Orthogonal Measurements Maximize All-Degree Born Rigidity among Equal-Trace Rank-One POVM Orbits in Dimension Five and Above

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Abstract

Under the explicit noncontextual homogeneous-response ansatz $p(wP_x) = wf(x)$, fix a finite rank-one POVM and rotate the entire measurement by every unitary in dimension d. How strongly does normalization on this restricted orbit force the scalar ray response toward a trace-one Hermitian quadratic, and which seed gives the best worst-degree stability? We solve the outcome-simple equal-trace design problem globally for all $d \geq 5$. For a weighted seed $\mathcal{F} = \{(w_i, x_i)\}_{i=1}^N$ on $\mathbb{CP}^{d-1}$, the orbit-normalization operator has squared singular value

$$g_k(\mathcal{F}) = \sum_{i,j} w_i w_j \frac{P_k^{(d-2,0)}(2|\langle x_i, x_j\rangle|^2 - 1)}{\binom{d+k-2}{k}}$$

on the projective harmonic sector of degree k. Thus the optimal L^2 inverse constant off the affine trace-one quadratic sector is $\gamma(\mathcal{F})^{-1/2}$, where $\gamma = \inf_{k\geq 2} g_k$. For $d \geq 3$, outcome-simple equal-trace seeds satisfy $g_k \to d^2/N$. Let $\mathfrak{F}_d$ denote all finite outcome-simple equal-trace rank-one seeds in dimension d. Combining the all-degree ceiling with the exact spectrum of an orthonormal basis and the forced simplex geometry at $N = d + 1$, we prove

$$\sup_{\mathcal{F}\in\mathfrak{F}_d} \gamma(\mathcal{F}) = d - \frac{6}{d+1}, \qquad d \geq 5,$$

with equality only for an orthonormal basis, up to unitary equivalence and relabeling. Distinct-outcome overcompleteness is therefore strictly detrimental within this outcome-simple equal-trace class in every dimension at least five. Applying the known harmonic characterization of projective designs, every weighted complex projective 2-design has $g_2 = 0$; hence every SIC, whenever it exists, is maximally non-rigid despite its tomographic advantages. These results optimize a response-field normalization problem, not state tomography, and do not derive state update, intrinsic randomness, or microscopic particle dynamics.

Keywords: Born rule; POVM; complex projective harmonics; Jacobi polynomials; measurement design; Gleason theorem; inverse stability.

1 Introduction

Gleason-type theorems constrain probability assignments by requiring consistent normalization across a declared family of measurements. For orthonormal projective measurements in complex dimension at least three, exact consistency forces a quadratic Born response [1]. Effect-valued and Parseval-frame extensions enlarge the event family and include qubits [2, 3, 5]. Moretti and Pastorello gave an L^2 proof using generalized complex spherical harmonics [4]. These results establish exact form under their respective hypotheses; they do not by themselves answer a measurement-design question: among restricted covariant families, which seed measurement most stably detects every non-Born component of an arbitrary response field?

For qubits, Caves, Fuchs, Manne, and Renes derived a spherical-harmonic zero criterion for several rotationally covariant POVM families and identified both trine rigidity and tetrahedral non-Born harmonics [3]. Quantitative inversion replaces the binary question "is a harmonic visible?" by the smallest squared singular value over all non-Born degrees. That objective differs sharply from tomography. Tomography reconstructs a state after assuming Born linearity, whereas the present inverse problem tests a general scalar response field containing harmonics of every degree. Projective designs can be optimal for the former while deliberately annihilating modes required by the latter [7, 8, 9].

This paper develops and optimizes the corresponding all-dimensional operator on $\mathbb{CP}^{d-1}$ under the noncontextual homogeneous-response ansatz $p(wP_x) = wf(x)$. The representation-theoretic ingredients—multiplicity-free decomposition, reproducing kernels, and Schur orthogonality—are standard and are not presented as a new form of Gleason's theorem. To our knowledge, the quantitative package established here has not previously been stated: the weighted seed-orbit singular-value formula as an all-degree condition number, its sharp inverse inequality, the high-degree effective-outcome ceiling, and the exact global optimizer theorem. In the natural class of finite, outcome-simple, equal-trace rank-one POVMs, orthonormal bases uniquely maximize the Born-rigidity gap for every $d \geq 5$, even though many overcomplete measurements are superior for tomography.

The proof is short only after the correct invariant has been isolated. It uses three regimes. A seed with $N = d$ outcomes is forced to be an orthonormal basis, whose complete spectrum has its first non-Born minimum in degree three. A seed with $N = d + 1$ outcomes is forced to be the regular projective simplex; its degree-three value is already strictly below the basis gap when $d \geq 5$. For $N \geq d + 2$, normalized Jacobi decay gives the universal high-degree ceiling $\gamma \leq d^2/N$, again strictly below the basis value. No numerical global optimization enters the theorem.

The closest boundaries are precise. Caves et al. give the exact zero/nonzero harmonic-moment criterion for restricted rotationally covariant qubit POVMs, but not an all-dimensional quantitative gap or global design optimum [3]. Moretti–Pastorello provide the L^2 projective-harmonic Gleason machinery, not a finite seed-orbit condition number [4]. Scott, Roy–Scott, and Saini et al. optimize tomography or finite-dimensional completeness, whereas the present infimum runs over every non-Born harmonic degree [7, 8, 9]. Projective-design annihilation of low harmonics is known; the contribution here is its consequence for, and contrast with, the optimized all-degree inverse objective.

Our contributions are:

1. the exact global optimum and uniqueness theorem for every $d \geq 5$ within finite outcome-simple equal-trace rank-one seeds;
2. the weighted $\mathbb{CP}^{d-1}$ seed-orbit singular-value formula and its sharp L^2 inverse inequality;
3. for $d \geq 3$, the asymptotic effective-outcome ceiling, including the equal-trace limit $g_k \to d^2/N$ for distinct rays;
4. application of the known projective-design moment identity to the all-degree obstruction $g_2 = 0$, placing SIC and mutually unbiased-basis designs on the opposite side of the tomography/rigidity distinction;
5. an explicit boundary analysis showing exactly which assumptions carry the theorem and which low-dimensional or weighted problems remain open.

The result concerns one-step response fields after the quantum event space and the rotated measurement family have been specified. It neither identifies a microscopic mechanism nor rules out deterministic, chaotic, or infinite-dimensional implementations. Such a mechanism is constrained only when it induces the response interface studied here.

2 Weighted seed POVMs and the orbit transform

Let $X_d = \mathbb{CP}^{d-1}$ carry normalized invariant measure μ, and let $G = U(d)$ carry normalized Haar measure. A ray $x \in X_d$ is represented by a unit vector, with $P_x = |x\rangle\langle x|$ independent of phase.

Definition 2.1 (Rank-one seed)**.** A weighted rank-one POVM seed is

$$\mathcal{F} = \{(w_i, x_i)\}_{i=1}^N, \qquad w_i > 0, \qquad \sum_{i=1}^N w_i P_{x_i} = I_d.$$

It is *equal trace* if $w_i = d/N$ for all i. It is *outcome-simple* if the rays x_i are pairwise distinct.

The response convention in (1) is homogeneous in the effect scale: the response attached to wP_x is written as $wf(x)$. For unequal weights this is a substantive modelling assumption and is not inferred from normalization alone. In the equal-trace optimization class it fixes one common scale for every outcome.

Taking traces gives $\sum_i w_i = d$. Outcome simplicity removes proportional duplicate effects. Such duplicates are classical refinements of one ray outcome and can be restored after coarsening; see Section 7.2.

For $f \in L^2(X_d)$ define the orbit-normalization transform

$$(T_{\mathcal{F}} f)(U) = \sum_{i=1}^N w_i f(U x_i), \qquad U \in G. \tag{1}$$

The expression is meaningful on L^2 equivalence classes, even though isolated point evaluation is not.

Lemma 2.2 (Bounded pullback)**.** *For fixed $x \in X_d$, $J_x f(U) = f(Ux)$ defines a well-defined linear isometry from $L^2(X_d)$ into $L^2(G)$. Consequently $T_{\mathcal{F}}$ is bounded and $\|T_{\mathcal{F}}\| \le d$.*

Proof. The pushforward of Haar measure under $U \mapsto Ux$ is μ. Hence

$$\int_G |f(Ux)|^2 \, \mathrm{d}U = \int_{X_d} |f(y)|^2 \, \mathrm{d}\mu(y).$$

This identity both removes dependence on representatives and proves the isometry. The triangle inequality and $\sum_i w_i = d$ give the operator bound. □

The affine trace-one quadratic sector is

$$\mathcal{B}_{\text{aff}} = \{q_A(x) = \langle x, Ax\rangle : A = A^*, \ \operatorname{Tr} A = 1\}.$$

For every $q_A \in \mathcal{B}_{\text{aff}}$, POVM completeness gives $T_{\mathcal{F}} q_A = 1$. This sector is sometimes called generalized Born form; A is not assumed positive. The inverse problem asks how far a general f can be from this affine space when $T_{\mathcal{F}} f$ is close to one.

3 Exact projective-harmonic spectrum

The unitary representation on $L^2(X_d)$ has the multiplicity-free decomposition

$$L^2(X_d) = \widehat{\bigoplus}_{k=0}^{\infty} \mathcal{H}_k, \tag{2}$$

where $\mathcal{H}_k$ has highest weight $(k, 0, \dots, 0, -k)$ and

$$D_{k,d} := \dim \mathcal{H}_k = \binom{d+k-1}{k}^2 - \binom{d+k-2}{k-1}^2.$$

The constant sector is $\mathcal{H}_0$ and $\mathcal{H}_1$ is the space of traceless Hermitian quadratics. Let

$$Q_{k,d}(t) = \frac{P_k^{(d-2,0)}(2t-1)}{P_k^{(d-2,0)}(1)} = \frac{P_k^{(d-2,0)}(2t-1)}{\binom{d+k-2}{k}}. \tag{3}$$

The reproducing kernel of $\mathcal{H}_k$ is $K_k(x,y) = D_{k,d}Q_{k,d}(|\langle x,y\rangle|^2)$.

Theorem 3.1 (Seed-orbit spectrum)**.** *For every weighted rank-one seed $\mathcal{F}$ and every $f_k \in \mathcal{H}_k$,*

$$\|T_{\mathcal{F}} f_k\|^2_{L^2(G)} = g_k(\mathcal{F})\|f_k\|^2_{L^2(X_d)}, \tag{4}$$

where

$$\boxed{g_k(\mathcal{F}) = \sum_{i,j=1}^{N} w_i w_j Q_{k,d}(|\langle x_i, x_j\rangle|^2).} \tag{5}$$

Images of different harmonic sectors are orthogonal. Moreover $g_k \geq 0$, $g_0 = d^2$, and POVM completeness is equivalent to the degree-one moment condition $g_1 = 0$.

Proof. Use the group action $(Uh)(x) = h(U^{-1}x)$. For fixed k, put $m_k = \sum_i w_i K_k(x_i, \cdot) \in \mathcal{H}_k$. The reproducing property and covariance give

$$(T_{\mathcal{F}} f_k)(U) = \langle f_k, U m_k\rangle.$$

Schur orthogonality in the irreducible space $\mathcal{H}_k$ yields

$$\int_G |\langle f_k, U m_k\rangle|^2 \, \mathrm{d}U = \frac{\|f_k\|^2\|m_k\|^2}{D_{k,d}}.$$

Using the reproducing kernel,

$$\frac{\|m_k\|^2}{D_{k,d}} = \sum_{i,j} w_i w_j Q_{k,d}(|\langle x_i, x_j\rangle|^2),$$

which proves (4)–(5) and nonnegativity. Inequivalent irreducible sectors have orthogonal matrix coefficients. Finally, $Q_{0,d} = 1$ gives $g_0 = (\sum_i w_i)^2 = d^2$. Since $\mathcal{H}_1$ consists of $x \mapsto \mathrm{Tr}(A_0 P_x)$ with $\mathrm{Tr}\, A_0 = 0$, the degree-one moment vanishes exactly when $\sum_i w_i P_{x_i}$ is scalar; its trace fixes that scalar as I_d. □

Definition 3.2 (Born-rigidity gap)**.** The all-degree gap of a seed is

$$\gamma(\mathcal{F}) = \inf_{k\geq 2} g_k(\mathcal{F}). \tag{6}$$

Theorem 3.3 (Sharp inverse inequality)**.** *If $\gamma(\mathcal{F}) > 0$, then every real $f \in L^2(X_d)$ satisfies*

$$\boxed{\mathrm{dist}_{L^2}(f, \mathcal{B}_{\mathrm{aff}}) \leq \gamma(\mathcal{F})^{-1/2}\|T_{\mathcal{F}} f - 1\|_{L^2(G)}.} \tag{7}$$

The constant is optimal. It remains optimal when sharpness is tested by sufficiently small probability-valued perturbations of the constant response $1/d$.

Proof. Write $f = c + f_1 + \sum_{k\geq 2} f_k$ according to (2). Orthogonality and theorem 3.1 give

$$\|T_{\mathcal{F}} f - 1\|_2^2 = |dc - 1|^2 + \sum_{k\geq 2} g_k \|f_k\|_2^2.$$

The nearest trace-one Hermitian quadratic is $1/d + f_1$, so

$$\operatorname{dist}(f, \mathcal{B}_{\mathrm{aff}})^2 = |c - 1/d|^2 + \sum_{k \geq 2} \|f_k\|_2^2.$$

The normalized zonal function is a unitary matrix coefficient, so $|Q_{k,d}(t)| \leq 1$ and therefore $0 \leq g_k \leq (\sum_i w_i)^2 = d^2$. Hence $\gamma \leq d^2$, and comparison proves (7). A unit vector in a sector whose g_k attains or approaches the infimum proves optimality. Each fixed-sector harmonic is continuous and bounded; multiplying it by a sufficiently small scalar makes $1/d + th$ take values in $[0, 1]$ without changing the ratio of the two norms. □

For $d \geq 3$ and an outcome-simple finite seed, Proposition 4.2 below shows that the spectral tail has a positive limit. Hence exact rigidity is equivalent to $g_k > 0$ for every $k \geq 2$, and then automatically implies a positive quantitative gap. The qubit endpoint has a separate parity structure.

4 The high-degree ceiling

The decisive global constraint comes from degrees tending to infinity rather than from a low-order design equation.

Lemma 4.1 (Normalized Jacobi decay)**.** *Fix $d \geq 3$ and $0 \leq t < 1$. Then*

$$Q_{k,d}(t) \longrightarrow 0 \qquad (k \to \infty).$$

Proof. At $t = 0$, the endpoint identity gives

$$Q_{k,d}(0) = \frac{(-1)^k}{\binom{d+k-2}{k}} \longrightarrow 0.$$

For $0 < t < 1$, the Darboux asymptotic for Jacobi polynomials at a fixed interior point gives $P_k^{(d-2,0)}(2t-1) = O_{d,t}(k^{-1/2})$, whereas $P_k^{(d-2,0)}(1) = \binom{d+k-2}{k} = \Theta_d(k^{d-2})$ [12, 13]. Their ratio tends to zero. □

Proposition 4.2 (Effective-outcome ceiling)**.** *Let $d \geq 3$, and let the distinct rays in a weighted seed have aggregate weights β_s after proportional duplicate effects are coarsened under the homogeneous-response ansatz. Then*

$$\lim_{k \to \infty} g_k(\mathcal{F}) = \sum_s \beta_s^2. \tag{8}$$

In particular, for an outcome-simple equal-trace seed with N rays,

$$\boxed{\lim_{k \to \infty} g_k(\mathcal{F}) = \frac{d^2}{N}, \qquad \gamma(\mathcal{F}) \leq \frac{d^2}{N}.} \tag{9}$$

Proof. In (5), pairs on the same ray have overlap one and contribute $Q_{k,d}(1) = 1$. Every pair of distinct rays has overlap strictly below one and vanishes by Lemma 4.1. Because the seed contains only finitely many fixed overlaps, pointwise Jacobi decay suffices and the limit passes through the double sum. Grouping identical rays proves (8). Equal weights d/N and distinct rays give $N(d/N)^2 = d^2/N$. □

Equation (9) is an all-degree obstruction. Adding distinct outcomes lowers the spectral plateau even when all low-degree moments appear well conditioned.

5 Global optimization in dimension five and above

Let $\mathfrak{F}_d$ be the union, over all finite N, of outcome-simple equal-trace rank-one POVM seeds in dimension d, modulo unitary equivalence and outcome relabeling. Define

$$\Gamma_d = \sup_{\mathcal{F}\in\mathfrak{F}_d} \gamma(\mathcal{F}).$$

We first record the two smallest outcome counts.

Lemma 5.1 (Minimal and minimally overcomplete frames)**.** *Every equal-trace rank-one seed has $N \geq d$.*

(i) If $N = d$, the seed is an orthonormal basis and every weight equals one.

(ii) If $N = d+1$, the seed is, up to unitary and phase equivalence, the regular projective simplex. Its weights are $d/(d+1)$ and

$$|\langle x_i, x_j\rangle|^2 = \frac{1}{d^2} \quad (i \neq j).$$

Proof. The frame operator has rank at most N, proving $N \geq d$. At $N = d$, the square synthesis matrix is a scalar multiple of a unitary; unit norms fix the scalar and give an orthonormal basis.

For $N = d+1$, let X be the $d \times (d+1)$ matrix of unit columns. Tightness gives $XX^* = (d+1)I_d/d$. Thus the Gram matrix $G = X^*X$ equals $(d+1)/d$ times a rank-d orthogonal projection. If u spans its kernel, the diagonal equations

$$1 = G_{ii} = \frac{d+1}{d}(1 - |u_i|^2)$$

give $|u_i|^2 = 1/(d+1)$. Rephasing the columns makes u constant, and the off-diagonal entries then have modulus $1/d$. ☐

For an orthonormal basis, all off-diagonal overlaps vanish, so

$$g_k^{\mathrm{ONB}}(d) = d\left[1 + (d-1)\frac{(-1)^k}{\binom{d+k-2}{k}}\right]. \tag{10}$$

Even degrees exceed d. Among odd degrees $k \geq 3$, the binomial denominator increases, so the minimum occurs at $k = 3$:

$$\boxed{\gamma_{\mathrm{ONB}}(d) = d - \frac{6}{d+1}.} \tag{11}$$

For the regular $d+1$ simplex, substitution in (5) gives its complete spectrum

$$g_k^{\mathrm{simp}}(d) = \frac{d^2}{d+1}\left[1 + dQ_{k,d}\left(\frac{1}{d^2}\right)\right]. \tag{12}$$

Only its degree-three value is needed for the global theorem:

$$g_3^{\mathrm{simp}}(d) = \frac{(d+3)(d^4 - 4d^3 + 7d^2 + 2d - 8)}{d^4}. \tag{13}$$

Theorem 5.2 (Global equal-trace optimum)**.** *For every complex dimension $d \geq 5$,*

$$\boxed{\Gamma_d = d - \frac{6}{d+1}.} \tag{14}$$

The maximizer is unique: equality holds only for an orthonormal-basis seed, up to a unitary and outcome relabeling.

Proof. An orthonormal basis attains the right-hand side by (11). We show that every outcome-simple overcomplete seed is strictly worse.

If $N = d + 1$, Lemma 5.1 forces the regular simplex and $\gamma \le g_3^{\mathrm{simp}}$. Direct simplification gives

$$g_3^{\mathrm{simp}}(d) - \gamma_{\mathrm{ONB}}(d) = -\frac{(d+2)(d+3)(d^3 - 5d^2 + d + 4)}{d^4(d+1)} < 0 \tag{15}$$

for $d \ge 5$. Indeed the cubic is positive at $d = 5$ and strictly increasing thereafter.

If $N \ge d + 2$, Proposition 4.2 gives

$$\gamma(\mathcal{F}) \le \frac{d^2}{N} \le \frac{d^2}{d+2}.$$

The remaining comparison is exact:

$$\frac{d^2}{d+2} - \gamma_{\mathrm{ONB}}(d) = -\frac{2(d^2 - 2d - 6)}{(d+1)(d+2)} < 0 \tag{16}$$

for $d \ge 5$. Therefore every overcomplete seed has smaller gap. At $N = d$, Lemma 5.1 gives the claimed uniqueness. □

Corollary 5.3 (Overcompleteness penalty)**.** *For $d \ge 5$, adding any distinct ray outcome to an equal-trace orthonormal seed necessarily worsens the optimal all-degree inverse constant. Quantitatively,*

$$\gamma(\mathcal{F}) \le \begin{cases} g_3^{\mathrm{simp}}(d), & N = d+1, \\ d^2/N, & N \ge d+2, \end{cases}$$

and both upper bounds are strictly below (11)*.*

6 Projective designs are rigidity nulls

The global optimizer theorem points in the opposite direction from familiar tomographic design principles.

Proposition 6.1 (Design obstruction)**.** *Let $p_i = w_i/d$, so that $\sum_i p_i = 1$. If the normalized weighted ray measure $\{(p_i, x_i)\}$ is a complex projective t-design, then*

$$g_k(\mathcal{F}) = 0, \qquad 1 \le k \le t.$$

In particular, every weighted complex projective 2-design has $\gamma(\mathcal{F}) = 0$.

Proof. A weighted projective t-design integrates every projective polynomial of bidegree at most (t, t) exactly. Every nonconstant $h \in \mathcal{H}_k$, $k \le t$, has Haar integral zero, hence its weighted seed moment vanishes. In the notation of theorem 3.1, $m_k = 0$, and therefore $g_k = \|m_k\|^2 / D_{k,d} = 0$. □

SIC-POVMs, in every dimension where they exist, are tight complex projective 2-designs [7]. Complete sets of mutually unbiased bases, when they exist, are weighted projective 2-designs [8]. Both therefore have zero Born-rigidity gap. This does not contradict their tomographic performance. A tomographic frame operator acts on the finite-dimensional state sector $\mathcal{H}_0 \oplus \mathcal{H}_1$ after Born linearity is assumed. The orbit-normalization operator instead tries to reject every sector $\mathcal{H}_k$ with $k \ge 2$. A design that deliberately cancels $\mathcal{H}_2$ is useful for quadrature and state reconstruction but fatal to this inverse problem.

The distinction can be summarized as

$$\boxed{\text{tomographic isotropy on the Born sector} \not\Rightarrow \text{rigidity against non-Born response fields.}}$$

The tetrahedral qubit example is known from Caves et al. [3]; Proposition 6.1 places it in an all-dimensional design hierarchy.

7 Boundary cases, refinements, and scope

7.1 Why dimension five appears

The proof of theorem 5.2 changes character at low dimension. For $d = 2$, an orthonormal projective basis has an infinite odd-harmonic kernel, while the trine has positive gap; orthogonality is therefore not optimal. In dimensions three and four, the comparison (15) reverses sign, so the minimally overcomplete simplex is not eliminated by its degree-three mode. Determining the global optimizer there requires a separate all-degree Jacobi analysis. The present theorem makes no claim about those two exceptional dimensions.

7.2 Outcome refinements

Remark 7.1 (Duplicate rays). Under the homogeneous-response ansatz, proportional duplicate effects are operationally coarsened by summing their weights. Uniformly splitting every orthonormal-basis effect can reproduce the same coarsened seed with arbitrarily many labels and would make literal uniqueness by outcome count meaningless. The optimizer is therefore stated on outcome-simple representatives. Labeled refinements with independent or context-dependent response rules are outside the model. For a seed with duplicates inside the homogeneous model, (8) shows that the relevant high-degree quantity is the squared aggregate weight $\sum_s \beta_s^2$ of distinct rays.

7.3 Unequal weights

The spectral theorem is fully weighted, but theorem 5.2 is not. Unequal weights change both the asymptotic ceiling and the geometry at fixed outcome count. Since each aggregate weight obeys $0 < \beta_s \leq 1$ and $\sum_s \beta_s = d$, one always has $\sum_s \beta_s^2 \leq d$, with equality only for d unit weights. This alone does not exclude weighted overcomplete seeds whose ceiling lies between (11) and d. Global weighted optimization remains open.

7.4 What the inverse theorem does and does not say

For approximate normalization, orthogonal projection produces a trace-one Hermitian quadratic, not automatically a positive density operator. A separate positivity-repair estimate is required for a finite-error physical-state conclusion. At zero defect, a nonnegative response field in the affine sector does correspond to a density operator: a negative eigenvalue would make the continuous quadratic negative on an open set.

The theorem assumes the complex projective event space, rank-one effects, a fixed seed and its full unitary orbit, and a reproducible one-step response field. It does not derive Hilbert space, the measurement update rule, tensor products, relativistic causality, or intrinsic randomness. A pendulum, rolling-circle hierarchy, deterministic chaotic system, or stochastic field model is covered only after it supplies this operational response map. The result then identifies the measurement family giving the strongest average normalization test within the stated class; it does not identify the ontology behind individual outcomes.

8 Reproducibility and proof boundary

The central optimizer theorem is analytic. Its proof depends on the exact identities (10), (13), (15), and (16), plus normalized Jacobi decay. The accompanying script independently:

1. constructs $Q_{k,d}$ symbolically from Jacobi polynomials;
2. verifies the orthonormal-basis and simplex degree-three formulas;

3. factors both comparison differences exactly;
4. checks representative rational spectra and the $d \geq 5$ sign threshold;
5. emits a machine-readable verification record.

The script is a replay of algebra, not an oracle used to bridge a missing proof. No local or remote branch-and-bound computation is part of theorem 5.2.

9 Discussion

Three lessons follow. First, restricted Born rigidity has a natural condition number: the smallest non-Born harmonic energy of the seed. Second, high harmonic degrees impose a simple but powerful outcome-count ceiling. Third, measurement overcompleteness is objective-dependent. It can improve state reconstruction while weakening—or completely destroying—the ability of rotated normalization identities to rule out general non-Born response fields.

The global theorem is deliberately narrower than a universal Born derivation. Within its class it is exact: for $d \geq 5$, no finite outcome-simple equal-trace rank-one POVM orbit has a larger all-degree gap than a projective orthonormal basis, and every distinct-outcome overcomplete seed in that class is strictly worse. Applying the known projective-design identity gives the complementary exact statement that every projective 2-design has a degree-two nullspace.

Natural next problems are the global equal-trace optimizers in $d = 3, 4$, the unequal-weight problem, finite deterministic unitary designs that retain quantitative control for bandlimited response fields, and causal extensions in which the response may adapt across a measurement history. Those problems should not be conflated with the theorem proved here.

10 Conclusion

We introduced the all-degree harmonic gap for a weighted rank-one POVM orbit on complex projective space and proved its sharp inverse meaning. The high-degree limit converts the number of distinct equal-trace outcomes into a universal ceiling. Together with the forced geometry of the two smallest frames, this solves the global design problem in every dimension $d \geq 5$: orthonormal bases uniquely maximize covariant Born rigidity, with gap $d - 6/(d+1)$.

The result also formalizes a broad separation from tomography. Weighted projective 2-designs—including SICs wherever they exist—annihilate a non-Born harmonic sector and have zero rigidity gap. Thus more symmetric or more informative measurements are not automatically stronger normalization tests. The relevant objective must be stated before a measurement can be called optimal.

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