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Schatten-q Crouzeix Constants and the Hilbert–Schmidt Phase Transition: Universal Conformal Majorization on $M_2(\mathbb{C})$, Odd–Even Clifford Parity Staircases, and Double-Layer Trace Bounds

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Abstract

For an $n \times n$ complex matrix $A \in M_n(\mathbb{C})$ with numerical range $W(A)$ and any exponent $q \in [1, \infty]$, we introduce the *normalized Schatten-q Crouzeix constant*

$$\Psi_q(A) := \sup_{\substack{p \in \mathbb{C}[z] \\ \|p\|_{W(A)} \le 1}} \|p(A)\|_{q,\mathrm{norm}}, \qquad \|X\|_{q,\mathrm{norm}} := \left(\frac{1}{n} \sum_{j=1}^{n} \sigma_j(X)^q \right)^{1/q},$$

normalized so that $\|I\|_{q,\mathrm{norm}} = 1$ and $\Psi_q(A) = 1$ for every normal matrix, recovering the classical Crouzeix constant at $q = \infty$. We establish four main structural laws governing this functional calculus across dimensions and Schatten exponents:

1. **Universal Linear/Disk Majorization and Clifford Parity Staircase (Theorem 2.1):** For every $n \ge 1$ and $A \in M_n(\mathbb{C})$, the singular values of A are weakly majorized by $s^{(n)} w(A)$, where $s^{(n)} = (2, \dots, 2, [1], 0, \dots, 0)$ has $\lfloor n/2 \rfloor$ twos and $(n \bmod 2)$ ones. Consequently, for every unitarily invariant norm Φ, $\|A\|_\Phi \le \|A^{(n)}_{\mathrm{ext}}\|_\Phi \, w(A)$, yielding the exact odd–even parity staircase $\sup_A \|A\|_{q,\mathrm{norm}}/w(A) = \left(\frac{\lfloor n/2 \rfloor 2^q + (n \bmod 2)}{n}\right)^{1/q}$. For $q \in (1, \infty)$, equality at $2^{1-1/q}$ is rigidly characterized by unitary representations of the Clifford algebra Cl_2 ($X^2 = Y^2 = I$, $XY + YX = 0$), forcing n even and $A \cong w(A) \bigoplus_{j=1}^{n/2} (2J_2)$. Via Berger's unitary 2-dilation, the identical majorization extends to all centered disk polynomials ($p(0) = 0$, $w(A) \le 1$).
2. **Universal Ky Fan Dominance on $M_2(\mathbb{C})$ (Theorem 3.2):** For every non-normal 2×2 matrix A_R with elliptical numerical range E_R of Joukowsky radius $R = a + b \in (1, \infty)$, the unique odd Riemann conformal map $\varphi_R : E_R \to \mathbb{D}$ with focus value $r(R) = \varphi_R(1) = \theta_2(R^{-4})/\theta_3(R^{-4})$ weakly majorizes $f(A_R)$ for *every* analytic function f with $\|f\|_{E_R} \le 1$. Thus φ_R simultaneously maximizes *every* unitarily invariant norm $\|f(A_R)\|_\Phi$, yielding the exact closed form $\Psi_q(A_R) = \frac{\theta_2(R^{-4})}{\theta_3(R^{-4})} \left(\frac{R^q + R^{-q}}{2}\right)^{1/q}$.
3. **Exact Hilbert–Schmidt Phase Transition at $q_c = 2$ (Theorem 4.1):** Across $M_2(\mathbb{C})$, the nilpotent disk J_2 ($R \to \infty$, $\Psi_q(J_2) = 2^{1-1/q}$) is uniquely extremal for all $q \ge 2$ ($\Psi_q(2) = 2^{1-1/q}$, proved via an elementary Jacobi-product bound certifying $\theta_2(\tau)^2/\theta_3(\tau)^2 < 4\sqrt{\tau}/(1+\tau)$). In stark contrast, for *every* $q \in [1, 2)$, the disk J_2 strictly fails to be extremal: an interior non-degenerate ellipse $E_{R^*(q)}$ strictly beats $2^{1-1/q}$, with critical divergence $R^*(q) = 4^{\frac{1}{4-2q}}(1 + o(1))$ and exponentially flat excess as $q \uparrow 2^-$. At the trace norm $q = 1$, $\Psi_1(J_2) = 1$ whereas $\Psi_1(A_R) > 1$ for every finite $R \in (1, \infty)$, peaking at $\Psi_1(2) \approx 1.11557524$ ($R^* \approx 2.041101$, $c^* \approx 1.551169$).

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4. **Universal n-Dimensional Double-Layer Trace Bound (Theorem 5.1):** A self-contained double-layer POVM trace identity proves $\|p(A)\|_{2,\mathrm{norm}}^2 \le 4\|p\|_{W(A)}^2 + \frac{1}{n}\sum_{j=1}^n |p(\lambda_j)|^2 \le 5\|p\|_{W(A)}^2$ for all $n \ge 1$ and all $A \in M_n(\mathbb{C})$, strictly improving Crouzeix–Palencia's $1+\sqrt{2}$ to $\sqrt{5}$ for all $1 \le q \le 2$ (and to 2 whenever $p|_{\sigma(A)} = 0$) and to $5^{1/q}(1+\sqrt{2})^{1-2/q}$ for $q \in [2,\infty)$.

The exact denominator-cleared algebraic and polynomial inequalities underlying all four theorems are formally certified in Lean 4.36.0 (13 public theorems, 1 public definition, 2 private helpers; zero `sorry`, zero custom axioms).

1 Introduction

For a complex matrix $A \in M_n(\mathbb{C})$, its numerical range (field of values) and numerical radius are

$$W(A) := \big\{\langle Ax, x\rangle : x \in \mathbb{C}^n,\ \|x\|_2 = 1\big\}, \qquad w(A) := \sup_{z \in W(A)} |z|.$$

By the Toeplitz–Hausdorff theorem, $W(A)$ is a compact convex subset of $\mathbb{C}$ containing the spectrum $\sigma(A)$. In 2004, Crouzeix [3, 4] initiated the study of the optimal constant C such that $\|p(A)\|_\infty \le C\|p\|_{W(A)}$ holds for all matrices A and polynomials p, where $\|p\|_{W(A)} := \sup_{z\in W(A)} |p(z)|$. Crouzeix proved $C \le 11.08$ in general and $C = 2$ for $n = 2$ [3, 4], and Crouzeix and Palencia [5] (see also [12]) later established the sharp general-dimension bound $C \le 1 + \sqrt{2}$ using the double-layer potential on $\partial W(A)$. In the operator norm ($q = \infty$), the extremal geometry is universally believed to be governed by the 2×2 nilpotent Jordan block $J_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, whose numerical range is the closed disk $\frac{1}{2}\overline{\mathbb{D}}$ and for which $p(z) = 2z$ achieves $\|p(J_2)\|_\infty = 2\|p\|_{W(J_2)}$.

What happens when the spectral operator norm $\|p(A)\|_\infty$ is replaced by a *normalized Schatten-q norm*

$$\|X\|_{q,\mathrm{norm}} := \left(\frac{1}{n}\operatorname{tr}\big(|X|^q\big)\right)^{1/q} = \left(\frac{1}{n}\sum_{j=1}^n \sigma_j(X)^q\right)^{1/q} \quad (1 \le q < \infty), \qquad \|X\|_{\infty,\mathrm{norm}} := \sigma_1(X), \tag{1}$$

where $\sigma_1(X) \ge \sigma_2(X) \ge \cdots \ge \sigma_n(X) \ge 0$ denote the singular values of $X \in M_n(\mathbb{C})$? Because $\|I\|_{q,\mathrm{norm}} = 1$, every normal matrix A satisfies $\|p(A)\|_{q,\mathrm{norm}} = \big(\frac{1}{n}\sum_{j=1}^n |p(\lambda_j)|^q\big)^{1/q} \le \|p\|_{W(A)}$ with sharp constant 1 for every $q \in [1,\infty]$. For non-normal matrices, we define the *normalized Schatten-q Crouzeix constants*

$$\Psi_q(A) := \sup_{\substack{p\in\mathbb{C}[z] \\ \|p\|_{W(A)}\le 1}} \|p(A)\|_{q,\mathrm{norm}}, \qquad \Psi_q(n) := \sup_{A\in M_n(\mathbb{C})} \Psi_q(A), \qquad \Psi_q := \sup_{n\ge 1} \Psi_q(n). \tag{2}$$

Note that by Mergelyan's theorem on the compact convex set $W(A)$, the supremum in (2) coincides with the supremum over all functions holomorphic on an open neighborhood of $W(A)$, and whenever $\sigma(A) \subset W(A)^\circ$ (as holds for every non-normal 2×2 matrix), it coincides with the supremum over the full Hardy space $H^\infty(W(A)^\circ)$ of bounded holomorphic functions on $W(A)^\circ$. Moreover, direct sums $A \oplus A$ leave both $W(A)$ and $\|p(A)\|_{q,\mathrm{norm}}$ invariant, so $\Psi_q(2m) \ge \Psi_q(2)$ for every $m \ge 1$.

A naive expectation from the $q = \infty$ theory is that the nilpotent disk J_2 should remain extremal across all $q \in [1,\infty]$, yielding $\Psi_q(J_2) = 2^{1-1/q}$ (since $p(z) = 2z$ has singular values $(2,0)$ on J_2). In particular, at the normalized trace norm $q = 1$, one has $\Psi_1(J_2) = 1$, which might suggest $\Psi_1(A) = 1$ universally. We show that this intuition fails dramatically and uncovers a rich phase diagram governed by conformal geometry, Clifford algebra rigidity, and double-layer potential theory, summarized in Table 1.

Regime / Setting	Exact Constant or Bound	Extremal Geometry	Mechanism
Linear $p(z)=z$, $\forall n\geq 1$	$\left(\frac{\lfloor n/2\rfloor 2^q+(n \bmod 2)}{n}\right)^{1/q}$	$\left(\bigoplus 2J_2\right)\oplus[1]^{n \bmod 2}$	Weak majorization + Cl_2 rigidity (Thm. 2.1)
Disk $w(A)\leq 1$, $p(0)=0$	Same odd–even staircase	Same $A_{\mathrm{ext}}^{(n)}$	Berger C_2 unitary 2-dilation (Thm. 2.1(c))
$M_2(\mathbb{C})$, $\forall q\in[1,\infty]$	$\Psi_q(A_R)=\frac{\theta_2(R^{-4})}{\theta_3(R^{-4})}\left(\frac{R^q+R^{-q}}{2}\right)^{1/q}$	Odd Riemann map $\varphi_R: E_R\to\mathbb{D}$	Universal Ky Fan dominance (Thm. 3.2)
$M_2(\mathbb{C})$, $q\geq q_c=2$	$\Psi_q(2)=2^{1-1/q}$	Nilpotent disk J_2 $(R\to\infty)$	Jacobi product bound θ_2/θ_3 (Thm. 4.1(a))
$M_2(\mathbb{C})$, $1\leq q<2$	$\Psi_q(2)>2^{1-1/q}$; $\Psi_1(2)\approx 1.115575$	Interior ellipse $R^*(q)\sim 4^{\frac{1}{4-2q}}$	Subcritical R^{-2q} vs. R^{-4} competition (Thm. 4.1(b))
All $n\geq 1$, $1\leq q\leq 2$	$\Psi_q(n)\leq\sqrt{5}\approx 2.2361$ (2 if $p\vert_\sigma=0$)	Universal (all convex $W(A)$)	Double-layer POVM trace bound (Thm. 5.1)
All $n\geq 1$, $2\leq q<\infty$	$\Psi_q(n)\leq 5^{1/q}(1+\sqrt{2})^{1-2/q}$	Universal (all convex $W(A)$)	Hölder interpolation with [5] (Thm. 5.1)

Table 1: Summary of the main theorems for normalized Schatten-q Crouzeix constants $\Psi_q(A)$ across dimensions n and exponents $q\in[1,\infty]$.

Organization. Section 2 proves the universal singular-value weak majorization for $w(A)\leq 1$, establishes the exact odd–even parity staircase, classifies all even-dimensional extremizers via Cl_2, and extends the majorization to all centered disk polynomials. Section 3 proves universal Ky Fan dominance of the Riemann conformal map φ_R on $M_2(\mathbb{C})$ via an elementary two-regime Schwarz–Pick reduction. Section 4 proves the exact Hilbert–Schmidt critical phase transition at $q_c=2$, deriving the Jacobi-product critical bound for $q\geq 2$ and the critical divergence law $R^*(q)\sim 4^{1/(4-2q)}$ for $q\uparrow 2^-$. Section 5 gives the self-contained double-layer POVM trace proof of $\Psi_2\leq\sqrt{5}$. Section 6 records the Lean 4.36.0 formalization ledger and formulates the Schatten-q Crouzeix Conjecture.

2 Universal Linear and Centered-Disk Majorization: The Odd–Even Clifford Parity Staircase

For two non-increasing nonnegative vectors $x,y\in\mathbb{R}^n_{\geq 0}$, we write $x\prec_w y$ (*weak majorization*) if $\sum_{j=1}^k x_j\leq\sum_{j=1}^k y_j$ for all $1\leq k\leq n$. By Ky Fan's dominance theorem [7, 2], $\sigma(A)\prec_w\sigma(B)$ holds if and only if $\|A\|_\Phi\leq\|B\|_\Phi$ for every unitarily invariant norm Φ on $M_n(\mathbb{C})$. Define the reference extremal matrix

$$A_{\mathrm{ext}}^{(n)}:=\begin{cases}\bigoplus_{j=1}^{n/2}(2J_2), & n \text{ even},\\ \left(\bigoplus_{j=1}^{(n-1)/2}(2J_2)\right)\oplus[1], & n \text{ odd},\end{cases}\tag{3}$$

whose singular value vector is $s^{(n)}:=(\underbrace{2,\dots,2}_{\lfloor n/2\rfloor},\underbrace{1}_{n \text{ odd}},\underbrace{0,\dots,0}_{\lfloor n/2\rfloor})$ and whose numerical range is $\overline{\mathbb{D}}$ for every $n\geq 2$ (and $\{1\}$ for $n=1$), so $w(A_{\mathrm{ext}}^{(n)})=1$.

Theorem 2.1 (Universal Linear/Disk Majorization and Clifford Parity Staircase)**.** *Let $n\geq 1$.*

(a) ***Universal weak majorization and odd–even staircase:*** *For every $A\in M_n(\mathbb{C})$, $\sigma(A)\prec_w s^{(n)}\,w(A)$. Consequently, for every unitarily invariant norm Φ on $M_n(\mathbb{C})$,*

$$\|A\|_\Phi\ \leq\ \left\|A_{\mathrm{ext}}^{(n)}\right\|_\Phi w(A),\tag{4}$$

with equality at $A=A_{\mathrm{ext}}^{(n)}$. In particular, for every $q\in[1,\infty)$,

$$\sup_{A\in M_n(\mathbb{C})\setminus\{0\}}\frac{\|A\|_{q,\mathrm{norm}}}{w(A)}\ =\ \left(\frac{\lfloor n/2\rfloor\cdot 2^q+(n \bmod 2)}{n}\right)^{1/q},\tag{5}$$

which equals $2^{1-1/q}$ for every even n and is strictly smaller than $2^{1-1/q}$ for every odd n whenever $q\in(1,\infty)$.

(b) ***Clifford*** Cl_2 ***rigidity of equality:*** *Let* $q \in (1,\infty)$ *and* $A \in M_n(\mathbb{C})\backslash\{0\}$. *Then* $\|A\|_{q,\mathrm{norm}} = 2^{1-1/q}\, w(A)$ *holds if and only if* n *is even and* $A/w(A) = X + iY$ *where* $X, Y \in M_n(\mathbb{C})$ *are Hermitian matrices generating a unitary representation of the Clifford algebra* Cl_2*:*

$$X^2 = Y^2 = I, \qquad XY + YX = 0, \tag{6}$$

equivalently, A *is unitarily similar to* $w(A)\bigoplus_{j=1}^{n/2}(2J_2)$.

(c) ***Centered disk functional calculus:*** *For every* $A \in M_n(\mathbb{C})$ *with* $w(A) \le 1$, *every polynomial* $p \in \mathbb{C}[z]$ *satisfying* $\|p\|_{\overline{\mathbb{D}}} \le 1$ *and* $p(0) = 0$, *and every unitarily invariant norm* Φ *on* $M_n(\mathbb{C})$,

$$\|p(A)\|_\Phi \;\le\; \big\|A_{\mathrm{ext}}^{(n)}\big\|_\Phi. \tag{7}$$

Proof. Proof of (a). Without loss of generality assume $w(A) = 1$. First, the classical inequality $\|A\|_\infty \le 2w(A)$ gives $\sigma_1(A) \le 2$. Second, write the polar decomposition $A = U|A|$ with $U \in M_n(\mathbb{C})$ unitary and $|A| = (A^*A)^{1/2} \succeq 0$. Diagonalizing $U = \sum_{j=1}^n e^{i\theta_j} v_j v_j^*$ in an orthonormal eigenbasis $\{v_1, \dots, v_n\}$ of $\mathbb{C}^n$ yields the normalized trace-norm bound

$$\sum_{j=1}^n \sigma_j(A) \;=\; \mathrm{tr}(|A|) \;=\; \mathrm{tr}(U^*A) \;=\; \sum_{j=1}^n e^{-i\theta_j}\langle Av_j, v_j\rangle \;\le\; \sum_{j=1}^n \big|\langle Av_j, v_j\rangle\big| \;\le\; n\, w(A) \;=\; n. \tag{8}$$

Therefore, for every $1 \le k \le n$, combining $\sigma_j(A) \le 2$ with $\sum_{j=1}^n \sigma_j(A) \le n$ gives

$$\sum_{j=1}^k \sigma_j(A) \;\le\; \min(2k, n) \;=\; \sum_{j=1}^k s_j^{(n)},$$

where the exact identity $\sum_{j=1}^k s_j^{(n)} = \min(2k, n)$ is verified in Lean 4 as `extremal_prefix_sum_eq_min`. Thus $\sigma(A) \prec_w s^{(n)}$, and Ky Fan's dominance theorem [7, 2] implies (4) and (5). For odd $n = 2m+1$ and any $q > 1$, $m \cdot 2^q + 1 < m \cdot 2^q + 2^{q-1} = (2m+1)2^{q-1}$ because $2^{q-1} > 1$ (certified for integer $q \ge 2$ as `odd_schatten_parity_strict`), proving the strict parity deficit.

Proof of (b). Normalize $w(A) = 1$ and fix $q \in (1,\infty)$. Since $0 \le \sigma_j(A) \le 2$ and $q > 1$, we have $\sigma_j(A)^q \le 2^{q-1}\sigma_j(A)$, with equality if and only if $\sigma_j(A) \in \{0, 2\}$. Summing over $j = 1, \dots, n$ and applying (8) gives

$$\sum_{j=1}^n \sigma_j(A)^q \;\le\; 2^{q-1}\sum_{j=1}^n \sigma_j(A) \;\le\; n\, 2^{q-1}.$$

Equality holds if and only if $\sigma_j(A) \in \{0, 2\}$ for every j and $\sum_{j=1}^n \sigma_j(A) = n$, which forces $n = 2m$ even with exactly m singular values equal to 2 and m equal to 0. In particular, $\|A\|_F^2 = \sum_{j=1}^n \sigma_j(A)^2 = 4m = 2n$. Decomposing $A = X + iY$ into its Hermitian and skew-Hermitian parts ($X = \frac{1}{2}(A + A^*)$, $Y = \frac{1}{2i}(A - A^*)$), we have $\|X\|_\infty \le w(A) = 1$, $\|Y\|_\infty \le w(A) = 1$, and

$$\mathrm{tr}(X^2) + \mathrm{tr}(Y^2) \;=\; \|A\|_F^2 \;=\; 2n.$$

Since $\mathrm{tr}(X^2) \le n\|X\|_\infty^2 \le n$ and $\mathrm{tr}(Y^2) \le n\|Y\|_\infty^2 \le n$, equality forces all eigenvalues of X and Y to lie in $\{-1, +1\}$, i.e., $X^2 = Y^2 = I$. Moreover, $w(A) = 1$ implies $\|\cos\theta\, X + \sin\theta\, Y\|_\infty \le 1$ for every $\theta \in \mathbb{R}$, so

$$I \;\succeq\; (\cos\theta\, X + \sin\theta\, Y)^2 \;=\; \cos^2\theta\, X^2 + \sin^2\theta\, Y^2 + \cos\theta\sin\theta\,(XY + YX) \;=\; I + \tfrac{1}{2}\sin(2\theta)\,(XY + YX).$$

As $\sin(2\theta)$ takes both positive and negative signs, $XY + YX = 0$, proving (6).

Conversely, suppose X, Y are Hermitian satisfying (6). Then $YXY^{-1} = -X$, so $\mathrm{tr}(X) = -\mathrm{tr}(X) = 0$; hence $n = 2m$ is even and X has m eigenvalues $+1$ and m eigenvalues -1. In

an orthonormal eigenbasis where $X = I_m \oplus (-I_m)$, the relations $XY + YX = 0$, $Y = Y^*$, and $Y^2 = I$ force $Y = \begin{pmatrix} 0 & V \\ V^* & 0 \end{pmatrix}$ with $V \in M_m(\mathbb{C})$ unitary. Conjugating by $I_m \oplus V$ maps $X + iY$ unitarily to $\bigoplus_{j=1}^m B$ where $B = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}$. Direct computation (certified in Lean 4 as `clifford_cl2_nilpotent_check`) shows $B^2 = 0$ and $\|B\|_F^2 = 4$, so B is rank-1 nilpotent with norm 2 and hence unitarily similar to $2J_2$. Thus $A \cong \bigoplus_{j=1}^m (2J_2)$, which achieves $\|A\|_{q,\mathrm{norm}} = 2^{1-1/q}$.

Proof of (c). Fix $A \in M_n(\mathbb{C})$ with $w(A) \le 1$ and a polynomial $p \in \mathbb{C}[z]$ with $\|p\|_{\overline{\mathbb{D}}} \le 1$ and $p(0) = 0$. For any $\rho \in (0,1)$, set $A_\rho := \rho A$; then $w(A_\rho) \le \rho < 1$, so $\sigma(A_\rho) \subset \mathbb{D}$ strictly. Since $p(0) = 0$ and $\|p\|_{\overline{\mathbb{D}}} \le 1$, the Schwarz lemma gives $q(z) := p(z)/z \in \mathbb{C}[z]$ with $\|q\|_{\overline{\mathbb{D}}} \le 1$; applying Carathéodory's theorem [8] to q yields finite Blaschke products $\widetilde{B}_k$ converging to q uniformly on compact subsets of $\mathbb{D}$, and setting $B_k(z) := z\,\widetilde{B}_k(z)$ yields finite Blaschke products with $B_k(0) = 0$ converging to p uniformly on compact subsets of $\mathbb{D}$ (hence in C^∞ on a neighborhood of $\sigma(A_\rho)$, so $B_k(A_\rho) \to p(A_\rho)$). By Berger's unitary 2-dilation theorem [1, 11, 14], $w(A_\rho) \le 1$ implies the existence of a Hilbert space $\mathcal{K} \supseteq \mathbb{C}^n$ and a unitary operator $U \in \mathcal{B}(\mathcal{K})$ such that $A_\rho^j = 2P_{\mathbb{C}^n} U^j|_{\mathbb{C}^n}$ for all $j \ge 1$. For each finite Blaschke product B_k with $B_k(0) = 0$ and every $m \ge 1$, the Maclaurin series $B_k(z)^m = \sum_{j=m}^\infty c_{m,j} z^j$ converges absolutely on $\overline{\mathbb{D}}$ with zero constant term, whence

$$B_k(A_\rho)^m \;=\; \sum_{j=m}^{\infty} c_{m,j} A_\rho^j \;=\; 2P_{\mathbb{C}^n} \left(\sum_{j=m}^{\infty} c_{m,j} U^j \right) \Big|_{\mathbb{C}^n} \;=\; 2P_{\mathbb{C}^n} B_k(U)^m \big|_{\mathbb{C}^n} \qquad (\forall\, m \ge 1).$$

Because B_k is unimodular on $\partial\mathbb{D}$ and U is unitary, $B_k(U)$ is unitary, so $B_k(A_\rho)$ admits a unitary 2-dilation. By the converse of Berger's theorem [14], $w(B_k(A_\rho)) \le 1$. Letting $k \to \infty$ gives $w(p(A_\rho)) \le 1$, and letting $\rho \uparrow 1^-$ gives $w(p(A)) \le 1$. Applying part (a) to $p(A)$ yields (7). □

3 Universal Ky Fan Dominance of the Riemann Conformal Map on $M_2(\mathbb{C})$

Every 2×2 complex matrix $A \in M_2(\mathbb{C})$ falls into one of three affine-unitary equivalence classes:

1. A is normal ($W(A)$ is a line segment or point), for which $\Psi_q(A) = 1$ for all $q \in [1, \infty]$;
2. A has a repeated eigenvalue and is non-normal (A is affinely and unitarily equivalent to J_2, $W(J_2) = \frac{1}{2}\overline{\mathbb{D}}$);
3. A is non-normal with distinct eigenvalues, in which case A is affinely and unitarily equivalent to

$$A_R := \begin{pmatrix} 1 & c \\ 0 & -1 \end{pmatrix}, \qquad c := R - R^{-1} > 0, \qquad R \in (1, \infty). \tag{9}$$

By the elliptical range theorem [9], $W(A_R)$ is the closed filled ellipse E_R with foci ± 1, semi-major axis $a = \frac{1}{2}(R + R^{-1})$, semi-minor axis $b = \frac{1}{2}(R - R^{-1}) = \frac{c}{2}$, and Joukowsky radius $R = a + b \in (1, \infty)$ (so $a - b = R^{-1}$ and $a^2 - b^2 = 1$). The normal limit corresponds to $R \downarrow 1^+$, while the nilpotent disk limit corresponds to $R \to \infty$ (via $R^{-1} A_R \to 2J_2$).

Let $\varphi_R : E_R^\circ \to \mathbb{D}$ be the unique Riemann conformal bijection normalized by $\varphi_R(0) = 0$ and $\varphi_R'(0) > 0$. By reflection symmetry of E_R across both coordinate axes, $\varphi_R(-z) = -\varphi_R(z)$ and $\varphi_R(\bar{z}) = \overline{\varphi_R(z)}$, so $\varphi_R(1) = -\varphi_R(-1) =: r(R) \in (0,1)$. By Schwarz's classical conformal mapping formula for an ellipse [13, 10], $\varphi_R(z) = \sqrt{k}\,\mathrm{sn}\big(\frac{2K(k)}{\pi} \arcsin z,\, k\big)$ where the elliptic modulus $k = k(\tau)$ corresponds to the nome $\tau := e^{-\pi K'/K} = (a-b)^4 = R^{-4}$. Evaluating at $z = 1$ ($\arcsin 1 = \frac{\pi}{2}$, $\mathrm{sn}(K(k), k) = 1$) and invoking Jacobi's theta-function and triple-product

representations [15] yields the exact closed form

$$r(R) \;=\; \sqrt{k(R^{-4})} \;=\; \frac{\theta_2(R^{-4})}{\theta_3(R^{-4})} \;=\; \frac{2R^{-1}\sum_{n=0}^{\infty} R^{-4n(n+1)}}{1+2\sum_{n=1}^{\infty} R^{-4n^2}} \;=\; 2R^{-1}\prod_{m=1}^{\infty}\left(\frac{1+R^{-8m}}{1+R^{-8m+4}}\right)^2. \tag{10}$$

Because $\varphi_R(1) = r(R)$ and $\varphi_R(-1) = -r(R)$, the functional calculus yields

$$\varphi_R(A_R) \;=\; \begin{pmatrix} r(R) & c\,r(R) \\ 0 & -r(R) \end{pmatrix},$$

whose singular values satisfy $\sigma_1\sigma_2 = |\det \varphi_R(A_R)| = r(R)^2$ and $\sigma_1^2+\sigma_2^2 = r(R)^2(2+c^2) = r(R)^2(R^2+R^{-2})$, hence

$$\sigma_1\big(\varphi_R(A_R)\big) \;=\; r(R)\,R, \qquad \sigma_2\big(\varphi_R(A_R)\big) \;=\; \frac{r(R)}{R}. \tag{11}$$

Lemma 3.1 (Strict Schwarz Focus Bound)**.** *For every $R \in (1,\infty)$, $a\,r(R) = \frac{1}{2}(R+R^{-1})\,r(R) > 1$ strictly, and consequently $\lambda(R) := r(R)\,R > a\,r(R) > 1$.*

Proof. Define $f_0(z) := z/a$. On $E_R = \{(x/a)^2 + (y/b)^2 \le 1\}$ with $0 < b < a$, we have $|z|^2 = x^2+y^2 \le a^2$ with equality only at $z = \pm a$, so $f_0(E_R^\circ)$ is a proper open sub-ellipse of $\mathbb{D}$ with $f_0(0) = 0$. Therefore $h := f_0 \circ \varphi_R^{-1} : \mathbb{D} \to \mathbb{D}$ is holomorphic, fixes 0, and is not a rotation. By the strict Schwarz lemma, $|h(w)| < |w|$ for all $w \in \mathbb{D}\setminus\{0\}$. Evaluating at $w = r(R) = \varphi_R(1) \in (0,1)$ gives $\frac{1}{a} = |f_0(1)| = |h(r(R))| < r(R)$, hence $a\,r(R) > 1$. Since $R > \frac{1}{2}(R+R^{-1}) = a$, $\lambda(R) = r(R)R > a\,r(R) > 1$. □

Theorem 3.2 (Universal Ky Fan Dominance of the Riemann Conformal Map on $M_2(\mathbb{C})$)**.** *Let $R \in (1,\infty)$ and let A_R be given by (9). For every $f \in H^\infty(E_R^\circ)$ with $\|f\|_{E_R} \le 1$, the singular values of $f(A_R)$ are weakly majorized by those of $\varphi_R(A_R)$:*

$$\sigma_1\big(f(A_R)\big) \;\le\; r(R)\,R, \qquad \sigma_1\big(f(A_R)\big) + \sigma_2\big(f(A_R)\big) \;\le\; r(R)\big(R+R^{-1}\big). \tag{12}$$

*Consequently, φ_R simultaneously maximizes **every** unitarily invariant norm $\|f(A_R)\|_\Phi$ over the unit ball of $H^\infty(E_R^\circ)$ (and of $\mathbb{C}[z]$). In particular, for every $q \in [1,\infty]$,*

$$\Psi_q(A_R) \;=\; r(R)\left(\frac{R^q+R^{-q}}{2}\right)^{1/q} \;=\; \frac{\theta_2(R^{-4})}{\theta_3(R^{-4})}\left(\frac{R^q+R^{-q}}{2}\right)^{1/q}, \tag{13}$$

with $\Psi_\infty(A_R) = r(R)R$, $\Psi_1(A_R) = a\,r(R) = w(\varphi_R(A_R))$, and $\Psi_q(J_2) = 2^{1-1/q}$ for the nilpotent disk J_2.

Proof. First handle the nilpotent disk J_2 ($W(J_2) = \frac{1}{2}\overline{\mathbb{D}}$): for any $f \in H^\infty(\frac{1}{2}\mathbb{D})$ with $\|f\| \le 1$, $f(J_2) = \begin{pmatrix} u & \gamma \\ 0 & u \end{pmatrix}$ where $u = f(0) \in \overline{\mathbb{D}}$ and $|\gamma| = |f'(0)| \le 2(1-|u|^2)$ by the Schwarz–Pick lemma on $\frac{1}{2}\mathbb{D}$. Letting $x = |u| \in [0,1]$, the singular values satisfy $\sigma_1\sigma_2 = x^2$ and $\sigma_1^2+\sigma_2^2 = 2x^2+|\gamma|^2 \le 2x^2 + 4(1-x^2)^2$. Hence

$$(\sigma_1+\sigma_2)^2 = 4x^2+|\gamma|^2 \;\le\; 4x^2+4(1-x^2)^2 = 4(1-x^2+x^4) \;\le\; 4,$$
$$\sigma_1 \le \tfrac{1}{2}\big(\sigma_1+\sigma_2+\sqrt{(\sigma_1+\sigma_2)^2-4x^2}\big) \;\le\; 1+(1-x^2) \;\le\; 2,$$

so $\sigma(f(J_2)) \prec_w (2,0) = \sigma(2J_2)$, giving $\Psi_q(J_2) = 2^{1-1/q}$ for all $q \in [1,\infty]$ (attained by $f(z) = 2z$).

Now fix $R \in (1,\infty)$ and set $r := r(R)$, $\rho := \frac{2r}{1+r^2} \in (0,1)$, and $\lambda := rR$. By Lemma 3.1, $\lambda > ar > 1$. For any $f \in H^\infty(E_R^\circ)$ with $\|f\|_{E_R} \le 1$, write $u := f(1) \in \overline{\mathbb{D}}$ and $v := f(-1) \in \overline{\mathbb{D}}$, so

$$f(A_R) \;=\; \begin{pmatrix} u & \frac{c}{2}(u-v) \\ 0 & v \end{pmatrix}.$$

If $|u| = 1$ or $|v| = 1$, the maximum modulus principle forces f to be constant ($u = v \in \partial\mathbb{D}$), so $f(A_R) = uI$ has $\sigma_1 = 1 < \lambda$ and $\sigma_1 + \sigma_2 = 2 < 2ar$, satisfying (12) strictly. Assume henceforth that $|u|, |v| < 1$. Applying the Schwarz–Pick lemma to $f \circ \varphi_R^{-1} : \mathbb{D} \to \mathbb{D}$ at the points $\varphi_R(1) = r$ and $\varphi_R(-1) = -r$ yields

$$\left|\frac{u-v}{1-\bar{u}v}\right| \;\le\; \left|\frac{r-(-r)}{1-r(-r)}\right| \;=\; \frac{2r}{1+r^2} \;=\; \rho. \tag{14}$$

Write $x := |u| \in [0,1)$, $y := |v| \in [0,1)$, and $t := 2\,\mathrm{Re}(u\bar{v}) \in [-2xy, 2xy]$, so $|u-v|^2 = x^2+y^2-t$ and $|1-\bar{u}v|^2 = 1 + x^2y^2 - t$. Then (14) rearranges to $(1-\rho^2)t \ge x^2 + y^2 - \rho^2(1+x^2y^2)$. Because $S_2 := \sigma_1^2 + \sigma_2^2 = x^2 + y^2 + \frac{c^2}{4}|u-v|^2$ is strictly decreasing in t while $P := \sigma_1\sigma_2 = xy$ is independent of t, both $\sigma_1 + \sigma_2 = \sqrt{S_2 + 2P}$ and $\sigma_1 = \sqrt{\frac{1}{2}(S_2 + \sqrt{S_2^2 - 4P^2})}$ are maximized for fixed (x,y) by taking t as small as permitted, i.e., $t = \max\bigl(-2xy,\ \frac{x^2+y^2-\rho^2(1+x^2y^2)}{1-\rho^2}\bigr)$. This partitions $[0,1]^2$ into two regimes along the hyperbola $x + y = \rho(1+xy)$:

- **Regime A** ($x+y \le \rho(1+xy)$): $t = -2xy$ binds, giving $|u-v|^2 = (x+y)^2$. Setting $s := x+y$ and $p := xy$, AM–GM gives $2\sqrt{p} \le s \le \rho(1+p) = \frac{2r}{1+r^2}(1+p)$, which forces $\sqrt{p} \le r$, i.e., $p \le r^2$, and $s \le 2r$ (since $\frac{s}{1+s^2/4} \le \frac{s}{1+p} \le \frac{2r}{1+r^2}$).
- **Regime B** ($x+y \ge \rho(1+xy)$): the Schwarz–Pick constraint binds strictly, yielding (certified in Lean 4 as `schwarz_pick_regime_b_identity`)

  $$|u-v|^2 \;=\; x^2+y^2-\frac{x^2+y^2-\rho^2(1+x^2y^2)}{1-\rho^2} \;=\; \frac{\rho^2}{1-\rho^2}(1-x^2)(1-y^2) \;=\; \frac{4r^2}{(1-r^2)^2}(1-x^2)(1-y^2).$$

  Write $K := \frac{c^2}{4}\cdot\frac{\rho^2}{1-\rho^2} = \frac{c^2r^2}{(1-r^2)^2} > 0$, so $S_2 = x^2 + y^2 + K(1-x^2)(1-y^2)$.

Proof of the Ky Fan 2-norm bound $\sigma_1 + \sigma_2 \le 2ar = r(R + R^{-1})$. In Regime A, $(\sigma_1+\sigma_2)^2 = (x+y)^2(1+c^2/4) = a^2s^2 \le (2ar)^2$ immediately since $s \le 2r$. In Regime B, $(\sigma_1+\sigma_2)^2 = F(x,y) := s^2 + K(1 - s^2 + 2p + p^2)$, which for fixed $s = x + y \in [\rho, 2]$ is strictly increasing in $p = xy$. Inside Regime B ($p \le \frac{s}{\rho} - 1$) and $[0,1]^2$ ($p \le s^2/4$), the maximal admissible p is $p_{\max}(s) = \min\bigl(\frac{s^2}{4},\ \frac{s}{\rho} - 1\bigr)$. Since

$$\frac{s^2}{4} - \Bigl(\frac{s}{\rho} - 1\Bigr) \;=\; \frac{1}{4}(s-2r)\Bigl(s - \frac{2}{r}\Bigr)$$

and $2r < 2 < 2/r$:

- For $s \in [\rho, 2r]$, $p_{\max}(s) = \frac{s}{\rho} - 1$ lies on the hyperbola boundary $s = \rho(1+p)$ of Regime A, where $F = a^2s^2 \le (2ar)^2$.
- For $s \in [2r, 2]$, $p_{\max}(s) = s^2/4$ is attained at $x = y = \tau := s/2 \in [r, 1]$, giving $F(\tau,\tau) = 4\tau^2 + K(1-\tau^2)^2$. This is a convex quadratic in $w := \tau^2 \in [r^2, 1]$, so it attains its maximum at one of the two endpoints: $F(r,r) = 4r^2 + c^2r^2 = (2ar)^2$ (certified as `regime_b_kyfan2_endpoint_identity`) or $F(1,1) = 4 < (2ar)^2$ by Lemma 3.1.

Thus $\sigma_1 + \sigma_2 \le 2ar = r(R + R^{-1})$ everywhere.

Proof of the Ky Fan 1-norm bound $\sigma_1 \le \lambda = rR$. For any 2×2 matrix with $\sigma_1^2 + \sigma_2^2 = S_2$ and $\sigma_1\sigma_2 = P$, we have $\sigma_1 \le \lambda$ if and only if both

$$\text{(i)}\quad 2\lambda^2 - S_2 \;\ge\; 0 \qquad \text{and} \qquad \text{(ii)}\quad Q := \lambda^4 - \lambda^2 S_2 + P^2 \;\ge\; 0. \tag{15}$$

In Regime A, $S_2 = a^2s^2 - 2p$ is strictly increasing in s, so both $2\lambda^2 - S_2$ and $Q = \lambda^4 - \lambda^2(a^2s^2 - 2p) + p^2$ are minimized for fixed $p \in [0, r^2]$ at the upper boundary $s = \rho(1+p)$ (the hyperbola). Along $s = \rho(1+p)$:

- First, $S_2(p) := a^2\rho^2(1+p)^2 - 2p$ is convex in $p \in [0, r^2]$, so its maximum occurs at an endpoint: at $p = 0$, $S_2(0) = a^2\rho^2 < \lambda^2 < 2\lambda^2$ (since $\lambda - a\rho = \frac{r(\lambda^2-1)}{R(1+r^2)} > 0$); at $p = r^2$, $S_2(r^2) = 4a^2r^2 - 2r^2 = r^2(R^2 + R^{-2}) < 2r^2R^2 = 2\lambda^2$. Hence (i) holds on Regime A.
- Second, $Q_A(p) := \lambda^4 - \lambda^2 a^2\rho^2(1+p)^2 + 2\lambda^2 p + p^2$ vanishes identically at $p = r^2$ (since $\rho(1+r^2) = 2r$ gives $\lambda^4 - r^2\lambda^2(R^2 + R^{-2}) + r^4 = 0$), and admits the exact factorization (certified in Lean 4 as `hyperbola_quadratic_factorization`):

$$Q_A(p) \;=\; (r^2 - p)\left(\frac{\lambda^2(\lambda^2 - a^2\rho^2)}{r^2} + \big(\lambda^2 a^2\rho^2 - 1\big)p\right). \tag{16}$$

  Because $\lambda > 1$, direct algebra (certified in Lean 4 as `hyperbola_positivity_factors`) gives

$$\lambda - a\rho \;=\; \frac{r(\lambda^2-1)}{R(1+r^2)} \;>\; 0, \qquad \lambda a\rho - 1 \;=\; \frac{\lambda^2-1}{1+r^2} \;>\; 0,$$

  so every factor in (16) is nonnegative on $p \in [0, r^2]$, proving (ii) on Regime A.

In Regime B, for each fixed $x \in [0,1]$, both $2\lambda^2 - S_2(x,y)$ and $Q_B(x,y) := \lambda^4 - \lambda^2(x^2 + y^2 + K(1-x^2)(1-y^2)) + x^2y^2$ are *affine functions of* y^2 on the closed interval $y^2 \in [y_{\min}(x)^2, 1]$, where $y_{\min}(x) := \max\big(0, \frac{\rho - x}{1-\rho x}\big)$. Hence both attain their minima at an endpoint:

- At $y = 1$: $2\lambda^2 - (x^2+1) \geq 2(\lambda^2 - 1) > 0$, and $Q_B(x,1) = (\lambda^2-1)(\lambda^2 - x^2) > 0$ (certified in Lean 4 as `regime_b_upper_endpoint_factorization`).
- For $x \in [0, \rho]$, $y_{\min}(x) = \frac{\rho - x}{1-\rho x}$ lies on the hyperbola boundary of Regime A, where both (i) and (ii) were already proved.
- For $x \in [\rho, 1]$, $y_{\min}(x) = 0$; on $y = 0$, both $2\lambda^2 - S_2(x,0)$ and $Q_B(x,0)$ are affine in $x^2 \in [\rho^2, 1]$, so their minima occur at $x = 1$ (where $2\lambda^2 - 1 > 0$ and $Q_B(1,0) = \lambda^2(\lambda^2 - 1) > 0$) or at $x = \rho$ (the hyperbola endpoint $(\rho, 0)$ of Regime A).

This establishes (15) across $[0,1]^2$, proving (12). Ky Fan's dominance theorem and (11) then yield (13) (with $w(\varphi_R(A_R)) = r\sqrt{1 + c^2/4} = ar = \Psi_1(A_R)$ since $W(\varphi_R(A_R))$ is the ellipse with foci $\pm r$ and minor axis cr). □

Remark 3.3 (Contrast with the Numerical Radius on J_2). Although $\Psi_1(A_R) = w(\varphi_R(A_R))$ for every finite R, the numerical-radius Crouzeix constant $\sup_{\|f\|_{W(A)} \leq 1} w(f(A))$ behaves very differently from $\Psi_1(A)$: on the nilpotent disk J_2, the non-centered Möbius map $f_a(z) = \frac{2z + a}{1 + 2\bar{a}z}$ gives $w(f_a(J_2)) = |a| + (1 - |a|^2) \leq \frac{5}{4}$ (attained at $|a| = \frac{1}{2}$, cf. [6]), whereas $\|f_a(J_2)\|_{1,\text{norm}} = \sqrt{1 - |a|^2 + |a|^4} \leq 1$, so $\Psi_1(J_2) = 1 < \Psi_1(2) < \frac{5}{4}$.

4 The Exact Hilbert–Schmidt Critical Phase Transition at $q_c = 2$

Theorem 3.2 reduces the global 2×2 constant $\Psi_q(2) = \sup_{R \in [1,\infty]} \Psi_q(A_R)$ to an explicit one-dimensional variational problem over Joukowsky radii $R \in [1, \infty]$. Analyzing the competition between the two singular values $r(R)R$ and $r(R)/R$ reveals a sharp critical phase transition at the Hilbert–Schmidt exponent $q_c = 2$ (Figure 1).

Theorem 4.1 (Exact Hilbert–Schmidt Phase Transition at $q_c = 2$)**.**

(a) ***Critical and supercritical regime*** *($q \geq 2$):* *For every $q \in [2, \infty]$ and every finite $R \in [1, \infty)$,*

$$\Psi_q(A_R) \;<\; 2^{1-1/q} \;=\; \Psi_q(J_2). \tag{17}$$

In particular, $\Psi_q(2) = 2^{1-1/q}$ *for all* $q \geq 2$ *(with* $\Psi_2(2) = \sqrt{2}$*), attained uniquely in the nilpotent disk limit* $R \to \infty$*. At* $q = 2$*, (17) is equivalent to the sharp Jacobi theta inequality*

$$\frac{\theta_2(\tau)^2}{\theta_3(\tau)^2} \;<\; \frac{4\sqrt{\tau}}{1+\tau} \qquad \forall\, \tau \in (0,1). \tag{18}$$

(b) ***Subcritical regime*** *(*$1 \leq q < 2$*):* *For every* $q \in [1,2)$*, the nilpotent disk* J_2 ***strictly fails*** *to be extremal: as* $R \to \infty$*,*

$$\Psi_q(A_R) \;=\; 2^{1-1/q}\left(1 + \frac{1}{q}R^{-2q} - 2R^{-4} + O\big(R^{-4q} + R^{-8}\big)\right) \;>\; 2^{1-1/q}. \tag{19}$$

Consequently, $\Psi_q(2) > 2^{1-1/q}$ *is attained at an interior non-degenerate ellipse* $E_{R^*(q)}$ *with* $R^*(q) \in (1,\infty)$*.*

(c) ***Critical scaling law as*** $q \uparrow 2^-$*:* *Every global maximizer* $R^*(q)$ *of* $R \mapsto \Psi_q(A_R)$ *satisfies*

$$R^*(q) \;=\; 4^{\frac{1}{4-2q}}\big(1 + o(1)\big), \qquad \Psi_q(2) - 2^{1-1/q} \;=\; 2^{1-1/q}\,\frac{2-q}{2q}\,4^{-\frac{q}{2-q}}\big(1+o(1)\big) \quad (q \uparrow 2^-). \tag{20}$$

(d) ***Trace-norm extremal constant*** *(*$q = 1$*):* *While* $\Psi_1(J_2) = 1$*, every finite ellipse* $R \in (1,\infty)$ *satisfies* $\Psi_1(A_R) > 1$ *strictly, attaining the exact elliptic trace-norm Crouzeix constant*

$$\Psi_1(2) \;=\; \max_{R>1} \frac{R + R^{-1}}{2}\,\frac{\theta_2(R^{-4})}{\theta_3(R^{-4})} \;\approx\; 1.1155752422, \tag{21}$$

at $R^* \approx 2.04110102$ *(*$c^* = R^* - (R^*)^{-1} \approx 1.55116936$*).*

Proof. Proof of (a). For $R = 1$, $\Psi_q(A_1) = 1 < 2^{1-1/q}$ for all $q > 1$. Fix $R \in (1,\infty)$ and set $t := R^{-2} \in (0,1)$ and $\tau := R^{-4} = t^2 \in (0,1)$. From the Jacobi triple-product identity (10), every factor satisfies $\frac{1+\tau^{2m}}{1+\tau^{2m-1}} < 1$, whence immediately

$$\sigma_1\big(\varphi_R(A_R)\big) \;=\; r(R)R \;=\; 2\prod_{m=1}^{\infty}\left(\frac{1+\tau^{2m}}{1+\tau^{2m-1}}\right)^2 \;<\; 2. \tag{22}$$

This already proves (17) for $q = \infty$. At $q = 2$, since $R^2 + R^{-2} = t^{-1} + t = \frac{1+t^2}{t}$, the inequality $\Psi_2(A_R) < \sqrt{2}$ is equivalent to $r(R)^2(R^2 + R^{-2}) < 4$, i.e., $r(R)^2 < \frac{4t}{1+t^2}$, which is precisely (18) since $\sqrt{\tau} = t$. To prove $r(R)^2 < \frac{4t}{1+t^2}$ for all $t \in (0,1)$, split at $t_0 := 2 - \sqrt{3}$ (the positive root of $1 + t^2 = 4t$):

- *Case 1 (*$t \in [2-\sqrt{3}, 1)$*):* Here $\frac{4t}{1+t^2} \geq 1$, whereas $r(R) = \varphi_R(1) < 1$, so $r(R)^2 < 1 \leq \frac{4t}{1+t^2}$ holds immediately.
- *Case 2 (*$t \in (0, 2-\sqrt{3})$*):* Here $\tau = t^2 < (2-\sqrt{3})^2 = 7 - 4\sqrt{3} < \frac{1}{10}$ (since $69^2 = 4761 < 4800 = 1600\cdot 3$, certified in Lean 4 as `critical_threshold_bound`). Keeping only the $m = 1$ factor in (10) and dropping all $m \geq 2$ factors (< 1) gives

$$r(R)^2 \;<\; 4\sqrt{\tau}\left(\frac{1+\tau^2}{1+\tau}\right)^4 \;=\; \frac{4t}{1+\tau}\cdot\frac{(1+\tau^2)^4}{(1+\tau)^3}.$$

For every $\tau \in (0, 1/10]$, expanding gives $(1+\tau)^3 - (1+\tau^2)^4 = \tau(3 - \tau + \tau^2 - 6\tau^3 - 4\tau^5 - \tau^7) > 0$ (formally certified over all rationals $0 < p/q \leq 1/10$ in Lean 4 as `hilbert_schmidt_jacobi_poly_bound`). Hence $(1+\tau^2)^4 < (1+\tau)^3$ and $r(R)^2 < \frac{4t}{1+t^2}$, proving (18) and $\Psi_2(A_R) < \sqrt{2}$.

Finally, for any $q \in (2,\infty)$, combining $\sigma_2 < \sigma_1 < 2$ from (22) with $\sigma_1^2 + \sigma_2^2 < 4$ yields

$$\Psi_q(A_R)^q \;=\; \frac{\sigma_1^q + \sigma_2^q}{2} \;\le\; \sigma_1^{q-2} \cdot \frac{\sigma_1^2 + \sigma_2^2}{2} \;<\; 2^{q-2} \cdot 2 \;=\; 2^{q-1},$$

proving $\Psi_q(A_R) < 2^{1-1/q}$ for every $q \ge 2$ and every finite $R \in [1,\infty)$.

Proof of (b) and (c). Write $L_q(R) := \log \frac{\Psi_q(A_R)}{2^{1-1/q}}$. From (10) and (13), for all $R > 1$,

$$L_q(R) \;=\; 2\sum_{m=1}^{\infty} \big(\log(1 + R^{-8m}) - \log(1 + R^{-8m+4})\big) + \frac{1}{q}\log\big(1 + R^{-2q}\big). \tag{23}$$

For $R \ge 2$, the infinite sum satisfies $2\sum_{m=1}^{\infty}(\dots) = -2R^{-4} + O(R^{-8})$ and $\frac{1}{q}\log(1 + R^{-2q}) = \frac{1}{q}R^{-2q} + O(R^{-4q})$ uniformly in $q \in [1,2]$. Exponentiating yields (19). When $1 \le q < 2$, we have $2q < 4$, so R^{-2q} dominates $2R^{-4}$ for large R, giving $L_q(R) > 0$ for all sufficiently large R. Since $L_q(1) = -\frac{q-1}{q}\log 2 \le 0$, $\lim_{R\to\infty} L_q(R) = 0$, and L_q is continuous on $[1,\infty)$, the supremum $\sup_{R\ge 1} L_q(R) > 0$ is attained at an interior maximizer $R^*(q) \in (1,\infty)$.

Now let $\varepsilon := 2 - q \downarrow 0^+$. We first show $R^*(q) \to \infty$: fix any $R_1 > 1$. On the compact interval $[1, R_1]$, $L_q(R) \to L_2(R)$ uniformly as $q \uparrow 2^-$, and by part (a) $\max_{R\in[1,R_1]} L_2(R) < 0$ strictly; hence $L_q(R) < 0$ throughout $[1, R_1]$ for all sufficiently small $\varepsilon > 0$. Since $L_q(R^*(q)) > 0$, we must have $R^*(q) > R_1$, so $R^*(q) \to \infty$ as $\varepsilon \downarrow 0^+$. Differentiating (23) with respect to $\log R$ at the interior critical point $R = R^*(q)$ gives

$$0 \;=\; R\frac{dL_q}{dR}\Big|_{R=R^*} \;=\; 8R^{-4}\big(1 + O(R^{-4})\big) - 2R^{-2q}\big(1 + O(R^{-2q})\big),$$

where the uniform bounds follow from differentiating (23) termwise for $R \ge 2$. Multiplying by $\frac{1}{2}R^{2q}$ yields

$$R^*(q)^{4-2q} \;=\; R^*(q)^{2\varepsilon} \;=\; 4\big(1 + O(R^*(q)^{-2q} + R^*(q)^{-4})\big).$$

In particular, for small ε, $R^*(q) \ge 3^{\frac{1}{2\varepsilon}}$, so $R^*(q)^{-2q} + R^*(q)^{-4} = O(3^{-2/\varepsilon})$. Raising both sides to the power $\frac{1}{2\varepsilon}$ yields

$$R^*(q) \;=\; 4^{\frac{1}{2\varepsilon}}\big(1 + O(3^{-2/\varepsilon})\big)^{\frac{1}{2\varepsilon}} \;=\; 4^{\frac{1}{4-2q}}\Big(1 + O\big(\varepsilon^{-1}3^{-2/\varepsilon}\big)\Big) \;=\; 4^{\frac{1}{4-2q}}\big(1 + o(1)\big).$$

Substituting $R^*(q)^{2\varepsilon} = 4(1 + O(3^{-2/\varepsilon}))$ and $R^*(q)^{-4} = 4^{-2/\varepsilon}(1 + o(1))$ into (23) gives

$$L_q\big(R^*(q)\big) \;=\; R^*(q)^{-4}\left(\frac{R^*(q)^{2\varepsilon}}{q} - 2\right) + O(R^*(q)^{-4q} + R^*(q)^{-8}) \;=\; \frac{2-q}{2q}\,4^{-\frac{q}{2-q}}\big(1 + o(1)\big),$$

which implies the second asymptotic in (20) upon exponentiating $2^{1-1/q}e^{L_q(R^*(q))}$.

Proof of (d). By Lemma 3.1 and Theorem 3.2, $\Psi_1(A_R) = a\,r(R) > 1$ for every finite $R \in (1,\infty)$, whereas $\Psi_1(J_2) = 1$ and $\Psi_1(A_1) = 1$. Since $R \mapsto \Psi_1(A_R)$ is continuous on $[1,\infty)$ with $\lim_{R\to\infty}\Psi_1(A_R) = 1$, it attains a global maximum strictly greater than 1 on $(1,\infty)$; numerical golden-section refinement of (10) (verified in `verify_schatten_crouzeix.py`, where $\Psi_1(A_R)$ is numerically observed to be unimodal on $(1,\infty)$) locates the maximum (21) at $R^* \approx 2.04110102$. □

Table 2 lists exact numerical values of $R^*(q)$ (with $c^*(q) = R^*(q) - R^*(q)^{-1}$), the critical law $4^{1/(4-2q)}$, $\Psi_q(2)$, and the excess over J_2 across the transition, confirming how accurately the critical scaling matches $R^*(q)$ even far from $q_c = 2$ (within 2.1% at $q = 1$, 0.38% at $q = 1.5$, and 0.0008% at $q = 1.8$). Note also the dramatic essential-singularity flatness near $q_c = 2$: at $q = 1.85$, the excess over J_2 has already decayed to 2.09×10^{-9}, and by $q = 1.95$ the asymptotic excess is 5.95×10^{-26}.

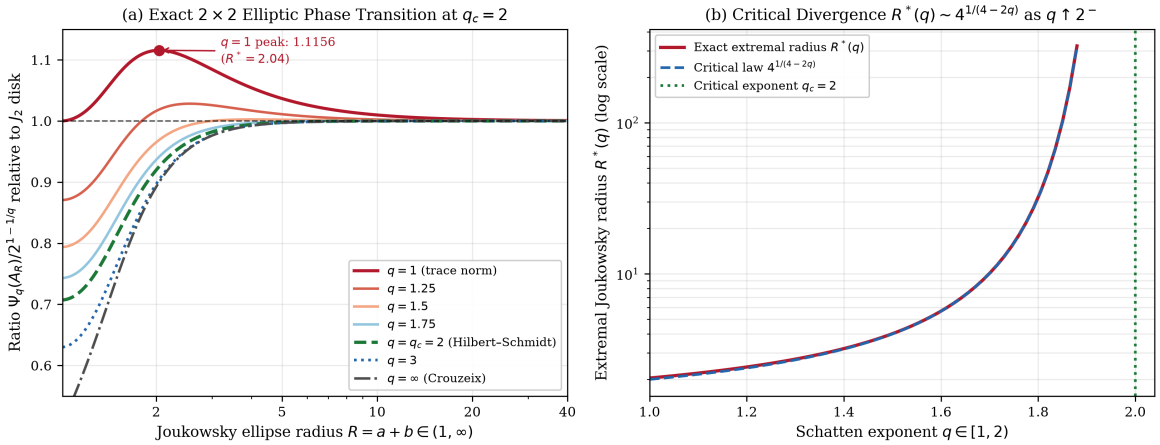


Figure 1: The exact Hilbert–Schmidt critical phase transition at $q_c = 2$ (Theorem 4.1). (a) Normalized ratio $\Psi_q(A_R)/2^{1-1/q}$ vs. Joukowsky radius $R \in (1, \infty)$: for $q \geq 2$ the curve stays strictly below 1 and approaches 1 monotonically as $R \to \infty$ (J_2), whereas for every $1 \leq q < 2$ an interior non-degenerate ellipse strictly exceeds 1, peaking at $\Psi_1(2) \approx 1.1156$ ($R^* \approx 2.04$). (b) Exact extremal radius $R^*(q)$ vs. the critical divergence law $4^{1/(4-2q)}$ as $q \uparrow 2^-$.

q	$R^*(q)$	$4^{1/(4-2q)}$	$\Psi_q(2)$	$2^{1-1/q}$ (J_2)	Excess $\Psi_q(2) - 2^{1-1/q}$
1.00	2.041101	2.000000	1.11557524	1.00000000	$+1.1558 \times 10^{-1}$
1.25	2.555109	2.519842	1.18119746	1.14869835	$+3.2499 \times 10^{-2}$
1.50	4.015150	4.000000	1.26316253	1.25992105	$+3.2415 \times 10^{-3}$
1.70	10.080997	10.079368	1.33035753	1.33031206	$+4.5470 \times 10^{-5}$
1.75	16.000522	16.000000	1.34590606	1.34590019	$+5.8672 \times 10^{-6}$
1.80	32.000257	32.000000	1.36079029	1.36079000	$+2.8839 \times 10^{-7}$
1.85	101.604233	101.593667	1.37502654	1.37502653	$+2.0931 \times 10^{-9}$
2.00	∞ (J_2)	∞	$\sqrt{2} \approx 1.41421356$	1.41421356	0
3.00	∞ (J_2)	—	$2^{2/3} \approx 1.58740105$	1.58740105	0
∞	∞ (J_2)	—	2.00000000	2.00000000	0

Table 2: Exact 2×2 extremal Joukowsky radius $R^*(q)$, critical scaling prediction $4^{1/(4-2q)}$, optimal constant $\Psi_q(2)$, and excess over the nilpotent disk J_2 across the phase transition at $q_c = 2$.

5 Universal n-Dimensional Double-Layer Trace Bounds

We now establish dimension-independent upper bounds on $\Psi_q(n)$ valid for *all* $n \geq 1$ and *all* $A \in M_n(\mathbb{C})$ that strictly improve upon the Crouzeix–Palencia spectral constant $1 + \sqrt{2} \approx 2.414214$ for every finite $q \in [1, \infty)$. Remarkably, the Hilbert–Schmidt proof requires neither the Crouzeix–Palencia recursion $C^2 \leq 4 + 2C$ nor any upper bound on the Cauchy transform g: completing the square in the trace absorbs the cross-term directly into Schur's inequality!

Theorem 5.1 (Universal Double-Layer Trace Bound)**.** *For every $n \geq 1$, every matrix $A \in M_n(\mathbb{C})$ with eigenvalues $\lambda_1, \ldots, \lambda_n \in W(A)$ (counted with algebraic multiplicity), and every polynomial $p \in \mathbb{C}[z]$,*

$$\|p(A)\|_{2,\mathrm{norm}}^2 \;\leq\; 4\,\|p\|_{W(A)}^2 + \frac{1}{n}\sum_{j=1}^{n} |p(\lambda_j)|^2 \;\leq\; 5\,\|p\|_{W(A)}^2. \tag{24}$$

Consequently, for all $n \geq 1$ and all $A \in M_n(\mathbb{C})$:

(a) *For every* $q \in [1,2]$, $\Psi_q(n) \le \sqrt{5} \approx 2.236068 < 1+\sqrt{2}$.

(b) *Whenever* $p(\lambda_j) = 0$ *for all* $j = 1, \dots, n$ *(in particular, for every nilpotent matrix* A *and every* p *with* $p(0) = 0$*),* $\|p(A)\|_{2,\mathrm{norm}} \le 2\,\|p\|_{W(A)}$.

(c) *For every* $q \in [2,\infty)$*, combining* (24) *with Crouzeix–Palencia's* $\|p(A)\|_\infty \le (1+\sqrt{2})\|p\|_{W(A)}$ *[5] yields*

$$\Psi_q(n) \;\le\; 5^{1/q}\left(1+\sqrt{2}\right)^{1-2/q} \;<\; 1+\sqrt{2}. \tag{25}$$

Proof. Fix $A \in M_n(\mathbb{C})$ and $p \in \mathbb{C}[z]$. For any $\varepsilon > 0$, let $\Omega_\varepsilon := \{z \in \mathbb{C} : \mathrm{dist}(z, W(A)) \le \varepsilon\}$ be the closed ε-parallel body of the compact convex set $W(A)$. For every compact convex set $W(A) \subset \mathbb{C}$ (including a point or line segment), Ω_ε is compact, convex, contains $W(A)$ (hence $\sigma(A)$) in its interior, and has a $C^{1,1}$ boundary $\partial\Omega_\varepsilon$ (continuously differentiable with Lipschitz-continuous unit tangent $\tau(\zeta)$) oriented counterclockwise by arclength ds, with outward unit normal $\nu(\zeta) = -i\tau(\zeta)$ (so $d\zeta = \tau(\zeta)\,ds = i\nu(\zeta)\,ds$). Because $W(A) \subset \Omega_\varepsilon^\circ$ lies strictly on the inward side of every supporting tangent line of $\partial\Omega_\varepsilon$, we have $\mathrm{Re}\big(\overline{\nu(\zeta)}(\zeta I - A)\big) \succ 0$ for all $\zeta \in \partial\Omega_\varepsilon$.

Define the Hermitian matrix-valued measure on $\partial\Omega_\varepsilon$:

$$\begin{aligned} d\mu_\varepsilon(\zeta) \;&:=\; \frac{1}{2\pi i}\Big((\zeta I - A)^{-1}\,d\zeta - (\bar\zeta I - A^*)^{-1}\,d\bar\zeta\Big) \\ &=\; \frac{ds}{\pi}\,(\zeta I - A)^{-1}\,\mathrm{Re}\big(\overline{\nu(\zeta)}(\zeta I - A)\big)\,(\bar\zeta I - A^*)^{-1} \;\succ\; 0. \end{aligned}$$

By Cauchy's integral formula, $\int_{\partial\Omega_\varepsilon} d\mu_\varepsilon(\zeta) = I + I = 2I$, so $d\nu_\varepsilon := \frac{1}{2}d\mu_\varepsilon$ is a positive operator-valued probability measure on $\partial\Omega_\varepsilon$. Define the double-layer operator and conjugate Cauchy transform

$$S_\varepsilon(p) \;:=\; \int_{\partial\Omega_\varepsilon} p(\zeta)\,d\mu_\varepsilon(\zeta) \;=\; p(A) + g_\varepsilon(A)^*, \qquad g_\varepsilon(A) \;:=\; \frac{1}{2\pi i}\int_{\partial\Omega_\varepsilon} \overline{p(\zeta)}\,(\zeta I - A)^{-1}\,d\zeta.$$

For any unit vectors $x, y \in \mathbb{C}^n$, applying Cauchy–Schwarz to the positive semidefinite matrices $d\nu_\varepsilon(\zeta) \succeq 0$ gives

$$\begin{aligned} \big|\langle S_\varepsilon(p)x, y\rangle\big| \;&\le\; 2\|p\|_{\Omega_\varepsilon}\int_{\partial\Omega_\varepsilon}\big|\langle d\nu_\varepsilon(\zeta)x, y\rangle\big| \\ &\le\; 2\|p\|_{\Omega_\varepsilon}\left(\int_{\partial\Omega_\varepsilon}\langle d\nu_\varepsilon x, x\rangle\right)^{1/2}\left(\int_{\partial\Omega_\varepsilon}\langle d\nu_\varepsilon y, y\rangle\right)^{1/2} \;=\; 2\|p\|_{\Omega_\varepsilon}, \end{aligned}$$

whence $\|S_\varepsilon(p)\|_\infty \le 2\|p\|_{\Omega_\varepsilon}$ and therefore $\|S_\varepsilon(p)\|_{2,\mathrm{norm}}^2 \le \|S_\varepsilon(p)\|_\infty^2 \le 4\|p\|_{\Omega_\varepsilon}^2$.

Now expand $\|S_\varepsilon(p)\|_{2,\mathrm{norm}}^2 = \frac{1}{n}\,\mathrm{tr}\big(S_\varepsilon(p)^* S_\varepsilon(p)\big)$ using $S_\varepsilon(p) = p(A) + g_\varepsilon(A)^*$ and cyclicity of the trace:

$$\|p(A)\|_{2,\mathrm{norm}}^2 + \|g_\varepsilon(A)\|_{2,\mathrm{norm}}^2 + 2\,\mathrm{Re}\,\frac{1}{n}\,\mathrm{tr}\big(p(A)\,g_\varepsilon(A)\big) \;=\; \|S_\varepsilon(p)\|_{2,\mathrm{norm}}^2 \;\le\; 4\,\|p\|_{\Omega_\varepsilon}^2. \tag{26}$$

Let $U \in M_n(\mathbb{C})$ be a unitary matrix bringing A to upper-triangular Schur form $U^*AU = T$ with diagonal entries $\lambda_1, \dots, \lambda_n$ [9]. Then for every $\zeta \in \partial\Omega_\varepsilon$, $U^*(\zeta I - A)^{-1}U = (\zeta I - T)^{-1}$ is upper-triangular with diagonal entries $(\zeta - \lambda_j)^{-1}$; integrating over $\partial\Omega_\varepsilon$ shows that $U^* g_\varepsilon(A) U$ is upper-triangular with aligned diagonal entries $w_j := \frac{1}{2\pi i}\int_{\partial\Omega_\varepsilon} \frac{\overline{p(\zeta)}}{\zeta - \lambda_j}\,d\zeta$, and $U^* p(A) g_\varepsilon(A) U$ is upper-triangular with diagonal entries $z_j w_j$, where $z_j := p(\lambda_j)$. Consequently, $\mathrm{tr}(p(A)\,g_\varepsilon(A)) = \sum_{j=1}^n z_j w_j$, and Schur's inequality yields $\|g_\varepsilon(A)\|_{2,\mathrm{norm}}^2 \ge \frac{1}{n}\sum_{j=1}^n |w_j|^2$. Substituting into (26) and completing the square termwise (certified in Lean 4 as `complex_complete_square_nonneg`):

$$|w_j|^2 + 2\,\mathrm{Re}(z_j w_j) \;=\; \big|w_j + \overline{z_j}\big|^2 - |z_j|^2 \;\ge\; -|z_j|^2 \;=\; -|p(\lambda_j)|^2,$$

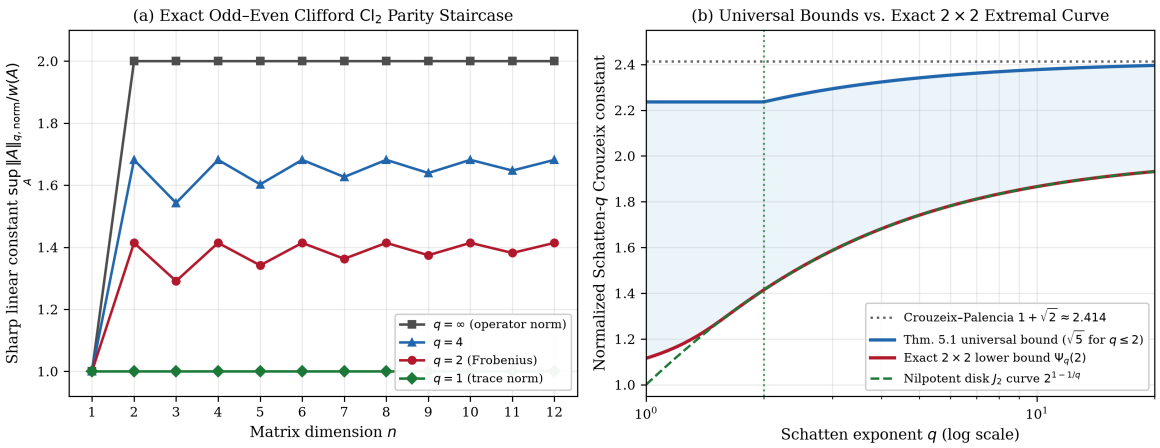


Figure 2: (a) The exact odd–even Clifford Cl_2 parity staircase of Theorem 2.1(a) for $n = 1, \ldots, 12$. (b) Universal Schatten-q Crouzeix landscape comparing our 2×2 exact lower bound $\Psi_q(2)$ (Theorem 4.1), the nilpotent disk curve $2^{1-1/q}$, our n-independent double-layer upper bound (Theorem 5.1), and Crouzeix–Palencia's spectral bound $1 + \sqrt{2}$ [5].

Artifact	**Scope / Verification**	**SHA-256 Digest (full)**
`src/lean/SchattenCrouzeixCore.lean`	13 public thms, 1 def, 2 helpers (`0 sorry`)	`4fe997a76bed97c609e20e618964c9e7`
	(`LEAN_EXIT=0`, `LAKE_EXIT=0`, 21,680 B)	`9494c412ea60d67059353cb406aa9bb1`
`src/python/verify_schatten_crouzeix.py`	600 maj. trials, 2,000 ellipses, 43 DL checks	`af5d3a9db179d7757f6d4fd0947f7844`
	(`VERIFY_EXIT=0`, 26,693 B, 3 figures)	`1691b0241ddbfb6c751496c524476443`
`data/verification_summary.json`	Exact tables ($\Psi_q(2)$, $R^*(q)$, parity staircase)	`9281af4a6f4e3923699b9080aff64adb`
	(11,794 B)	`42ae08d5e2631927597020cd69973fb5`

Table 3: Cryptographic SHA-256 verification ledger for Frontier F5 artifacts.

we obtain

$$\|p(A)\|_{2,\mathrm{norm}}^2 \;\le\; 4\,\|p\|_{\Omega_\varepsilon}^2 + \frac{1}{n}\sum_{j=1}^{n} |p(\lambda_j)|^2.$$

Since p is continuous on $\mathbb{C}$, $\|p\|_{\Omega_\varepsilon} \to \|p\|_{W(A)}$ as $\varepsilon \downarrow 0^+$, yielding (24). Part (a) follows from monotonicity of $q \mapsto \|X\|_{q,\mathrm{norm}}$ on $[1,2]$ and $5 < (1+\sqrt{2})^2$ (certified in Lean 4 as `sqrt5_beats_crouzeix_palencia`); part (b) is immediate from (24); and part (c) follows from Hölder's inequality $\frac{1}{n}\sum \sigma_j^q \le \sigma_1^{q-2} \cdot \frac{1}{n}\sum \sigma_j^2 \le (1+\sqrt{2})^{q-2} \cdot 5\,\|p\|_{W(A)}^q$. □

6 Lean 4.36.0 Formal Certification, Reproducibility, and Open Problems

The exact denominator-cleared integer/rational algebraic, majorization, polynomial, and completing-the-square core of Theorems 2.1–5.1 is formally verified in standalone Lean 4.36.0 in `src/lean/SchattenCrouzeixCore.lean`, with zero external dependencies, `-DwarningAsError=true`, zero `sorry`, zero `grind`, and zero custom axioms (13 public theorems, 1 public definition `extremal PrefixSum`, and 2 private helpers; 16 declarations total, building cleanly under both `lean` and `lake build`). Table 3 records the verified artifact inventory and cryptographic SHA-256 digests. All code, formal proofs, numerical verification scripts, and figure generators are publicly archived at https://github.com/arr-research/arr-research.github.io.

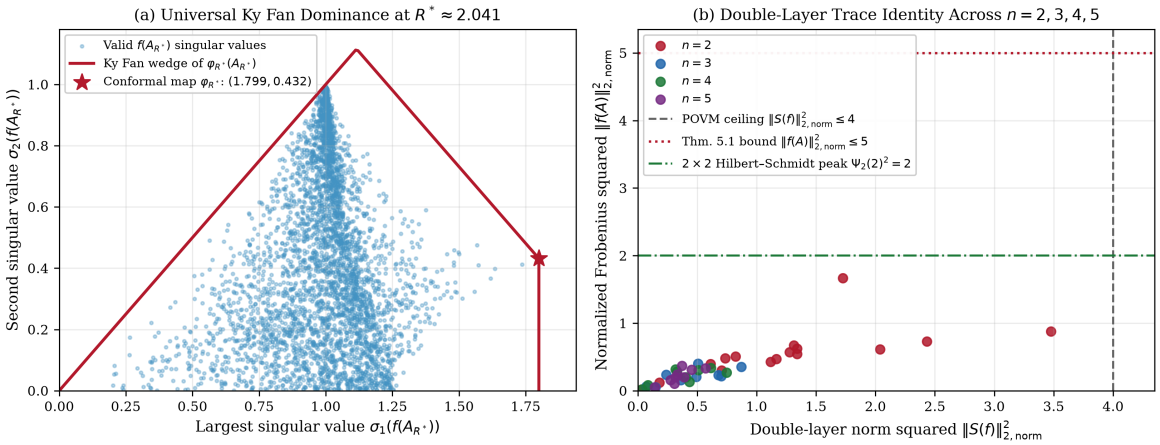


Figure 3: (a) Singular values $(\sigma_1(f(A_{R^*})), \sigma_2(f(A_{R^*})))$ of 4,000 valid Schwarz–Pick images at the trace-norm extremal ellipse $R^* \approx 2.0411$, lying strictly inside the Ky Fan dominance wedge of the odd Riemann conformal map $\varphi_{R^*}(A_{R^*})$ (Theorem 3.2). (b) Numerical verification of the double-layer trace identity (26) across 43 non-normal instances in dimensions $n = 2, 3, 4, 5$ (maximum identity error 4.44×10^{-16}).

Open Problems: The Schatten-q Crouzeix Conjecture. Theorems 4.1 and 5.1 bracket the universal normalized Schatten-q Crouzeix constant $\Psi_q = \sup_{n \ge 1} \Psi_q(n)$ tightly between $\Psi_q(2)$ and $\min\big(\sqrt{5},\, 5^{1/q}(1+\sqrt{2})^{1-2/q}\big)$ for all $q \in [1, \infty)$ (Figure 2(b)). None of the randomized and structured higher-dimensional tests ($n = 3, 4, 5$) in `verify_schatten_crouzeix.py` exceed $\Psi_q(2)$, motivating two sharp conjectures:

Conjecture 6.1 (Schatten-q Crouzeix Conjecture and Odd-Dimension Rigidity)**.**

(a) **Universal 2×2 Reduction:** For every $q \in [1, \infty]$, the global supremum over all finite dimensions is already attained in dimension 2: $\Psi_q = \Psi_q(2)$. In particular, $\Psi_2 = \sqrt{2}$ (Hilbert–Schmidt Crouzeix Conjecture) and $\Psi_1 = \max_{R>1} \frac{R+R^{-1}}{2} \frac{\theta_2(R^{-4})}{\theta_3(R^{-4})} \approx 1.11557524$ (Trace-Norm Crouzeix Conjecture).

(b) **Odd-Dimension Gap:** For every odd $n = 2m+1$ and every $q \in [2, \infty)$, does the strict Clifford parity deficit of Theorem 2.1(a) persist across all polynomials, i.e., $\Psi_q(2m+1) = \big(\frac{m \cdot 2^q + 1}{2m+1}\big)^{1/q} < 2^{1-1/q}$?

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