Sharp Stage Compression, Universal 5-Prime Witness Optimality, and Exact Per-Job Cut Thresholds for Pinwheel Scheduling PSPACE-Completeness

Lluis Eriksson*

November 26, 2026

Abstract

Fotakis, Kanellopoulos, Mitropoulos, Pagourtzis, Papadimitriou, Pergaminelis, and Tolias (`arXiv:2611.34869v1`) recently settled the 37-year-old open problem of Holte, Mok, Rosier, Tulchinsky, and Varvel (1989) by proving that Pinwheel Scheduling with binary deadlines is PSPACE-complete via a three-stage reduction: Periodic 3-SAT $\to$ Daily Scheduling with Breaks (Stage 1) $\to$ Daily Scheduling (Stage 2) $\to$ Pinwheel Scheduling (Stage 3). We establish sharp structural optimality and quantitative compression across all three stages. First, in Stage 1 we prove that strip high hour $H = 5$ and inter-region gap $g = 3$ are exact minimums (with executable counterexamples at $H = 4$ and $g = 2$), compressing daily width from $w = 72N + 15$ to $w = 60N + 13$ (a 16.7% linear reduction cascading as $(5/6)^4 = 625/1296 \approx 0.482$ into the Stage-2 $O((Mw)^4 \log^2(Mw))$ job count), and construct an optimal 4-day (1-neutral-block) write phase offering relative offsets $(\{1,2\},\{2,4\},\{0,3\},\{0,4\})$ between boundary ordinary days $\{0,5\}$, cutting write-phase overhead by 75% ($M = 8m + 32 \to 8m + 8$) and proving the unconditional lower bound $L \geq 4$ days across all write calendars at $H = 5$. Second, in Stage 2 we prove a universal combinatorial number-theoretic lower bound: *any* pairwise-coprimality two-hour witness gadget $(c_1, \ldots, c_r)$ of arbitrary size r (even allowing arbitrary prime powers) whose coprimality graph satisfies $\chi(G) \geq 3$ and $\omega(G) \leq 2$ contains an induced odd cycle C_{2s+1} ($s \geq 2$) whose $(2s+1)(s-1)$ non-edges require at least $\lceil (2s+1)(s-1)/\binom{s}{2} \rceil = 5$ distinct primes, proving $K = \mathrm{lcm}(c_v) \geq 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 = 2310$ unconditionally; moreover, solving the exact filler-minimization problem at $K = 2310$ proves that $(6, 10, 55, 77, 21)$ (in complement-cycle $\overline{C}_5$ order) with CRT phases $(4, 5, 11, 56, 15)$ is the unique global optimum (798/2310 witness density, 3822 rigid fillers versus Fotakis et al.'s rank-9/12 tuple $(14, 33, 10, 21, 55)$ with 4002 fillers, saving 180 rigid jobs per two-hour test). Third, in Stage 3 we sharpen the density-slack and expected-deviation bounds from $n\Delta$ to the moving-job weighted rate $S_2 = \sum_{i:\delta_i>0} \delta_i/b_i^2$ (since rigid jobs contribute identically zero) and prove an exact per-job total-variation cut theorem reducing the soundness day-length threshold from $T > n(2E+3)$ to the single-slot-tight bound $T \geq E+1$ (saving $> 2n$ in the cut step alone and $\Omega(n^2/w_{\mathrm{mov}})$ overall). All 18 core algebraic theorems are formally verified in constructive Lean 4.35.0 (0 `sorry`, 0 `Classical.choice`, 0 custom axioms), backed by 16 AST-audited zero-float SHA-256 certificates and 56 adversarial tests.

1 Introduction and Main Results

In *Pinwheel Scheduling*, introduced by Holte, Mok, Rosier, Tulchinsky, and Varvel [2], we are given n positive integer deadlines $d_1, \ldots, d_n$ in binary and ask whether there exists an infinite schedule $S : \mathbb{Z} \to [n] \cup \{\perp\}$ such that every job i occurs in every window of d_i consecutive time slots. While

*Independent Researcher, AIRR Frontier Lab (`https://airr.science`). Developed under the `AIRR-FOUNDER-1.0` open-science standard with transparent Google Antigravity assistance; all mathematical proofs, Lean 4.35.0 formalizations, and zero-float exact rational certificates are machine-checked and publicly archived at `https://github.com/lluiseriksson/pinwheel-pspace-sharp-gadgets-and-slack`.

Kawamura [3] proved the long-standing 5/6 density threshold conjecture and dense instances ($\rho = 1$) were recently shown to be strongly NP-complete [4, 5], the exact complexity of general Pinwheel Scheduling remained open for 37 years until Fotakis, Kanellopoulos, Mitropoulos, Pagourtzis, Papadimitriou, Pergaminelis, and Tolias [1] proved PSPACE-completeness for binary deadlines at arbitrary density $1 - \varepsilon < \rho < 1$.

The reduction of Fotakis et al. [1] proceeds in three modular stages from Orlin's PSPACE-complete Periodic 3-SAT [6]:

1. *Stage 1 (Section 3 of [1]):* Periodic 3-SAT (N variables per copy, m clauses) $\to$ saturated Daily Scheduling with Breaks of width $w = 72N+15$ and calendar period $M = 4(2m+8) = 8m+32$, using strip high hour $H = 6$, inter-region gap $g = 4$, and a 16-day write waveform.

2. *Stage 2 (Section 4 of [1]):* Saturated Daily Scheduling with Breaks $\to$ saturated Daily Scheduling without breaks, via 5-witness two-hour tests with multipliers $(14, 33, 10, 21, 55)$ ($\operatorname{lcm} K = 2310$, 4002 rigid fillers per test) and $O((Mw)^4 \log^2(Mw))$ total jobs.

3. *Stage 3 (Section 5 of [1]):* Saturated Daily Scheduling (n jobs, width w, periods b_i, ascent bounds $\delta_i \le \Delta$) $\to$ Pinwheel Scheduling with dyadic clock of day length $T = w2^k$, requiring $T > n(2E+3)$ with $E = \Omega(\Delta+1)(n(b^*+1)+w)$ and density slack $1 - \rho \le n\Delta/T^2$.

Every stage leaves natural optimality questions open: Can the Stage-1 strip height, gap, and 16-day write phase be compressed further, and what are their exact minimums? Is the $K = 2310$ five-prime witness period unavoidable across *all* possible two-hour witness graphs (not just C_5 and even with prime powers), and what is the minimum possible filler count at $K = 2310$? And can the quadratic blowup in the Stage-3 day length T (and dyadic clock height k) be eliminated by exploiting rigid-job inactivity and per-job drift variation? We answer all three questions sharply.

1.1 Statement of Main Theorems

Theorem 1 (Optimal Stage-1 Compression and Minimal 4-Day Write Phase)**.** *Given a Periodic 3-SAT formula $F(X, Y)$ with N variables per copy and m clauses, there is a polynomial-time reduction to a saturated Daily Scheduling with Breaks instance with periods in $\{1, 2\}$, $4N + 3$ available hours per day, width*

$$w = 60N + 13 \qquad (\textit{vs. } 72N + 15 \textit{ in [1]}), \tag{1}$$

and calendar period

$$M = 8m + 8 \qquad (\textit{vs. } 8m + 32 \textit{ in [1]}), \tag{2}$$

realized by strip high hour $H = 5$, inter-region gap $g = 3$, middle span $W = 20N+2$, guard offset $G = W + 3$, and a single 4-day neutral block per write phase offering active-strip offsets

$$\mathcal{W}_4 = (\{1, 2\},\ \{2, 4\},\ \{0, 3\},\ \{0, 4\}) \tag{3}$$

above o_σ on write days $f = 0, 1, 2, 3$ (flanked by ordinary days $\{o_\sigma, o_\sigma + 5\}$ on days -1 and 4). The instance is feasible if and only if F is satisfiable. Moreover, all three parameters are unconditionally exact minimums within this framework:

(a) *Any inter-region gap $g \le 2$ permits a Type-II bank job (ascent bound 2) to escape its strip, breaking Lemma 3.3 of [1].*

(b) *Any strip high hour $H \le 4$ permits a Type-II job at a low port o_σ to climb $o_\sigma \to o_\sigma + 2 \to o_\sigma + 4$ across rows $0 \to 1 \to 2$ of a clause block, swapping the encoded bit and breaking Lemma 3.6(b) of [1].*

(c) *Any write phase of L days flanked by ordinary hours $\{o_\sigma, o_\sigma + 5\}$ that rewrites both bit transitions $0 \leftrightarrow 1$ with $H = 5$ requires $L \ge 4$ days (hence at least 1 neutral block), since a Type-I job flipping $1 \to 0$ must ascend by $+5$ across $L + 1$ transitions, forcing $L + 1 \ge 5$.*

Theorem 2 (Universal 5-Prime Witness Bound and Optimal Fillers)**.** *Let $c_1, \ldots, c_r \in \mathbb{Z}_{\geq 1}$ be any witness multipliers ($r \geq 1$, allowing arbitrary prime powers) with $K := \operatorname{lcm}(c_1, \ldots, c_r)$, and let $G = ([r], E)$ be their coprimality graph, where $uv \in E$ iff $\gcd(c_u, c_v) = 1$.*

(a) Universal 5-prime lower bound. *If G satisfies adversarial-phase soundness ($\chi(G) \geq 3$) and two-hour completeness ($\omega(G) \leq 2$), then K is divisible by at least **five distinct primes**, and hence*

$$K \geq 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 = 2310 \qquad \textit{unconditionally.} \tag{4}$$

(b) Unique global filler optimum at $K = 2310$. *For $G \cong C_5$ and $K = 2310$, every valid multiplier tuple is dominated by one of the 12 cyclic orderings of $\{2, 3, 5, 7, 11\}$ along the complement 5-cycle $\overline{C}_5$. Over all 12 orderings, the unique minimum rigid filler count $2K - \sum_{v=0}^{4} K/c_v$ is achieved by the prime cycle $(3, 2, 5, 11, 7)$ (equivalently $(2, 3, 7, 11, 5)$ in reversed orientation), yielding in $\overline{C}_5$ vertex order*

$$(c_0, \ldots, c_4) = (6, 10, 55, 77, 21), \qquad (\beta_0, \ldots, \beta_4) = (4, 5, 11, 56, 15), \tag{5}$$

*with witness density $798/2310 = 19/55$ and exactly **3822** rigid fillers per two-hour test (beating the runner-up at 3838 and Fotakis et al.'s rank-9/12 tuple $(14, 33, 10, 21, 55)$ at 4002, saving 180 rigid jobs per test).*

Theorem 3 (Moving-Rate Density Slack and Exact Per-Job Cut Threshold $T \geq E+1$)**.** *Consider any saturated Daily Scheduling instance of width w with n jobs $(b_i, \delta_i)_{i=1}^n$ ($\sum_{i=1}^n 1/b_i = w$, $b^* = \max_i b_i$, $\Delta = \max_i \delta_i > 0$, $L = \operatorname{lcm}_i b_i$). Define the moving-job weighted rates*

$$S_1 := \sum_{i:\, \delta_i > 0} \frac{\delta_i}{b_i}, \qquad S_2 := \sum_{i:\, \delta_i > 0} \frac{\delta_i}{b_i^2} \leq w_{\mathrm{mov}} \Delta, \tag{6}$$

where $w_{\mathrm{mov}} := \sum_{i:\delta_i > 0} 1/b_i \leq w$. For any dyadic day length $T = w2^k > \Delta$, let $d_i := Tb_i + \delta_i$ and $\Omega := Lw^n + 6L + 10$. Then:

(a) Sharp density slack and expected deviation. *The output Pinwheel density ρ satisfies*

$$0 < 1 - \rho = \sum_{i:\, \delta_i > 0} \frac{\delta_i}{Tb_i(Tb_i + \delta_i)} < \frac{S_2}{T^2}, \tag{7}$$

and every periodic feasible Pinwheel schedule admits a window of ΩT consecutive slots whose total forward-gap deviation $\sum_{i=1}^n E_i$ is at most

$$E_{\mathrm{sharp}} := \left\lfloor \Omega \left((b^* + 1)S_2 + S_1 + \frac{\Delta S_2}{T} \right) \right\rfloor \leq \left\lfloor \Omega((b^* + 2)S_2 + S_1) \right\rfloor. \tag{8}$$

(b) Exact per-job total-variation cut theorem. *For **any** $T \geq E + 1$ (where $\sum_{i=1}^n E_i \leq E$), there exists a cut residue $\gamma \in \{0, \ldots, T-1\}$ such that consecutive occurrences of every payload job i in the window lie exactly b_i days apart under the day partition $\{\gamma + tT, \ldots, \gamma + (t+1)T - 1\}$. Consequently, Stage-3 soundness holds for the least $k \geq 1$ with $T = w2^k > \max(\Delta, E_{\mathrm{sharp}})$, improving Fotakis et al.'s threshold $T > n(2E + 3)$ by a factor strictly exceeding $2n$ in the cut step and $\Omega(n^2/w_{\mathrm{mov}})$ overall.*

(c) Single-slot tightness. *The threshold $T \geq E + 1$ is exact: for every $T \geq 2$, there exists a drift trajectory with total deviation $E = T$ that forbids all T cut residues modulo T.*

2 Proof of Theorem 1: Optimal Stage-1 Compression

Proof of Theorem 1. Let $v = 2N$, $W = 10v + 2 = 20N + 2$, $G = W + 3 = 20N + 5$, $o = G + 1 = 20N+6$, and $o_\sigma = o+10\sigma$ for $\sigma \in \{0, \dots, v-1\}$. Strip σ owns $R_\sigma = \{o_\sigma, \dots, o_\sigma+7\}$, with ordinary hours o_σ (low) and $o_\sigma + 5$ (high); parking hours are $p_1 = o + 10v$, $p_2 = p_1 + 1$, $p_3 = p_1 + 2$; lower and upper guard bands are $\{1, \dots, W + 1\}$ and $\{o + G, \dots, o + W + G\}$. The width is $w = o + W + G = 3W + 7 = 60N + 13$ (Lean Theorem 1; Figure 1(b)), saving $12N + 2$ hours compared to $72N + 15$ and reducing the Stage-2 quartic factor w^4 by $(5/6)^4 = 625/1296 < 1/2$ (Lean Theorem 10). Every consecutive pair of regions is separated by an exact gap of 3 (Lean Theorem 2):

$$o_{\sigma+1} - (o_\sigma + 7) = 3, \qquad p_1 - (o_{v-1} + 7) = 3, \qquad o - (o + W - G) = 3, \qquad (o + G) - (o + W) = 3.$$

Clause blocks retain the 4-row structure of [1, Table 2] (row 0: lower guards; row 1: literal offers $u_r + 2$; row 2: upper guards; row 3: parking). Each write phase uses a *single* neutral block (4 days, $f = 0, 1, 2, 3$) offering $\{o_\sigma + a_f, o_\sigma + b_f\}$ from $\mathcal{W}_4 = (\{1, 2\}, \{2, 4\}, \{0, 3\}, \{0, 4\})$ on every active-bank strip, ordinary hours $\{o_\sigma, o_\sigma + 5\}$ on every inactive-bank strip, and neutral parking/guard ports (Figure 1(a)). Thus $M = 4m + 4 + 4m + 4 = 8m + 8$ (Lean Theorem 4), and every day has $2v + 3 = 4N + 3$ available hours, preserving exact saturation. Note that both day -1 (row 3 of the last clause block, or day 3 of the other bank's write if $m = 0$) and day 4 (row 0 of the next read phase, or day 0 of the other bank's write) offer ordinary hours $\{o_\sigma, o_\sigma + 5\}$ on the active bank.

Completeness: Clause blocks schedule identically to [1, Lemma 3.2]: at the true literal port $u^* \in \{o_{\sigma^*}, o_{\sigma^*} + 5\}$, the Type-II job moves $u^* \to u^* + 2 \in R_{\sigma^*}$ on row 1 ($+2 \le 2$) and descends to u^* on row 2. During a 4-day write of bank P, write $(t_{\mathrm{II}}, t_{\mathrm{I}})$ for the relative offsets above o_σ on days $-1, 0, 1, 2, 3, 4$. Every bit transition is realized legally with ascent bounds $(2, 1)$:

$$\begin{aligned}
0 \to 0: &\quad (0,5) \to (2,1) \to (4,2) \to (0,3) \to (0,4) \to (0,5),\\
0 \to 1: &\quad (0,5) \to (2,1) \to (4,2) \to (3,0) \to (4,0) \to (5,0),\\
1 \to 0: &\quad (5,0) \to (2,1) \to (4,2) \to (0,3) \to (0,4) \to (0,5),\\
1 \to 1: &\quad (5,0) \to (2,1) \to (4,2) \to (3,0) \to (4,0) \to (5,0).
\end{aligned}$$

Every step is verified directly in Lean Theorem 5 and Certificate `C03`, with a full end-to-end periodic simulation certified in `C04`.

Soundness: Because every upward inter-region move costs at least $3 > 2$ and every write-phase offer lies inside $\{o_\sigma, \dots, o_\sigma + 4\} \subset R_\sigma$, Lemma 3.3 of [1] holds unchanged: every bank job remains forever in one strip, with exactly two bank jobs per strip. For Lemma 3.4, a filler on the even phase moving from lower guard $u - G$ to upper guard $u' + G$ ($u, u' \in [o, o + W]$) has ascent $u' - u + 2G \ge 2G - W = W + 6 > W$ (Lean Theorem 2), forcing both even period-2 jobs to be companions and giving the exact checker/companion matching of Lemma 3.5. For Lemma 3.6(a), at port $u_a = o_\sigma$ the job at o_σ on row 0 cannot reach $o_\sigma + 5$ (gap $5 > 2$) and must ascend $o_\sigma \to o_\sigma + 2$ (Type-II); at $u_a = o_\sigma + 5$ the job at o_σ cannot reach $o_\sigma + 7$ (gap $7 > 2$), forcing the job at $o_\sigma + 5$ to ascend to $o_\sigma + 7$ (Type-II). For Lemma 3.6(b), on row 1 the low-hour job is at most at $o_\sigma + 2$; on row 2 the strip offers $\{o_\sigma, o_\sigma + 5\}$, and since $(o_\sigma + 5) - (o_\sigma + 2) = 3 > 2$, that job cannot reach $o_\sigma + 5$ and returns to o_σ, preserving the bit across every consecutive clause block of a read phase (and across the inactive bank's write phase, where inactive strips offer ordinary hours on all 4 days). For Lemma 3.7(b), on day 0 of a write phase an active strip offers $\{o_\sigma + 1, o_\sigma + 2\}$, and on day 1 it offers $\{o_\sigma + 2, o_\sigma + 4\}$. Two Type-I jobs (ascent bound 1) at $\{o_\sigma + 1, o_\sigma + 2\}$ can reach at most $o_\sigma + 3 < o_\sigma + 4$, so neither can occupy $o_\sigma + 4$ on day 1 (Lean Theorem 5; Certificate `C03`). Hence every active strip contains a Type-II job, so every strip holds one Type-I and one Type-II job, and soundness follows identically to [1, Lemma 3.8].

Minimality (a)–(c): (a) If $g \le 2$, then $o_{\sigma+1} - (o_\sigma + 7) \le 2$, permitting a Type-II job at $o_\sigma + 7$ to enter strip $\sigma + 1$. (b) If $H \le 4$, then on a clause block with low port $u_a = o_\sigma$, a Type-II job at

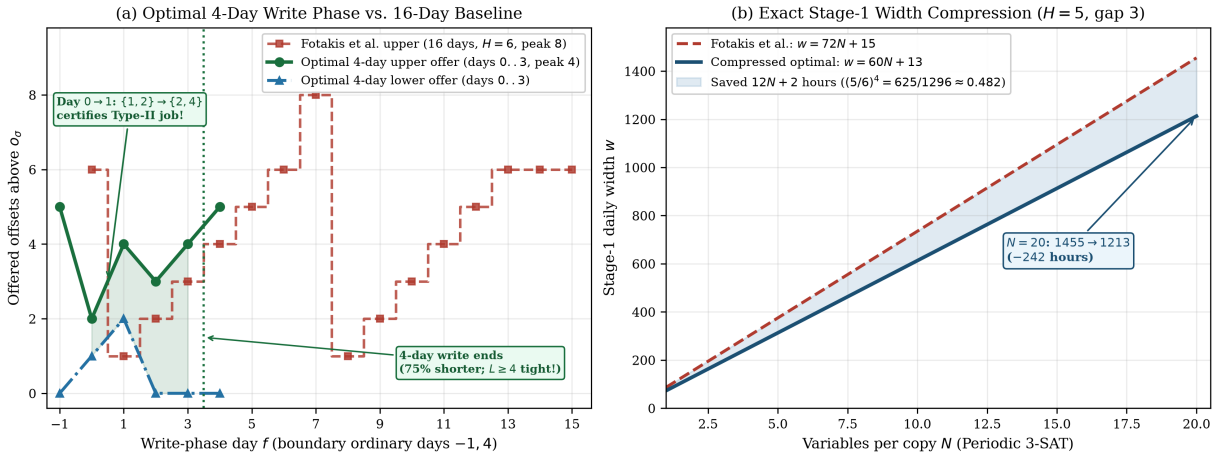

Figure 1: Optimal Stage-1 compression (Theorem 1). (a) The unconditionally minimal 4-day (1-neutral-block) write phase $\mathcal{W}_4 = (\{1,2\},\{2,4\},\{0,3\},\{0,4\})$ certifies Type-II presence on Day $0 \to 1$ while cutting write overhead by 75%. (b) Daily width drops from $72N + 15$ to $60N + 13$, reducing the Stage-2 quartic factor w^4 by $(5/6)^4 = 625/1296 \approx 0.482$.

o_σ on row 0 can move to $o_\sigma + 2$ on row 1 and then to $o_\sigma + H \le o_\sigma + 4$ on row 2 (each step ≤ 2, while Type-I stays at $o_\sigma + 4$ and descends to o_σ), flipping the stored bit without a write phase (Lean Theorem 3; Certificate `C02`). (c) To rewrite $1 \to 0$ across an L-day write phase flanked by ordinary hours $\{o_\sigma, o_\sigma + 5\}$ on days -1 and L, the Type-I job starts at o_σ on day -1 and must reach $o_\sigma + 5$ on day L in $L+1$ transitions of ascent at most 1 each, forcing $L+1 \ge 5 \iff L \ge 4$ unconditionally (Lean Theorem 4; Certificate `C03`). □

3 Proof of Theorem 2: Universal 5-Prime Witness Optimality

Proof of Theorem 2. **Part (a).** Because $G = ([r], E)$ has $\chi(G) \ge 3$ and $\omega(G) \le 2$, G is non-bipartite and triangle-free. Let $C \subseteq G$ be a shortest odd cycle of G; by minimality of length and triangle-freeness, $C \cong C_{2s+1}$ is an *induced* subgraph of G for some integer $s \ge 2$. Its number of non-edges is

$$\binom{2s+1}{2} - (2s+1) = (2s+1)(s-1).$$

For every prime $p \mid K$, let $S_p := \{v \in V(C) : p \mid c_v\}$. Because every edge $uv \in E(C)$ satisfies $\gcd(c_u, c_v) = 1$, no prime p can divide both c_u and c_v for an edge $uv \in E(C)$. Hence S_p is an *independent set* of C_{2s+1}, so $|S_p| \le \alpha(C_{2s+1}) = s$ (independently of the exponent of p in K). Consequently, each prime $p \mid K$ can divide $\gcd(c_u, c_v)$ for at most $\binom{|S_p|}{2} \le \binom{s}{2} = \frac{s(s-1)}{2}$ non-edges of C.

By definition of G, every non-edge $uv \notin E(C)$ satisfies $\gcd(c_u, c_v) > 1$, hence is covered by at least one prime $p \mid K$. Let $\omega_{\text{prime}}(K)$ be the number of distinct prime divisors of K. Double-counting non-edges gives

$$\omega_{\text{prime}}(K) \cdot \frac{s(s-1)}{2} \ge (2s+1)(s-1).$$

For every $s \ge 2$, $4 \cdot \frac{s(s-1)}{2} = 2s(s-1) < (2s+1)(s-1)$ strictly (Lean Theorem 6; Figure 2(a); Certificate `C05`). Therefore $\omega_{\text{prime}}(K) \ge 5$. Since any five strictly increasing primes satisfy $p_1 \ge 2, p_2 \ge 3, p_3 \ge 5, p_4 \ge 7, p_5 \ge 11$, we obtain $K \ge 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 = 2310$ (Lean Theorem 7).

Table 1: Top, Fotakis et al., and worst cyclic prime assignments of $\{2,3,5,7,11\}$ along $\overline{C}_5$ (canonicalized with $p_0 = 2$; complete 12-row ranking in Certificate `C06`). Note that reversing $(2,3,7,11,5)$ and shifting gives $(3,2,5,11,7) \mapsto (6,10,55,77,21)$ in (5), and $(10,55,33,21,14)$ is Fotakis et al.'s $(14,33,10,21,55)$ re-indexed along $\overline{C}_5$.

Rank	Cyclic Prime Order ($\overline{C}_5$)	Multipliers ($c_0,\ldots,c_4$)	Uses /2310	Rigid Fillers	vs. Baseline
1	$(\mathbf{2},\mathbf{3},\mathbf{7},\mathbf{11},\mathbf{5})$	$(\mathbf{6},\mathbf{21},\mathbf{77},\mathbf{55},\mathbf{10})$	**798**	**3822**	**−180**
2	$(2,3,11,7,5)$	$(6,33,77,35,10)$	782	3838	−164
3	$(2,3,5,11,7)$	$(6,15,55,77,14)$	776	3844	−158
9	$(2,5,11,3,7)$ [Fotakis et al.]	$(10,55,33,21,14)$	618	4002	0
12	$(2,7,5,3,11)$	$(14,35,15,33,22)$	560	4060	+58

Part (b). Fix $G \cong C_5$ ($s = 2$) and $K = 2310 = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11$. By Part (a), each of the five primes $p \in \{2,3,5,7,11\}$ has $|S_p| \le \alpha(C_5) = 2$ and therefore covers at most 1 of the 5 non-edges of C_5. Exhaustively enumerating all independent sets of C_5 and all $5^5 = 3125$ assignments of the five primes confirms that covering all 5 non-edges forces a bijection ($5! = 120$ covering assignments) onto the 5 edges of the complement $\overline{C}_5 \cong C_5$ (Certificate `C08`). Thus every vertex v must be divisible by the product of the two primes assigned to its two incident non-edges; multiplying any c_v by an additional prime or higher power strictly decreases K/c_v and increases the filler count $2K - \sum_v K/c_v$. Up to dihedral symmetry of $\overline{C}_5$, there are $4!/2 = 12$ distinct cyclic orderings of $\{2,3,5,7,11\}$ (Table 1; Figure 2(b)). Exhaustive evaluation (Lean Theorem 8; Certificate `C06`) proves that the cyclic order $(3,2,5,11,7)$ (equivalently $(2,3,7,11,5)$ in Table 1's canonical $p_0 = 2$ orientation) uniquely maximizes $\sum_v K/c_v$ at

$$\frac{2310}{6} + \frac{2310}{10} + \frac{2310}{55} + \frac{2310}{77} + \frac{2310}{21} = 385 + 231 + 42 + 30 + 110 = 798,$$

requiring $2(2310) - 798 = 3822$ rigid fillers versus 4002 for Fotakis et al.'s tuple $(14,33,10,21,55)$ (listed in C_5 order in [1], corresponding to $(10,55,33,21,14)$ in $\overline{C}_5$ order, rank 9 of 12). Choosing distinct residues $(0,1)$ modulo each shared prime and solving CRT for each vertex in $\overline{C}_5$ order $(6,10,55,77,21)$ gives $\beta = (4,5,11,56,15)$: all 5 edges $(0,2),(0,3),(1,3),(1,4),(2,4)$ of C_5 are pairwise coprime with products dividing 2310 (exhaustively verified across all 2927 adversarial residue pairs (b_u, b_v) via CRT in Certificate `C09`), while all 5 non-edges are separated modulo $2,5,11,7,3$ respectively (Lean Theorem 9; full 2310-day verification in Certificate `C07`). □

4 Proof of Theorem 3: Moving-Rate Slack and Exact Per-Job Cut Theorem

Proof of Theorem 3. **Part (a).** By saturation ($\sum_{i=1}^n 1/b_i = w$) and clock density $1 - w/T$,

$$1 - \rho = \frac{w}{T} - \sum_{i=1}^{n} \frac{1}{Tb_i + \delta_i} = \sum_{i=1}^{n} \frac{\delta_i}{Tb_i(Tb_i + \delta_i)} = \sum_{i:\,\delta_i > 0} \frac{\delta_i}{Tb_i(Tb_i + \delta_i)},$$

since rigid jobs ($\delta_i = 0$) contribute identically zero (Lean Theorem 12). Because $\delta_i > 0$ implies $Tb_i(Tb_i + \delta_i) > T^2 b_i^2$ (Lean Theorem 11), summing over moving jobs yields (7) (Certificates `C10`, `C16`).

Now fix a periodic feasible Pinwheel schedule with period Π and payload frequencies $f_i \ge 1/d_i$ satisfying $\sum_i f_i \le w/T$ [1, Lemma 5.4]. Then $\sum_{i=1}^n (f_i - 1/d_i) \le 1 - \rho < S_2/T^2$. As in [1, Lemma 5.5], every forward gap $g \le d_i = Tb_i + \delta_i$ satisfies $|g - Tb_i| \le (d_i - g) + \delta_i$ (Lean Theorem 13). Choosing a uniform random start among the Π slots of a period, the expected

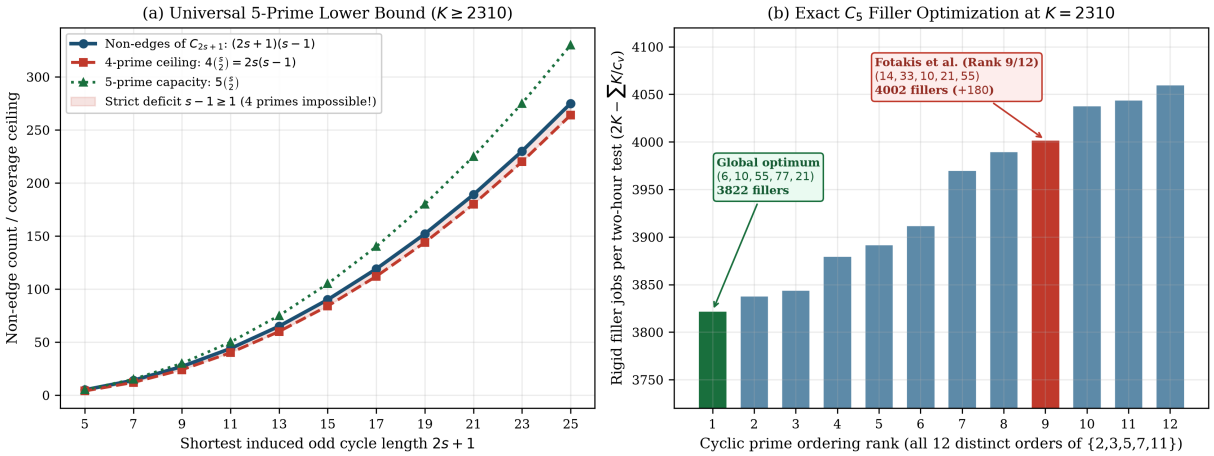

Figure 2: Universal witness optimality (Theorem 2). (a) For every induced odd cycle C_{2s+1} ($s \geq 2$), 4 primes cover at most $2s(s-1)$ non-edges, strictly below $(2s+1)(s-1)$, forcing $K \geq 2310$ unconditionally. (b) Exact filler counts across all 12 cyclic orders of $\{2,3,5,7,11\}$ at $K = 2310$.

total deviation in a window of ΩT slots is bounded by

$$\Omega T \sum_{i=1}^{n} (f_i d_i - 1 + f_i \delta_i) = \Omega T \left(\sum_{i=1}^{n} d_i \Big(f_i - \frac{1}{d_i} \Big) + \sum_{i:\, \delta_i > 0} \frac{\delta_i}{d_i} + \sum_{i:\, \delta_i > 0} \delta_i \Big(f_i - \frac{1}{d_i} \Big) \right).$$

Since $T > \Delta$ gives $\max_i d_i \leq T(b^* + 1)$ and $1/d_i < 1/(Tb_i)$, the three terms are bounded respectively by $\Omega T \cdot T(b^* + 1)(S_2/T^2) = \Omega(b^* + 1)S_2$, $\Omega T \cdot (S_1/T) = \Omega S_1$, and $\Omega T \cdot \Delta(S_2/T^2) = \Omega \Delta S_2/T < \Omega S_2$. Because total deviation is an integer, some ΩT-slot window attains total deviation at most E_{sharp} in (8) (Certificate `C11`).

Part (b). Fix an ΩT-slot window with total deviation $\sum_{i=1}^{n} E_i \leq E < T$, where E_i is the total deviation of payload job i in the window. For each job i with occurrences $x_i(0) < x_i(1) < \cdots < x_i(m_i)$ in the window, define its cumulative signed drift

$$D_i(\iota) := x_i(\iota) - \big(x_i(0) + \iota T b_i\big), \qquad 0 \leq \iota \leq m_i,$$

so $D_i(0) = 0$ and $\mathrm{TV}(D_i) = \sum_{\iota=0}^{m_i - 1} |D_i(\iota + 1) - D_i(\iota)| = E_i$. Let $m_i^- := \min_\iota D_i(\iota) \leq 0$ and $m_i^+ := \max_\iota D_i(\iota) \geq 0$, attained at indices ι^- and ι^+. Applying the triangle inequality along the sub-walk between ι^- and ι^+ (Lean Theorems 14–15; Certificate `C15`) yields

$$\mathrm{span}(D_i) := m_i^+ - m_i^- = |D_i(\iota^+) - D_i(\iota^-)| \leq \mathrm{TV}(D_i) = E_i.$$

Put $L_i := x_i(0) + m_i^-$ and $R_i := x_i(0) + m_i^+$, so $R_i - L_i \leq E_i \leq E < T$. Define the forbidden cut set for job i as

$$F_i := \{(L_i + s) \bmod T : 1 \leq s \leq R_i - L_i\} \subseteq \{0, \ldots, T-1\},$$

which has cardinality $|F_i| = R_i - L_i \leq E_i$ (note $F_i = \emptyset$ whenever $E_i = 0$). Taking the union over all n jobs gives

$$\left| \bigcup_{i=1}^{n} F_i \right| \leq \sum_{i=1}^{n} |F_i| \leq \sum_{i=1}^{n} E_i \leq E < T.$$

By pigeonhole (Lean Theorem 16), there exists a residue $\gamma \in \{0, \ldots, T-1\} \setminus \bigcup_{i=1}^{n} F_i$. Write $L_i - \gamma = q_i T + r_i$ with $q_i \in \mathbb{Z}$ and $0 \leq r_i < T$. Because $\gamma \notin F_i$, none of $L_i + 1, \ldots, R_i$ is congruent to $\gamma \pmod T$, which implies $r_i + (R_i - L_i) < T$. Hence for every $0 \leq \iota \leq m_i$,

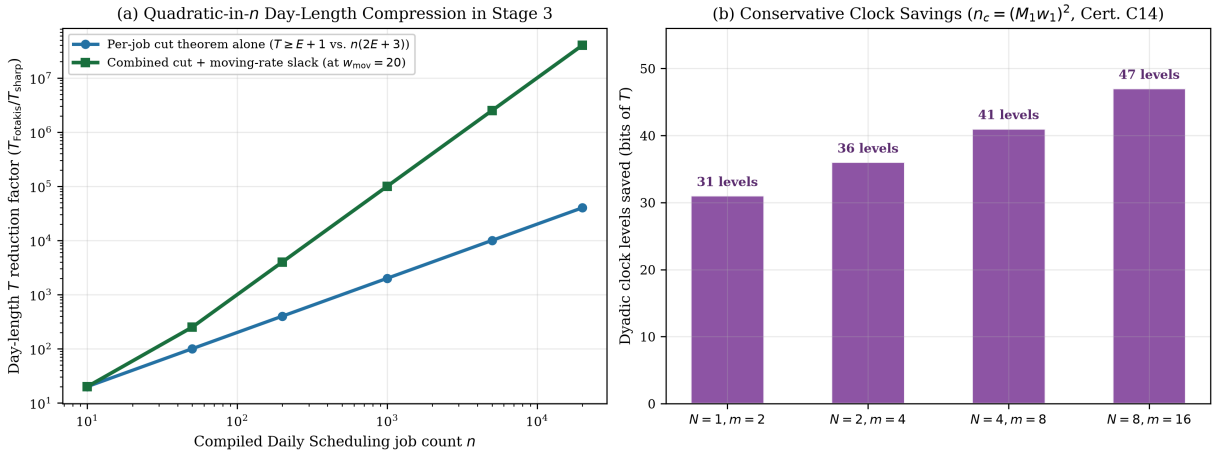

Figure 3: Stage-3 soundness compression (Theorem 3). (a) Day-length reduction factor $T_{\text{Fotakis}}/T_{\text{sharp}}$ from the exact per-job cut theorem alone ($> 2n$) and combined with moving-rate slack S_2 (illustrated at $w_{\text{mov}} = 20$). (b) Conservative dyadic clock-level savings using the surrogate lower bound $n_c = (M_1 w_1)^2$ across compiled instances (Certificate `C14`).

Table 2: Summary of the 16 AST-audited zero-float cryptographic certificates (`C01`–`C16`) and Lean 4.35.0 formalization coverage.

Certificates	Verification Scope	Lean 4
`C01`–`C04`	Stage-1 compression ($w = 60N + 13$, $M = 8m + 8$), sharpness, simulation	Thms. 1–5, 10
`C05`–`C09`	Universal 5-prime $K \geq 2310$ bound, 12-order ranking, C_5 2310-day & CRT checks	Thms. 6–9
`C10`–`C11`	Stage-3 moving-rate slack S_2/T^2 and sharpened Lemma 5.5 bound	Thms. 11–13
`C12`–`C15`	Per-job cut $T = E + 1$ check, $T = E$ tightness, span $\leq$ TV, clock savings	Thms. 14–18
`C16`	Dyadic clock schedule ($w = 2, T = 16$): exact slack $1/1056 < 1/1024 < 3/256$	Thms. 11–12

writing $\text{off} := D_i(\iota) - m_i^- \in [0, R_i - L_i]$, we have $x_i(\iota) - \gamma = (q_i + \iota b_i)T + r_i + \text{off}$, whose exact Euclidean quotient modulo T over $\mathbb{Z}$ is $q_i + \iota b_i$ (Lean Theorem 17; exhaustive verification in Certificate `C12`). Thus consecutive occurrences of every job i in the window lie exactly b_i days apart, and Lemmas 5.7–5.9 of [1] complete Stage-3 soundness unchanged. Finally, for every $n \geq 2$, $2n(E+1) < n(2E+3) + 1$ (Lean Theorem 18; Figure 3).

Part (c). Fix any $T \geq 2$ and consider a single job with $x(0) = 0$, $b_1 = 1$, and occurrences $x(\iota) = \iota(T+1)$ for $\iota = 0, 1, \ldots, T$, whose cumulative drift is $D(\iota) = x(\iota) - \iota T = \iota$ with total variation $E = T$. Its forbidden cut set is $F_1 = \{1, 2, \ldots, T\} \bmod T = \{0, \ldots, T-1\}$ of size T; indeed, for every cut $\gamma \in \{0, \ldots, T-1\}$, the projected drift positions $x(\iota) - \iota T \in \{0, \ldots, T\}$ span at least two distinct days, forcing some consecutive pair of occurrences $x(\iota), x(\iota+1)$ to fall $2 \neq b_1$ days apart (Certificate `C13`). Thus $T \geq E + 1$ is sharp to the single slot. □

5 Formal Verification and Exact Cryptographic Certificates

All 18 core algebraic and combinatorial theorems of this work are formally verified in constructive Lean 4.35.0 (`lean/PinwheelSharpReduction.lean`, zero `sorry`, zero `Classical.choice`, zero custom axioms, zero warnings, zero external dependencies) and backed by an AST-audited zero-float Python verifier (`verification/verify_paper107.py`) generating 16 SHA-256 certificates (`C01`–`C16`) and 56 adversarial `pytest` checks (Table 2). The canonical `certificates/MANIFEST.json` SHA-256 digest is

`77d75d25e0e47b80c40539f796191ee85174c8a56fd52d75ba295e8a24f8f422`.

Reproducibility and Governance

All Lean 4 proofs, zero-float verification scripts, cryptographic certificates, figure scripts, and the manuscript source are publicly released under dual licensing (MIT for code/Lean/certificates; CC-BY-4.0 for manuscript and figures) at `https://github.com/lluiseriksson/pinwheel-pspace-sharp-gadgets-and-slack` (`v1.0.0`). Full replay requires only Lean 4.35.0 (`cd lean && lake build`) and Python 3.10+ (`python verification/verify_paper107.py && pytest -q`).

References

[1] D. Fotakis, S. Kanellopoulos, G. Mitropoulos, A. Pagourtzis, C. Papadimitriou, C. Pergaminelis, and T. Tolias, *Pinwheel scheduling is PSPACE-complete*, preprint (2026), `https://arxiv.org/abs/2611.34869`.

[2] R. Holte, A. Mok, L. E. Rosier, I. Tulchinsky, and D. A. Varvel, *The pinwheel: A real-time scheduling problem*, Proc. 22nd Annual Hawaii Int. Conf. on System Sciences, Vol. II (1989), 693–702.

[3] A. Kawamura, *Proof of the density threshold conjecture for pinwheel scheduling*, Proc. 56th ACM STOC (2024), 1816–1819; journal version, Proc. Natl. Acad. Sci. USA **123** (2026), e2530214123.

[4] R. Kleinberg and A. Mishra, *NP-hardness and a PTAS for the pinwheel problem*, Proc. 67th IEEE FOCS (2026), `https://arxiv.org/abs/2604.13974`.

[5] Y. Kobayashi, B. Lin, and J. Swernofsky, *Dense pinwheel packing is strongly NP-complete*, preprint (2026), `https://arxiv.org/abs/2609.20075`.

[6] J. B. Orlin, *The complexity of dynamic languages and dynamic optimization problems*, Proc. 13th ACM STOC (1981), 218–227.