mathrm{Var}_p(g(Y)) \le \mathbb{E}_p\Big[\nabla g(Y)^\top (\nabla^2 U(Y))^{-1} \nabla g(Y)\Big] \le \mathbb{E}_p\Big[\nabla g(Y)^\top H^{-1} \nabla g(Y)\Big]. \tag{2}$$

2 Anisotropic Hubbard–Stratonovich Sub-Gaussian Concentration

We first record the exact Hubbard–Stratonovich Gaussian channel and its log-concave marginal potential.

Lemma 2.1 (Hubbard–Stratonovich Gaussian Channel)**.** *Let $J = J^\top \in \mathbb{R}^{n\times n}$, $h \in \mathbb{R}^n$, and let $D = \mathrm{diag}(d_1,\dots,d_n)$ be any diagonal matrix such that $A := J + D \succ 0$. Let $\sigma \sim \mu_{J,h}$ and $\xi \sim \mathcal{N}(0,A)$ be independent, and set $Y := A\sigma + \xi \in \mathbb{R}^n$. Then:*

(a) *The conditional law of σ given $Y = y$ is the product Rademacher distribution*

$$\mu(\sigma \mid Y = y) = \prod_{i=1}^n \frac{e^{\sigma_i(y_i+h_i)}}{2\cosh(y_i+h_i)}, \tag{3}$$

with conditional mean $\mathbb{E}[\sigma_i \mid Y = y] = \tanh(y_i + h_i)$ and conditional covariance $\mathrm{Cov}(\sigma \mid Y = y) = \mathrm{diag}(\mathrm{sech}^2(y_i+h_i))$.

(b) *The marginal Lebesgue density of Y is $p(y) = C^{-1}e^{-U(y)}$ with potential*

$$U(y) := \frac{1}{2}y^\top A^{-1} y - \sum_{i=1}^n \ln\big(2\cosh(y_i+h_i)\big), \qquad \nabla^2 U(y) = A^{-1} - \mathrm{diag}\big(\mathrm{sech}^2(y_i+h_i)\big). \tag{4}$$

In particular, whenever $0 \prec A \prec I_n$, $\nabla^2 U(y) \succeq A^{-1} - I_n \succ 0$ uniformly for all $y \in \mathbb{R}^n$.

Proof. Since $\sigma_i^2 = 1$, $\mu_{J,h}(\sigma) = \mu_{A,h}(\sigma) = Z(A,h)^{-1}\exp(\frac{1}{2}\sigma^\top A\sigma + \langle h,\sigma\rangle)$. The joint density of (σ, Y) at (σ, y) is proportional to

$$\exp\left(\frac{1}{2}\sigma^\top A\sigma + \langle h,\sigma\rangle - \frac{1}{2}(y-A\sigma)^\top A^{-1}(y-A\sigma)\right) = \exp\left(-\frac{1}{2}y^\top A^{-1}y + \langle y+h,\sigma\rangle\right),$$

where the quadratic term $\frac{1}{2}\sigma^\top A\sigma$ cancels identically. Summing over $\sigma \in \{-1,+1\}^n$ yields the marginal factor $\prod_{i=1}^n 2\cosh(y_i+h_i)$, proving both (3) and (4). Since $\frac{d^2}{dt^2}\ln\cosh(t) = \mathrm{sech}^2(t) \in (0,1]$, we have $\nabla^2 U(y) = A^{-1} - \mathrm{diag}(\mathrm{sech}^2(y_i+h_i)) \succeq A^{-1} - I_n$; when $0 \prec A \prec I_n$, $A^{-1} \succ I_n$, so $A^{-1} - I_n \succ 0$. □

Theorem 2.2 (Anisotropic Sub-Gaussian Concentration, Covariance Ceiling, and Field Stability)**.** *Let $J = J^\top \in \mathbb{R}^{n\times n}$, $h \in \mathbb{R}^n$, and let $D = \mathrm{diag}(d_1,\dots,d_n)$ satisfy $0 \prec A := J + D \prec I_n$.*

(a) ***Exact anisotropic sub-Gaussian CGF bound.*** *For every $w \in \mathbb{R}^n$,*

$$\ln \mathbb{E}_{\mu_{J,h}}[\exp\langle w, \sigma - \mathbb{E}\sigma\rangle] \le \frac{1}{2}\, w^\top (I_n - A)^{-1} w, \tag{5}$$

and consequently, for every $w \in \mathbb{R}^n \setminus \{0\}$ and every $t \ge 0$, $\mathbb{P}_{\mu_{J,h}}(\langle w, \sigma - \mathbb{E}\sigma\rangle \ge t) \le \exp\big(-t^2/(2\,w^\top(I_n-A)^{-1}w)\big)$ (while at $w = 0$ the centered statistic vanishes identically).

(b) ***Anisotropic and spectral-span covariance ceilings.*** *The spin covariance matrix* $\Sigma(J,h) := \mathrm{Cov}_{\mu_{J,h}}(\sigma)$ *satisfies*

$$\Sigma(J,h) \preceq (I_n - A)^{-1}. \tag{6}$$

In particular, whenever $\Delta_\lambda(J) = \lambda_{\max}(J) - \lambda_{\min}(J) < 1$, *letting* $D = (-\lambda_{\min}(J) + \varepsilon)I_n$ *with* $\varepsilon \to 0^+$ *yields*

$$\Sigma(J,h) \preceq \big((1 + \lambda_{\min}(J))I_n - J\big)^{-1} \preceq \frac{1}{1 - \Delta_\lambda(J)}\, I_n. \tag{7}$$

(c) ***External-field KL and TV stability.*** *For any second external field* $h_2 \in \mathbb{R}^n$, *setting* $\Delta h := h_2 - h$,

$$D_{\mathrm{KL}}(\mu_{J,h} \| \mu_{J,h_2}) \le \frac{1}{2}\, \Delta h^\top (I_n - A)^{-1} \Delta h, \qquad \|\mu_{J,h} - \mu_{J,h_2}\|_{\mathrm{TV}} \le \frac{1}{2}\sqrt{\Delta h^\top (I_n - A)^{-1} \Delta h}. \tag{8}$$

Proof. (a) Let $Y = A\sigma + \xi$ as in Lemma 2.1, and define the cumulant generating function $\Psi_Y(v) := \ln \mathbb{E}[\exp\langle v, Y - \mathbb{E}Y\rangle]$ for $v \in \mathbb{R}^n$. Since $\nabla^2 U(y) \succeq H := A^{-1} - I_n \succ 0$, Ψ_Y is finite and C^∞ on $\mathbb{R}^n$. For any fixed $v \in \mathbb{R}^n$, the Hessian $\nabla^2 \Psi_Y(v)$ equals the covariance matrix $\mathrm{Cov}_{p_v}(Y)$ under the exponentially tilted density $p_v(y) \propto \exp(\langle v, y\rangle - U(y))$. Because adding the linear term $-\langle v, y\rangle$ to $U(y)$ leaves the Hessian unchanged ($\nabla^2(U(y) - \langle v, y\rangle) = \nabla^2 U(y) \succeq H$), applying the Brascamp–Lieb inequality (2) to $g(y) = \langle u, y\rangle$ under p_v yields

$$u^\top \nabla^2 \Psi_Y(v)\, u = \mathrm{Var}_{p_v}(\langle u, Y\rangle) \le u^\top H^{-1} u \qquad \forall u, v \in \mathbb{R}^n.$$

Since $\Psi_Y(0) = 0$ and $\nabla \Psi_Y(0) = 0$, Taylor's theorem with integral remainder gives $\Psi_Y(v) = \int_0^1 (1 - s)\, v^\top \nabla^2 \Psi_Y(sv)\, v\, ds \le \frac{1}{2} v^\top H^{-1} v$ for all $v \in \mathbb{R}^n$. On the other hand, independence of σ and $\xi \sim \mathcal{N}(0, A)$ gives

$$\Psi_Y(v) = \ln \mathbb{E}_{\mu_{J,h}}[\exp\langle Av, \sigma - \mathbb{E}\sigma\rangle] + \frac{1}{2} v^\top A v.$$

Subtracting $\frac{1}{2} v^\top A v$ and setting $v := A^{-1} w$ (valid since $A \succ 0$), we observe the exact matrix identity

$$A^{-1}\big((A^{-1} - I_n)^{-1} - A\big)A^{-1} = A^{-1}\big(A(I_n - A)^{-1} - A\big)A^{-1} = \big((I_n - A)^{-1} - I_n\big)A^{-1} = (I_n - A)^{-1},$$

which proves (5). Applying Markov's inequality to $\exp(\lambda\langle w, \sigma - \mathbb{E}\sigma\rangle)$ and optimizing over $\lambda > 0$ gives the tail bound.

(b) Dividing (5) (at εw) by $\frac{1}{2}\varepsilon^2$ and letting $\varepsilon \to 0$ yields $w^\top \Sigma(J,h) w \le w^\top (I_n - A)^{-1} w$ for all $w \in \mathbb{R}^n$, proving (6). If $\Delta_\lambda(J) < 1$, then for every $\varepsilon \in (0, 1 - \Delta_\lambda(J))$ the uniform shift $D_\varepsilon := (-\lambda_{\min}(J) + \varepsilon)I_n$ satisfies $0 \prec J + D_\varepsilon \prec I_n$; letting $\varepsilon \to 0^+$ in (6) yields (7).

(c) Directly from (1),

$$D_{\mathrm{KL}}(\mu_{J,h} \| \mu_{J,h_2}) = \ln \frac{Z(J,h_2)}{Z(J,h)} - \langle \Delta h, \mathbb{E}_{\mu_{J,h}}[\sigma]\rangle = \ln \mathbb{E}_{\mu_{J,h}}\left[\exp\langle \Delta h, \sigma - \mathbb{E}_{\mu_{J,h}}[\sigma]\rangle\right],$$

so (5) with $w = \Delta h$ proves the first inequality in (8). Pinsker's inequality $\|\mu - \nu\|_{\mathrm{TV}} \le \sqrt{\frac{1}{2} D_{\mathrm{KL}}(\mu \| \nu)}$ yields the second. □

3 Elementary Spectral-Span Glauber Gap Floor

Let L_G be the continuous-time heat-bath (Glauber) generator on $\{-1,+1\}^n$ reversible with respect to $\mu = \mu_{J,h}$:

$$(L_G f)(\sigma) := \sum_{i=1}^{n} \big(\mathbb{E}_\mu[f(\sigma) \mid \sigma_{-i}] - f(\sigma)\big),$$

whose Dirichlet form is $\mathcal{E}_G(f,f) := -\langle f, L_G f\rangle_{L^2(\mu)} = \sum_{i=1}^{n} \mathbb{E}_\mu\big[\mathrm{Var}_\mu(f(\sigma) \mid \sigma_{-i})\big]$. The spectral gap of $-L_G$ is $\lambda_1(-L_G) := \inf\{\mathcal{E}_G(f,f)/\,\mathrm{Var}_\mu(f) : \mathrm{Var}_\mu(f) > 0\}$.

Theorem 3.1 (Universal Spectral-Span Glauber Spectral-Gap Floor)**.** *Let $J = J^\top \in \mathbb{R}^{n\times n}$, $h \in \mathbb{R}^n$, and let $D = \mathrm{diag}(d_1,\dots,d_n)$ satisfy $0 \prec A := J + D \prec I_n$. Then for every function $f : \{-1,+1\}^n \to \mathbb{R}$,*

$$\mathrm{Var}_\mu(f) \le \frac{1}{1-\lambda_{\max}(A)}\,\mathcal{E}_G(f,f), \tag{9}$$

and hence $\lambda_1(-L_G) \ge 1 - \lambda_{\max}(A)$. In particular, whenever $\Delta_\lambda(J) < 1$,

$$\lambda_1(-L_G) \ge 1 - \Delta_\lambda(J) > 0. \tag{10}$$

Proof. Let $Y = A\sigma + \xi$ as in Lemma 2.1, and set $g(y) := \mathbb{E}[f(\sigma) \mid Y = y]$. By the law of total variance,

$$\mathrm{Var}_\mu(f(\sigma)) = \mathbb{E}_Y\big[\mathrm{Var}(f(\sigma) \mid Y)\big] + \mathrm{Var}_Y(g(Y)). \tag{11}$$

For each fixed $y \in \mathbb{R}^n$, $\mu(\cdot \mid Y = y)$ is a product measure on $\{-1,+1\}^n$ (Lemma 2.1(a)), so the discrete Efron–Stein inequality gives

$$\mathrm{Var}(f(\sigma) \mid Y = y) \le \sum_{i=1}^{n} \mathbb{E}\big[\mathrm{Var}(f(\sigma) \mid \sigma_{-i}, Y = y) \mid Y = y\big]. \tag{12}$$

Next, for any $i \in [n]$, let $\sigma^{(i+)}$ and $\sigma^{(i-)}$ denote the spin configurations obtained from σ by setting coordinate i to $+1$ and -1, respectively, while keeping σ_{-i} fixed, and write $f(\sigma) = a_i(\sigma_{-i}) + \sigma_i D_i f(\sigma_{-i})$ where $D_i f(\sigma_{-i}) := \frac{1}{2}(f(\sigma^{(i+)}) - f(\sigma^{(i-)}))$. Differentiating $g(y) = \sum_\sigma f(\sigma) \prod_j \frac{e^{\sigma_j(y_j+h_j)}}{2\cosh(y_j+h_j)}$ with respect to y_i and using conditional independence of σ_i and σ_{-i} given $Y = y$ gives

$$\frac{\partial g(y)}{\partial y_i} = \mathrm{Cov}(f(\sigma), \sigma_i \mid Y = y) = \mathbb{E}[D_i f(\sigma_{-i}) \mid Y = y]\ \mathrm{Var}(\sigma_i \mid Y = y).$$

Since $\mathrm{Var}(f(\sigma) \mid \sigma_{-i}, Y = y) = (D_i f(\sigma_{-i}))^2\,\mathrm{Var}(\sigma_i \mid Y = y)$ and $\mathrm{Var}(\sigma_i \mid Y = y) = \mathrm{sech}^2(y_i + h_i) \le 1$, Cauchy–Schwarz yields

$$\left(\frac{\partial g(y)}{\partial y_i}\right)^2 \le \mathbb{E}[(D_i f)^2 \mid Y = y]\ \mathrm{Var}(\sigma_i \mid Y = y)^2 \le \mathbb{E}\big[\mathrm{Var}(f(\sigma) \mid \sigma_{-i}, Y = y) \mid Y = y\big].$$

Summing over $i \in [n]$ gives $\|\nabla g(y)\|_2^2 \le \sum_{i=1}^{n} \mathbb{E}[\mathrm{Var}(f(\sigma) \mid \sigma_{-i}, Y = y) \mid Y = y]$. Applying the Brascamp–Lieb inequality (2) to $g(Y)$ with $\nabla^2 U(y) \succeq A^{-1} - I_n \succeq \frac{1-\lambda_{\max}(A)}{\lambda_{\max}(A)} I_n$ gives

$$\mathrm{Var}_Y(g(Y)) \le \frac{\lambda_{\max}(A)}{1-\lambda_{\max}(A)}\,\mathbb{E}_Y\big[\|\nabla g(Y)\|_2^2\big] \le \frac{\lambda_{\max}(A)}{1-\lambda_{\max}(A)} \sum_{i=1}^{n} \mathbb{E}\big[\mathrm{Var}(f(\sigma) \mid \sigma_{-i}, Y)\big]. \tag{13}$$

Adding the expectation of (12) and (13) into (11) and observing $1 + \frac{\lambda_{\max}(A)}{1-\lambda_{\max}(A)} = \frac{1}{1-\lambda_{\max}(A)}$ yields

$$\mathrm{Var}_\mu(f) \le \frac{1}{1-\lambda_{\max}(A)} \sum_{i=1}^{n} \mathbb{E}\big[\mathrm{Var}(f(\sigma) \mid \sigma_{-i}, Y)\big] \le \frac{1}{1-\lambda_{\max}(A)} \sum_{i=1}^{n} \mathbb{E}\big[\mathrm{Var}(f(\sigma) \mid \sigma_{-i})\big],$$

where the final step uses conditional variance monotonicity $\mathbb{E}[\mathrm{Var}(f \mid \sigma_{-i}, Y)] \le \mathbb{E}[\mathrm{Var}(f \mid \sigma_{-i})]$ (additional conditioning reduces expected conditional variance). This proves (9). Taking $D = (-\lambda_{\min}(J) + \varepsilon) I_n$ with $\varepsilon \to 0^+$ establishes (10). □

4 Two-Sided Shifted Mean-Field Free-Energy Sandwich

For any $m \in (-1,1)^n$, define the naive mean-field (Gibbs–Bogoliubov) variational free energy

$$\mathcal{F}_{J,h}(m) := -\frac{1}{2} m^\top J m - \langle h, m\rangle - \sum_{i=1}^{n} H_2\left(\frac{1+m_i}{2}\right). \tag{14}$$

Theorem 4.1 (Shifted Mean-Field Convexity and Exact Free-Energy Sandwich)**.** *Let $J = J^\top \in \mathbb{R}^{n\times n}$ with $\operatorname{diag}(J) = 0$, $h \in \mathbb{R}^n$, and let $D = \operatorname{diag}(d_1, \dots, d_n)$ satisfy $0 \prec A := J + D \prec I_n$.*

(a) ***Unique shifted mean-field fixed point.*** *The shifted variational free energy $\mathcal{F}_{A,h}(m) := \mathcal{F}_{J,h}(m) - \frac{1}{2}\sum_{i=1}^n d_i m_i^2$ is $(1 - \lambda_{\max}(A))$-strongly convex on $(-1,1)^n$ and admits a unique global minimizer $m^* \in (-1,1)^n$, which is the unique solution of $m^* = \tanh(Am^* + h)$.*

(b) ***Exact two-sided free-energy sandwich.*** *With $m^* \in (-1,1)^n$ as in part* (a)*,*

$$-\mathcal{F}_{J,h}(m^*) \;\le\; \ln Z(J,h) \;\le\; -\mathcal{F}_{J,h}(m^*) - \frac{1}{2}\sum_{i=1}^{n} d_i\big(1 - (m_i^*)^2\big) - \frac{1}{2}\ln\det(I_n - A). \tag{15}$$

In particular, at $h = 0$ (where $m^* = 0$), the sandwich gap reduces to

$$0 \;\le\; \ln Z(J,0) - n\ln 2 \;\le\; -\frac{1}{2}\ln\det(I_n - A) - \frac{1}{2}\operatorname{tr}(A) = \frac{1}{2}\sum_{k=1}^{n}\big(-\ln(1-\lambda_k(A)) - \lambda_k(A)\big). \tag{16}$$

All first-order terms in A cancel in (16), and since $\operatorname{diag}(J) = 0$ implies $\operatorname{tr}(JD) = 0$, the upper bound expands at high temperature as $\frac{1}{4}\operatorname{tr}(A^2) + O(n\|A\|_{\mathrm{op}}^3) = \frac{1}{4}\operatorname{tr}(J^2) + \frac{1}{4}\operatorname{tr}(D^2) + O(n\|A\|_{\mathrm{op}}^3)$, decomposing into the exact second-order Ising expansion $\frac{1}{4}\operatorname{tr}(J^2)$ plus the strictly positive Hubbard–Stratonovich shift overhead $\frac{1}{4}\operatorname{tr}(D^2) > \frac{1}{4}\lambda_{\min}(J)^2$ (for any $J \neq 0$, where $\operatorname{diag}(J) = 0$ forces $\lambda_{\min}(J) < 0$ and $A = J + D \succ 0$ prevents $A \to J$).

Proof. (a) Extending H_2 continuously to $[0,1]$ via $H_2(0) = H_2(1) = 0$, the function $\mathcal{F}_{A,h}$ is continuous on the compact hypercube $[-1,1]^n$ and therefore attains a global minimum there. On the open cube $(-1,1)^n$, direct differentiation gives $\frac{\partial}{\partial m_i}\big(-H_2(\frac{1+m_i}{2})\big) = \operatorname{arctanh}(m_i)$ and $\frac{\partial^2}{\partial m_i^2}\big(-H_2(\frac{1+m_i}{2})\big) = \frac{1}{1-m_i^2} \ge 1$. Therefore

$$\nabla^2 \mathcal{F}_{A,h}(m) = \operatorname{diag}\left(\frac{1}{1-m_i^2}\right) - A \succeq I_n - A \succeq (1 - \lambda_{\max}(A)) I_n \succ 0 \qquad \forall m \in (-1,1)^n.$$

As $|m_i| \to 1$, $\operatorname{arctanh}(m_i) \to \pm\infty$ while $(Am + h)_i$ remains bounded, so no boundary point of $[-1,1]^n$ can be a minimizer; hence $\mathcal{F}_{A,h}$ attains a unique interior global minimum $m^* \in (-1,1)^n$ characterized by $\nabla\mathcal{F}_{A,h}(m^*) = \operatorname{arctanh}(m^*) - (Am^* + h) = 0$, i.e., $m^* = \tanh(Am^* + h)$.

(b) For the lower bound, let $\nu_{m^*}(\sigma) := \prod_{i=1}^n \frac{1+\sigma_i m_i^*}{2}$ be the product distribution on $\{-1,+1\}^n$ with mean m^*. By nonnegativity of KL divergence $D_{\mathrm{KL}}(\nu_{m^*}\|\mu_{J,h}) \ge 0$ and $\operatorname{diag}(J) = 0$ (so $\mathbb{E}_{\nu_{m^*}}[\sigma^\top J\sigma] = (m^*)^\top J m^*$),

$$\ln Z(J,h) = \mathbb{E}_{\nu_{m^*}}\left[\frac{1}{2}\sigma^\top J\sigma + \langle h, \sigma\rangle - \ln \nu_{m^*}(\sigma)\right] + D_{\mathrm{KL}}(\nu_{m^*}\|\mu_{J,h}) \ge -\mathcal{F}_{J,h}(m^*).$$

For the upper bound, set $\tilde{h} := Am^* + h = \operatorname{arctanh}(m^*)$ and note the algebraic identity

$$\frac{1}{2}\sigma^\top J\sigma + \langle h, \sigma\rangle = -\frac{1}{2}\operatorname{tr}(D) - \frac{1}{2}(m^*)^\top A m^* + \frac{1}{2}(\sigma - m^*)^\top A(\sigma - m^*) + \langle \tilde{h}, \sigma\rangle.$$

Using the Gaussian integral $\exp(\frac{1}{2}x^\top Ax) = \mathbb{E}_{G\sim\mathcal{N}(0,A)}[\exp\langle G, x\rangle]$ at $x = \sigma - m^*$ and summing over $\sigma \in \{-1,+1\}^n$ gives

$$Z(J,h) = e^{-\frac{1}{2}\operatorname{tr}(D)-\frac{1}{2}(m^*)^\top Am^*}\mathbb{E}_{G\sim\mathcal{N}(0,A)}\left[\exp\left(-\langle G, m^*\rangle + \sum_{i=1}^n \ln\bigl(2\cosh(G_i+\tilde{h}_i)\bigr)\right)\right].$$

Because $\frac{d^2}{dt^2}\ln(2\cosh t) = \operatorname{sech}^2 t \le 1$, Taylor's theorem yields $\ln(2\cosh(G_i+\tilde{h}_i)) \le \ln(2\cosh\tilde{h}_i) + G_i\tanh(\tilde{h}_i) + \frac{1}{2}G_i^2$. Since $\tanh(\tilde{h}_i) = m_i^*$, the linear term $\langle G, \tanh(\tilde{h}) - m^*\rangle$ vanishes identically, leaving

$$Z(J,h) \le \exp\left(-\frac{1}{2}\operatorname{tr}(D) - \frac{1}{2}(m^*)^\top Am^* + \sum_{i=1}^n \ln(2\cosh\tilde{h}_i)\right)\mathbb{E}_{G\sim\mathcal{N}(0,A)}\left[\exp\left(\frac{1}{2}\|G\|_2^2\right)\right].$$

For $0 \prec A \prec I_n$, the Gaussian expectation evaluates exactly to $\det(I_n - A)^{-1/2}$. Moreover, using $\ln(2\cosh\tilde{h}_i) - m_i^*\tilde{h}_i = H_2(\frac{1+m_i^*}{2})$ (since $\tilde{h}_i = \operatorname{arctanh}(m_i^*)$) and $\langle\tilde{h}, m^*\rangle = (m^*)^\top Am^* + \langle h, m^*\rangle$, the prefactor exponent equals

$$-\frac{1}{2}\operatorname{tr}(D) + \frac{1}{2}(m^*)^\top Am^* + \langle h, m^*\rangle + \sum_{i=1}^n H_2\left(\frac{1+m_i^*}{2}\right) = -\mathcal{F}_{J,h}(m^*) - \frac{1}{2}\sum_{i=1}^n d_i\bigl(1-(m_i^*)^2\bigr),$$

where we used $(m^*)^\top Am^* - \operatorname{tr}(D) = (m^*)^\top Jm^* - \sum_i d_i(1-(m_i^*)^2)$. Taking logarithms proves (15). At $h = 0$, $m^* = 0$ solves $m^* = \tanh(Am^*)$ uniquely, $\mathcal{F}_{J,0}(0) = -n\ln 2$, and $\operatorname{tr}(D) = \operatorname{tr}(A)$ (since $\operatorname{diag}(J) = 0$), yielding (16). Finally, expanding $-\ln(1-\lambda) - \lambda = \frac{1}{2}\lambda^2 + O(|\lambda|^3)$ and using $\operatorname{tr}(A^2) = \operatorname{tr}(J^2) + 2\operatorname{tr}(JD) + \operatorname{tr}(D^2) = \operatorname{tr}(J^2) + \operatorname{tr}(D^2)$ (as $\operatorname{diag}(J) = 0$) gives the stated second-order expansion; for any $J \neq 0$ with $\operatorname{diag}(J) = 0$, $\operatorname{tr}(J) = 0$ implies $\lambda_{\min}(J) < 0$, and if $u \in \mathbb{R}^n$ is a unit eigenvector of J for $\lambda_{\min}(J)$, then $u^\top(J+D)u > 0$ yields $\sum_{i=1}^n d_i u_i^2 > -\lambda_{\min}(J) > 0$, whence Cauchy–Schwarz and $\sum_{i=1}^n u_i^4 \le (\sum_{i=1}^n u_i^2)^2 = 1$ give $\operatorname{tr}(D^2) = \sum_{i=1}^n d_i^2 \ge (\sum_{i=1}^n d_i u_i^2)^2 / \sum_{i=1}^n u_i^4 > \lambda_{\min}(J)^2$. □

5 Projective Stability Across Temperatures and Magnetization Lipschitz Law

We finally prove joint coupling-and-field stability bounds valid at *all* temperatures (without assuming $\Delta_\lambda(J) < 1$), together with a high-temperature magnetization Lipschitz law.

Theorem 5.1 (Hyperbolic Projective Stability and Magnetization Lipschitz Law)**.** *Let (J_1, h_1) and (J_2, h_2) be two Ising specifications on $\{-1,+1\}^n$ with symmetric coupling matrices $J_1 = J_1^\top$, $J_2 = J_2^\top$ satisfying $\operatorname{diag}(J_1) = \operatorname{diag}(J_2) = 0$, $\Delta J := J_2 - J_1$, and $\Delta h := h_2 - h_1$, inducing Gibbs measures $\mu_1 := \mu_{J_1,h_1}$ and $\mu_2 := \mu_{J_2,h_2}$.*

(a) ***Universal projective, TV, and Jeffreys bounds (all temperatures).*** *Hilbert's projective distance satisfies*

$$d_H(\mu_1,\mu_2) \le \min\left\{\sum_{i\neq j}|\Delta J_{ij}|,\ \frac{n}{2}\Delta_\lambda(\Delta J)\right\} + 2\|\Delta h\|_1. \tag{17}$$

Furthermore, the Total Variation and Jeffreys divergences obey the sharp nonlinear hyperbolic bounds

$$\|\mu_1-\mu_2\|_{\mathrm{TV}} \le \tanh\left(\frac{d_H(\mu_1,\mu_2)}{4}\right), \qquad D_J(\mu_1,\mu_2) \le d_H(\mu_1,\mu_2)\tanh\left(\frac{d_H(\mu_1,\mu_2)}{4}\right). \tag{18}$$

(b) ***High-temperature magnetization Lipschitz law.*** *At fixed $J = J^\top$ admitting a diagonal shift $0 \prec A := J + D \prec I_n$, the mean magnetization map $m(J,h) := \mathbb{E}_{\mu_{J,h}}[\sigma]$ satisfies*

$$\|m(J,h_2) - m(J,h_1)\|_2 \le \|(I_n - A)^{-1}\|_{\mathrm{op}}\,\|h_2 - h_1\|_2 \le \frac{1}{1-\lambda_{\max}(A)}\|h_2 - h_1\|_2, \quad (19)$$

and in particular $\|m(J,h_2) - m(J,h_1)\|_2 \le (1-\Delta_\lambda(J))^{-1}\|h_2 - h_1\|_2$ whenever $\Delta_\lambda(J) < 1$.

Proof. (a) Directly from (1), $\ln(\mu_2(\sigma)/\mu_1(\sigma)) = \frac{1}{2}\sigma^\top \Delta J \sigma + \langle \Delta h, \sigma\rangle - \ln(Z_2/Z_1)$. Taking the oscillation over $\sigma, \tau \in \{-1,+1\}^n$ cancels the constant $\ln(Z_2/Z_1)$:

$$d_H(\mu_1,\mu_2) \le \frac{1}{2}\max_{\sigma,\tau}\big(\sigma^\top \Delta J \sigma - \tau^\top \Delta J \tau\big) + \max_{\sigma,\tau}\langle \Delta h, \sigma - \tau\rangle.$$

The second term equals $2\|\Delta h\|_1$. For the first term, since $|\sigma_i\sigma_j| = 1$ and $\mathrm{diag}(\Delta J) = 0$, we have $|\sigma^\top \Delta J \sigma| \le \sum_{i\neq j}|\Delta J_{ij}|$, giving $\frac{1}{2}(\sigma^\top \Delta J \sigma - \tau^\top \Delta J \tau) \le \sum_{i\neq j}|\Delta J_{ij}|$. Simultaneously, since $\|\sigma\|_2^2 = \|\tau\|_2^2 = n$, the Rayleigh quotient gives $\sigma^\top \Delta J \sigma \le n\lambda_{\max}(\Delta J)$ and $\tau^\top \Delta J \tau \ge n\lambda_{\min}(\Delta J)$, yielding $\frac{1}{2}(\sigma^\top \Delta J \sigma - \tau^\top \Delta J \tau) \le \frac{n}{2}\Delta_\lambda(\Delta J)$. Combining both bounds proves (17).

For (18), let $R := d_H(\mu_1,\mu_2)$. If $R = 0$ then $\mu_1 = \mu_2$ and (18) is trivial. Assume $R > 0$, let $S_+ := \{\sigma : \mu_1(\sigma) \ge \mu_2(\sigma)\}$ and $S_- := \{-1,+1\}^n \setminus S_+$, and set $a := \mu_1(S_+)$, $b := \mu_2(S_+)$, so $\delta := \|\mu_1 - \mu_2\|_{\mathrm{TV}} = a - b \in (0,1)$ and $0 < b < a < 1$. Choosing $\sigma_+ \in S_+$ with $\mu_1(\sigma_+)/\mu_2(\sigma_+) \ge a/b$ and $\sigma_- \in S_-$ with $\mu_1(\sigma_-)/\mu_2(\sigma_-) \le (1-a)/(1-b)$ gives

$$e^R \ge \frac{\mu_1(\sigma_+)/\mu_2(\sigma_+)}{\mu_1(\sigma_-)/\mu_2(\sigma_-)} \ge \frac{a(1-b)}{b(1-a)}.$$

Writing $s := a + b \in (\delta, 2-\delta)$ so $a = (s+\delta)/2$ and $b = (s-\delta)/2$, we have

$$\frac{a(1-b)}{b(1-a)} = \frac{(s+\delta)(2-s+\delta)}{(s-\delta)(2-s-\delta)} = \frac{(1+\delta)^2 - (s-1)^2}{(1-\delta)^2 - (s-1)^2} \ge \left(\frac{1+\delta}{1-\delta}\right)^2,$$

since $(1+\delta)^2 > (1-\delta)^2$ and $(s-1)^2 \ge 0$ (equality at $s = 1$). Taking square roots and solving for δ yields $\delta \le \frac{e^{R/2}-1}{e^{R/2}+1} = \tanh(R/4)$. Finally, letting $r(\sigma) := \ln(\mu_1(\sigma)/\mu_2(\sigma))$ and $c := \frac{1}{2}(\max r + \min r)$ (so $|r(\sigma) - c| \le R/2$ for all σ),

$$\begin{aligned} D_J(\mu_1,\mu_2) &= \sum_\sigma \big(\mu_1(\sigma) - \mu_2(\sigma)\big)\big(r(\sigma) - c\big) \\ &\le \frac{R}{2}\sum_\sigma |\mu_1(\sigma) - \mu_2(\sigma)| = R\,\|\mu_1 - \mu_2\|_{\mathrm{TV}} \le R\tanh\left(\frac{R}{4}\right). \end{aligned}$$

(b) Because $\nabla_h m(J,h) = \nabla_h^2 \ln Z(J,h) = \Sigma(J,h)$, integrating along $h_t := (1-t)h_1 + t h_2$ and applying Theorem 2.2(b) ($\|\Sigma(J,h_t)\|_{\mathrm{op}} \le \|(I_n - A)^{-1}\|_{\mathrm{op}} = (1 - \lambda_{\max}(A))^{-1}$) gives

$$\|m(J,h_2) - m(J,h_1)\|_2 = \left\| \int_0^1 \Sigma(J,h_t)(h_2 - h_1)\,dt \right\|_2 \le \|(I_n - A)^{-1}\|_{\mathrm{op}}\|h_2 - h_1\|_2,$$

proving (19). Letting $D = (-\lambda_{\min}(J) + \varepsilon)I_n$ with $\varepsilon \to 0^+$ yields the $\Delta_\lambda(J)$ bound. □

6 Computational Verification and Explicit Benchmarks

All four theorems were verified programmatically against exact 2^n partition functions and exact $2^n \times 2^n$ symmetrized Glauber generators via a self-contained Python suite (`ising_geo.py`, `verify_all.py`, and `test_ising_geo.py`; four automated unit tests passing in 0.43 s) across 240 deterministic spin-glass instances using NumPy `default_rng` streams.

In every suite below, each raw coupling matrix is drawn by symmetrizing an i.i.d. Gaussian matrix M via $J_{\mathrm{raw}} = \frac{1}{2}(M + M^\top)$ with zeroed diagonal ($\mathrm{diag}(J_{\mathrm{raw}}) = 0$), then rescaled to a target spectral span $\Delta_\lambda(J)$.

Table 1: Exact quantitative verification on trial 0 of each dimension block $n \in \{4, 5, 6, 7\}$ (seed `202643`, Section 6). Every exact covariance norm, Glauber spectral gap, and log-partition function lies strictly inside the closed-form bounds of Theorems 2.2–4.1.

Model	$\Delta_\lambda(J)$	$\|\Sigma\|_{\mathrm{op}}$	$(1-\Delta_\lambda)^{-1}$	$\lambda_1(-L_G)$	$1-\Delta_\lambda$	$-\mathcal{F}_{J,h}(m^*)$	$\ln Z$	UB (15)
`Ising_n4`	0.5368	1.0202	2.1587	0.7864	0.4632	3.2980	3.3653	3.7030
`Ising_n5`	0.3593	1.0638	1.5607	0.8572	0.6407	3.6370	3.6631	3.7884
`Ising_n6`	0.4947	1.1071	1.9790	0.8273	0.5053	4.6360	4.7066	5.1053
`Ising_n7`	0.6308	1.2913	2.7082	0.7202	0.3692	5.2774	5.3604	5.8542

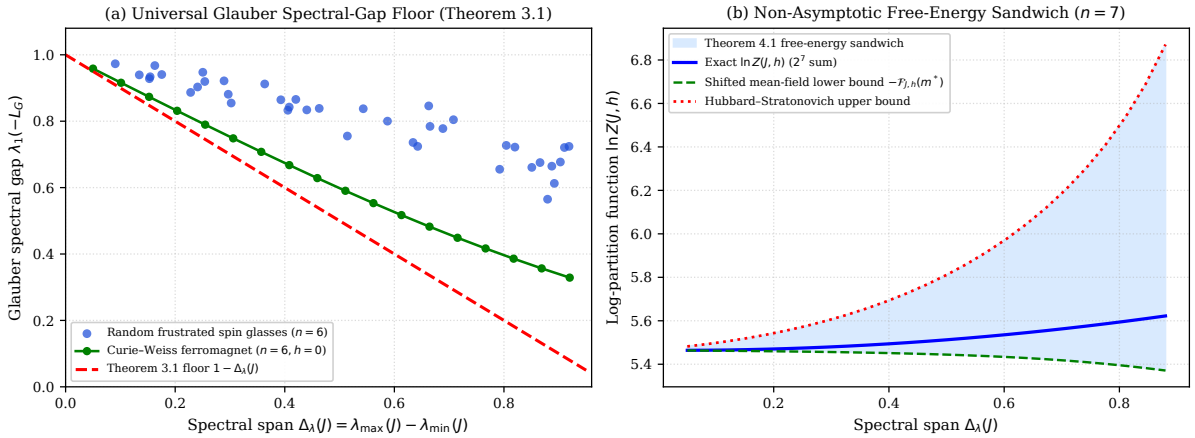


Figure 1: **(a)** Exact continuous-time Glauber spectral gaps $\lambda_1(-L_G)$ for 40 random frustrated $n = 6$ spin glasses (fresh seed `202642`) and the $n = 6$ Curie–Weiss ferromagnet ($h = 0$) compared with the universal Theorem 3.1 floor $1 - \Delta_\lambda(J)$. **(b)** Exact 2^7 log-partition function $\ln Z(J, h)$ vs. the two-sided shifted mean-field sandwich (15) as spectral span varies (standalone $n = 7$ spin glass, fresh seed `202643`).

- **Theorem 2.2 suite (seed `202641`, 60 models, 300 CGF checks):** Across $n \in \{4, 6, 8\}$ (20 spin glasses per n; $M_{ij} \sim \mathcal{N}(0, 0.35^2)$, rescaled to $\Delta_\lambda(J) \sim \mathrm{Unif}(0.2, 0.85)$, $h \sim \mathcal{N}(0, 0.6^2)$, uniform shift $d = -\lambda_{\min}(J) + 0.05(1 - \Delta_\lambda(J))$, 5 tilt vectors $w \sim \mathcal{N}(0, 0.5^2)$ and field shift $\Delta h \sim \mathcal{N}(0, 0.4^2)$ per model), the anisotropic CGF bound (5), covariance ceiling (6), and external-field KL/TV bounds (8) all held with 0.0 violation.

- **Theorem 3.1 suite (seed `202642`, 60 exact Glauber generators):** Across $n \in \{4, 5, 6\}$ (20 frustrated spin glasses each; $M_{ij} \sim \mathcal{N}(0, 0.4^2)$, rescaled to $\Delta_\lambda(J) \sim \mathrm{Unif}(0.15, 0.90)$, $h \sim \mathcal{N}(0, 0.7^2)$), exact diagonalization of $D_\mu^{1/2}(-L_G)D_\mu^{-1/2}$ verified $\lambda_1(-L_G) \geq 1 - \Delta_\lambda(J)$ with 0.0 violation (Figure 1(a)).

- **Theorem 4.1 suite (seed `202643`, 60 spin glasses):** Looping over $n \in \{4, 5, 6, 7\}$ (15 instances each; $M_{ij} \sim \mathcal{N}(0, 0.4^2)$, rescaled to $\Delta_\lambda(J) \sim \mathrm{Unif}(0.2, 0.85)$, $h \sim \mathcal{N}(0, 0.5^2)$, uniform shift $d = -\lambda_{\min}(J) + 0.02(1 - \Delta_\lambda(J))$), both the lower and upper bounds in (15) held with 0.0 violation across all 60 instances (Table 1, Figure 1(b)).

- **Theorem 5.1 suite (seed `202644`, 60 Ising pairs):** Across $n \in \{4, 6, 8\}$ (20 pairs each; J_1 from $M_{1,ij} \sim \mathcal{N}(0, 0.35^2)$ rescaled to $\Delta_\lambda(J_1) \sim \mathrm{Unif}(0.2, 0.8)$, perturbation ΔJ zero-diagonal symmetrized from $M_{2,ij} \sim \mathcal{N}(0, 0.2^2)$, $h_1 \sim \mathcal{N}(0, 0.5^2)$, $\Delta h \sim \mathcal{N}(0, 0.3^2)$), the Hilbert projective bound (17), hyperbolic TV and Jeffreys bounds (18), and magnetization Lipschitz law (19) held with 0.0 violation (Figure 2(b)).

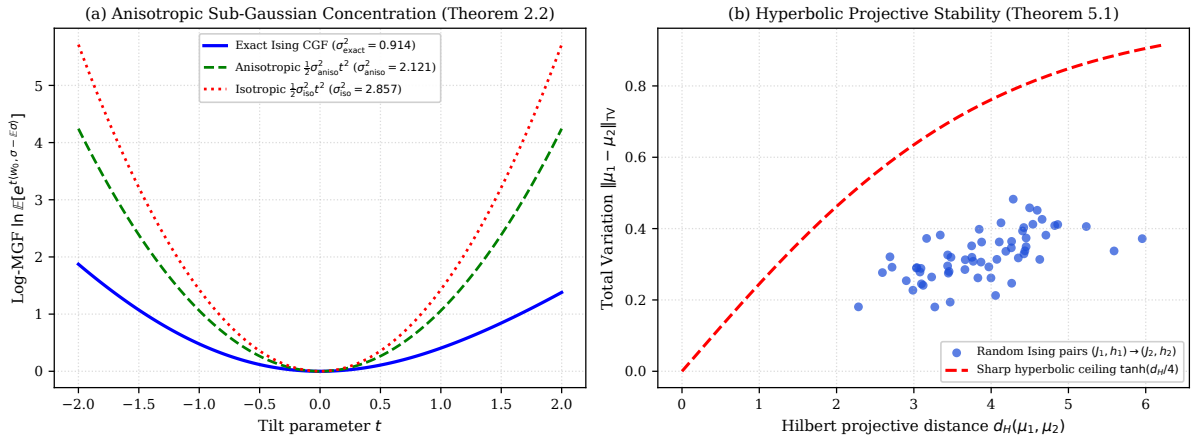


Figure 2: **(a)** Exact log-MGF $\ln \mathbb{E}[e^{t\langle w_0, \sigma - \mathbb{E}\sigma\rangle}]$ along a unit direction $\|w_0\|_2 = 1$ on a standalone $n = 7$ spin glass ($\Delta_\lambda = 0.65$, $h \sim \mathcal{N}(0, 0.5^2)$, shift $d = -\lambda_{\min}(J) + 0.02(1 - 0.65)$, fresh seed `202641`) vs. the anisotropic Hubbard–Stratonovich ceiling $\frac{1}{2}w^\top (I_n - A)^{-1} w$ and isotropic span ceiling $\frac{1}{2(1-\Delta_\lambda)}\|w\|_2^2$ (Theorem 2.2(a)). **(b)** Total Variation vs. Hilbert projective distance across 60 random $n = 6$ Ising pairs (fresh seed `202644`) bounded by the sharp hyperbolic ceiling $\tanh(d_H/4)$ (Theorem 5.1(a)).

7 Discussion and Open Directions

The four theorems established above show that combining diagonal shift invariance $A = J + D \in (0, I_n)$ with the Hubbard–Stratonovich channel $Y = A\sigma + \xi$ yields a unified, elementary proof framework for sub-Gaussian concentration, Glauber spectral gaps, mean-field free-energy sandwiches, and measure stability on arbitrary frustrated Ising spin systems. Three natural directions remain open:

1. **Optimal non-uniform diagonal shifts:** Choosing a non-uniform diagonal matrix $D = \operatorname{diag}(d_1, \ldots, d_n)$ via semidefinite programming to minimize $\lambda_{\max}(J + D)$ subject to $J + D \succ 0$ can strictly improve upon the uniform spectral span $\Delta_\lambda(J)$ on heterogeneous-degree graphs, tightening all four bounds simultaneously.

2. **TAP Onsager correction in the free-energy upper bound:** Replacing the global quadratic ceiling $\ln\cosh(G_i + \tilde{h}_i) \le \ln\cosh \tilde{h}_i + G_i \tanh \tilde{h}_i + \frac{1}{2}G_i^2$ by local curvature $\operatorname{sech}^2(\tilde{h}_i)$ with controlled third-order remainder could tighten (15) toward the Thouless–Anderson–Palmer (TAP) free energy in strong external fields.

3. **Potts and multi-spin p-body extensions:** Extending the Brascamp–Lieb tilted-CGF proof of Theorem 2.2 and the eight-line Glauber gap proof of Theorem 3.1 to q-state Potts models and p-spin tensor Hamiltonians is an appealing target.

Reproducibility and Data Availability

All analytical verifications, unit tests, and figure generation scripts are openly archived under the MIT license (version `v1.0.2`) at:
`https://github.com/lluiseriksson/ising-hubbard-spectral-geometry`.

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