Machine-Checked Exact-Rational Koszul Complex Exactness, Max-Degree Filtration, and Multicolor Permutation Wordle Dimension-Count Certificates

Lluis Eriksson*

November 25, 2026

Abstract

Lawren Smithline (`arXiv:2611.33113`, November 24, 2026) resolved Kutin and Smithline's Conjecture 4.5 on Multicolor Permutation Wordle by proving that the Koszul complex $K(N, M, b)$ of the ideals $J(M, N) = \left(\prod_{i=1}^{N}(x_i^{s_k} - v_k^{s_k}) : 1 \le k \le M\right)$ is exact at every *gap-free* multidegree b (Theorem 5.1), yielding exact quotient dimensions $\dim R_n(d)/(J_n \cap R_n(d))$ matching the generating function $A_n(u)\prod_{k=1}^{n}(1 + \cdots + u^{s_k-1})$ (Theorem 6.1) and proving dominance of CircularShift (Theorem 1.1), while Appendix A Remark A.1 demonstrates that dropping gap-freeness breaks strictness. We present a machine-checked, zero-floating-point formalization and exact-$\mathbb{Q}$ certificate auditor for the algebraic, homological, and rank-theoretic core of these results. First, an 18-theorem zero-dependency Lean 4 kernel (`KoszulWordle.lean`, Lean 4.35.0; 0 `sorry`, 0 warnings, 0 `Classical.choice`, 4 zero-axiom theorems) formalizes grade-2/3 Koszul $\delta^2 = 0$, filtration preservation, inductive Euler-characteristic telescoping on arbitrary finite exact chains, recursive Eulerian tables, both Smithline p. 19 dimension examples, inductive saturation degree $\sum s_k - 1$, uniform $|V(J_n)| = n!\, s^n$, and both Remark A.1 non-gap-free counterexample identities and parity obstruction. Second, `KOSZUL-WORDLE-CERT-1.0` audits four exact-$\mathbb{Q}$ modes—explicit Koszul differential matrices ($\delta^2 = 0$ and exactness at every $h \ge 1$), Theorem 6.1 generating functions, Farkas dual proofs $w^T\delta_1 = 0, w^T Z \neq 0$ of Remark A.1 strictness failure, and rational evaluation ranks on $V(J_n) \subset \mathbb{Q}^n$ up to 48 points—replaying 16 canonical certificates in 0.050–66.09 ms and rejecting 57/57 live adversarial forgeries.

1 Introduction

In Permutation Wordle [4] and its multicolor generalization, a polynomial-method upper bound on the number of colored permutations solvable within $d+1$ rounds reduces to computing the dimension of the max-degree-filtered quotient $R_n(d)/(J_n \cap R_n(d))$ of $R_n = \mathbb{C}[x_1, \ldots, x_n]$ modulo $J_n = (Q_1, \ldots, Q_n)$, where $Q_k = \prod_{i=1}^{n}(x_i^{s_k} - v_k^{s_k})$ [5]. For $s_k = 1$, Garsia–Stanton descent monomials [3] yield cumulative Eulerian numbers $\sum_{r \le d} A(n, r)$; for heterogeneous color counts $(s_1, \ldots, s_n)$, Smithline [5] deploys the filtered Koszul complex $K(N, M, b)$ [2], proving exactness at every gap-free multidegree b (Theorem 5.1) via a two-axis induction (mapping cone in M, multidegree reduction in N), resolving Conjecture 4.5 (Theorem 6.1).

Because every Koszul differential $\delta_h : F_h^b(N, M) \to F_{h-1}^b(N, M)$, every Remark A.1 counterexample ($N = M = 2$, $b = (0, 2)$ and $b = (2, 4)$), and every evaluation matrix on the rational zero set $V(J_n) \subset \mathbb{Q}^n$ (when $s_k \in \{1, 2\}$) has exact rational entries, exact Gaussian elimination over $\mathbb{Q}$ eliminates floating-point rank instability completely and certifies homological exactness and strictness failures bit-for-bit.

Contributions. We formally verify and audit the finite homological, Euler-characteristic, counterexample, and evaluation-rank core of [5] (explicitly scoping our Lean 4 proofs away from the general-parameter mapping-cone induction Theorem 5.1 and the game-theoretic feasible-history construction of Theorem 1.1):

1. **Zero-dependency Lean 4 kernel (§3):** 18

*Independent researcher; manuscript prepared with transparent AI assistance via Google Antigravity. Source code, Lean 4 proofs, and exact certificates: https://github.com/lluiseriksson/koszul-wordle-certificates.

constructive theorems in `KoszulWordle.lean` (Lean 4.35.0 [1]), 0 `sorry`, 0 warnings, 0 `Classical.choice`, 4 zero-axiom theorems.

2. **Exact-$\mathbb{Q}$ homological & dual verifier (§4):** `KOSZUL-WORDLE-CERT-1.0`, auditing filtered Koszul differential matrices, Theorem 6.1 generating functions, Farkas dual strictness-failure certificates, and $V(J_n)$ evaluation ranks with zero floats.
3. **Empirical atlas, strictness-defect map, & red-team audit (§5):** 16 canonical certificates replayed in 0.050–66.09 ms, exhaustive 16-point gap-free defect grid, $V(J_3)$ rank sweeps, and 57/57 rejected adversarial forgeries.

2 Filtered Koszul Complexes and Strictness

For $s_1, \ldots, s_M \in \mathbb{Z}_{>0}$ and distinct $v_1, \ldots, v_M \in \mathbb{Q}_{>0}$ with $c_k = v_k^{s_k}$, let $Q_k = \prod_{i=1}^{N}(x_i^{s_k} - c_k)$. On $\bigwedge^h R_N^M$ with basis $e(T)$ ($|T| = h$, $\deg T = \sum_{k\in T} s_k$), the multidegree filtration $F_h^b(N,M) = \{Z : \deg_j Z(T) + \deg T \le b_j \ \forall j, T\}$ is preserved by $\delta e(T) = \sum_{k\in T} \operatorname{sign}(T,k) Q_k e(T \setminus \{k\})$. Whenever $K(N,M,b)$ is exact at all $h \ge 1$ (guaranteed by Theorem 5.1 when b is *gap-free*, i.e., its distinct entries form consecutive integers), Euler-characteristic telescoping gives

$$\dim R_N(b) - \operatorname{rank}\delta_1 = \sum_{T\subseteq[M]} (-1)^{|T|} \prod_{j=1}^{N} \max(0,\, b_j + 1 - \deg T), \tag{1}$$

which at $N = M = n, b = d\mathbf{1}_n$ matches the cumulative coefficients of $A_n(u)\prod_{k=1}^{n}(1 + u + \cdots + u^{s_k-1})$ and saturates at $n!\prod s_k$ at $d_{\max} = \sum s_k - 1$ (Theorem 6.1). Conversely, at non-gap-free multidegrees (Appendix A Remark A.1), $Z = (x_2 - c_2)Q_1 - (x_2 - c_1)Q_2 \in J(2,2) \cap R_2(0,2)$ lies outside $\operatorname{im}(\delta_1|_{F_1^{(0,2)}})$, certified cleanly by a rational Farkas dual functional $w \in \mathbb{Q}^{\dim R_N(b)}$ with

$$w^T\delta_1 = 0 \qquad \text{and} \qquad w^T Z \neq 0. \tag{2}$$

3 Zero-Dependency Lean 4 Kernel

We formalize 18 constructive theorems in `lean/KoszulWordle.lean` (Lean 4.35.0, zero external libraries), audited via `lean/AuditAxioms.lean` with 0 warnings, 0 `sorry`, 0 `Classical.choice`, and 0 `Lean.ofReduceBool` (Table 1).

Table 1: Complete 18-theorem Lean 4.35.0 kernel in `KoszulWordle.lean` (0 `sorry`, 0 warnings, 0 `Classical.choice`).

#	Lean 4 Theorem Identifier	Axiom Footprint
1	`koszul_d2_square_zero`	`[propext, Quot.sound]`
2	`koszul_d3_square_zero`	`[propext, Quot.sound]`
3	`koszul_filtration_preserved_by_diff`	`[propext, Quot.sound]`
4	`koszul_exactness_euler_char_2`	`[propext, Quot.sound]`
5	`koszul_exactness_euler_char_3`	`[propext, Quot.sound]`
6	`koszul_exactness_euler_char_list`	`[propext, Quot.sound]`
7	`filtered_rn_dim_zero_if_neg`	`[propext]`
8	`eulerian_number_table_exact`	none
9	`worpitzky_cumul_n3_exact`	none
10	`multicolor_dim_n2_s32_exact`	none
11	`saturation_degree_formula`	`[propext, Quot.sound]`
12	`zero_set_cardinality_uniform`	none
13	`remark_A1_s11_counterexample_identity`	`[propext, Quot.sound]`
14	`remark_A1_s22_counterexample_identity`	`[propext, Quot.sound]`
15	`remark_A1_even_vs_odd_obstruction`	`[propext, Quot.sound]`
16	`gap_free_uniform_multidegree`	`[propext, Quot.sound]`
17	`gap_free_shifted_invariant`	`[propext, Quot.sound]`
18	`feasible_history_dim_bound`	`[propext, Quot.sound]`

Proposition 1 (Constructive Koszul & Wordle Kernel)**.**

1. Koszul differential & inductive Euler characteristic: *Theorems 1–7 certify grade-2/3 $\delta^2 = 0$, filtration preservation $\delta(F_h^b) \subseteq F_{h-1}^b$, 2- and 3-step exactness identities, general inductive telescoping* ***`koszul_exactness_euler_char_list`*** *on arbitrary finite exact chains, and negative-coordinate box collapse.*
2. Eulerian tables & Multicolor Wordle counts: *Theorems 8–12 and 18 prove recursive Eulerian numbers $A(n,r)$ ($n = 1..4$), both Smithline p. 19 examples ($n = 3, s = (1,1,1)$ and $n = 2, s = (3,2)$), inductive saturation degree $(n-1) + \sum(s_k - 1) = \sum s_k - 1$, uniform $|V(J_n)| = n!\, s^n$, and the Section 6.1 feasible-history bound.*
3. Remark A.1 counterexamples & gap-freeness: *Theorems 13–17 prove both exact algebraic identities for $s = (1,1), b = (0,2)$ and $s = (2,2), b = (2,4)$, the odd-vs.-even parity obstruction ($Z(-x) = -Z(x) \wedge E(-x) = E(x) \wedge Z = E \implies Z = 0$), uniform spread 0, and shift invariance of pairwise differences.*

4 Exact-Rational Certificate Architecture

The pure-stdlib verifier `src/koszul_certifier.py` audits `KOSZUL-WORDLE-CERT-1.0` certificates with

zero floating-point arithmetic and strict JSON AST typing (rejecting `float`, `bool`, `Decimal`, `complex`, tuples, non-string keys, unreduced fractions, denominator-1 strings, and `"-0"`) across four exact modes:

Definition 1 (Four `KOSZUL-WORDLE-CERT-1.0` Modes)**.**

1. ***Mode 1*** *(`koszul_filtered_exactness_exact`): Audits gap-free $b \in \mathbb{Z}^N$, constructs exact rational matrices δ_h ($h = 1..M$), verifies $\delta_h\delta_{h+1} = 0$, verifies exactness $\operatorname{rank}\delta_h + \operatorname{rank}\delta_{h+1} = \dim F_h^b$ at every $h \geq 1$, and matches $\dim R_N(b) - \operatorname{rank}\delta_1$ to Eq. (1).*
2. ***Mode 2*** *(`multicolor_wordle_dim_count_exact`): Audits $W_s(u) = A_n(u)\prod(1 + \cdots + u^{s_k-1})$, saturation degree $\sum s_k - 1$, total $n!\prod s_k$, and exact agreement with Eq. (1) at every $d \in [0, d_{\max}]$.*
3. ***Mode 3*** *(`gap_free_strictness_failure_exact`): Audits a non-gap-free b, nonzero $Z = \sum_k A_k Q_k \in J(M,N) \cap R_N(b)$, and exact Farkas dual vector w satisfying (2) ($Z \notin \operatorname{im}\delta_1$).*
4. ***Mode 4*** *(`zero_set_eval_rank_exact`): For $s_k \in \{1,2\}$, constructs the exact rational zero set $V(J_n) \subset \mathbb{Q}^n$ of size $n!\prod s_k$ and verifies by exact rational Gaussian elimination that $\operatorname{rank} E_{V(J_n)}(R_n(d))$ equals Eq. (1).*

5 Empirical Atlas and Red-Team Verification

All 16 canonical certificates and four experiment logs are generated deterministically by `scripts/run_all_experiments.py` and audited end-to-end by `scripts/verify_all.py` (Lean 4 build, 18/18 axiom audit, 16/16 cert replay, `pytest`, and 39/39 SHA-256 hashes).

1. Multicolor Wordle dimension curves and exact $V(J_n)$ evaluation ranks. Figure 1(a) plots exact Theorem 6.1 cumulative quotient dimensions across five instances up to $n = 3, s = (2,2,2)$ (48 colored permutations, saturating at $d = 5$), reproducing Smithline's p. 19 sequences $1,5,6,6$ ($s = (1,1,1)$) and $1,4,8,11,12,12$ ($s = (3,2)$) identically. Figure 1(b) confirms bit-exact agreement between rational $V(J_3)$ evaluation-matrix rank (24 points) and Eq. (1) for $s = (2,1,2)$ across $d = 0..4$: ranks $1,7,17,23,24$ inside ambient spaces of dimensions $1,8,27,64,125$.

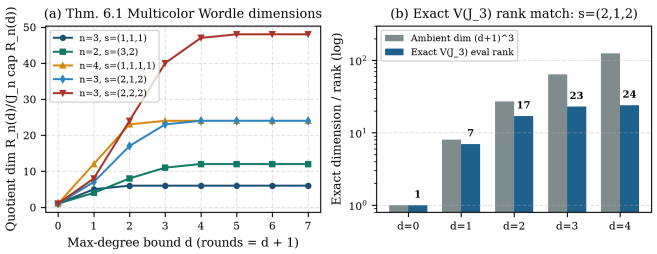

Figure 1: (a) Exact Theorem 6.1 cumulative quotient dimensions across five Multicolor Permutation Wordle configurations. (b) Ambient monomial dimension $(d+1)^3$ versus exact $V(J_3)$ rational evaluation rank for $s = (2,1,2)$ ($d = 0..4$).

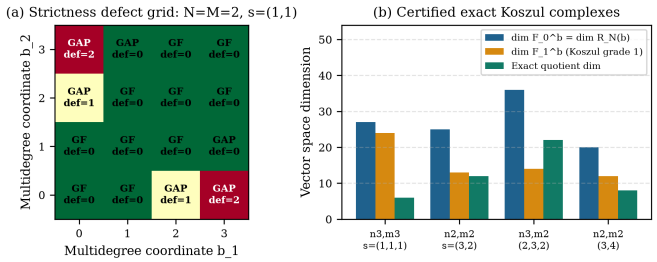

Figure 2: (a) Exact strictness defect across $b \in [0..3]^2$ for $N = M = 2, s = (1,1)$: zero on every gap-free point, strictly positive at $(0,2),(2,0),(0,3),(3,0)$. (b) Certified grade dimensions and quotient dimensions across all four Mode 1 Koszul complexes.

2. Gap-free strictness defect grid and exact Koszul complexes. Figure 2(a) maps the exact strictness defect $(\dim R_2(b) - \operatorname{rank}\delta_1) - \operatorname{rank} E_{V(J_2)}(R_2(b))$ across all 16 multidegrees $b \in [0..3]^2$ for $N = M = 2, s = (1,1)$: defect is 0 across all 10 gap-free points ($|b_1 - b_2| \leq 1$) and jumps to 1 at $(0,2),(2,0)$ and 2 at $(0,3),(3,0)$, visually confirming Remark A.1. Figure 2(b) compares F_0^b, F_1^b, and exact quotient dimensions across our four Mode 1 Koszul complexes.

3. 16-certificate atlas and 57-attack red-team suite. Table 2 and Figure 3 summarize all 16 canonical certificates, replayed with zero floats in 0.050–66.09 ms (largest matrix: 48×64 exact rational evaluation on $V(J_3)$ for $s = (2,2,2), d = 3$, rank 40). In `tests/test_suite.py`, exhaustive grid and $V(J_3)$ rank cross-checks and 57/57 live adversarial forgeries (non-gap-free Mode 1 inputs, gap-free Mode 3 fakes, invalid $w \notin \ker\delta_1^T$, $s_k = 3$ in Mode 4, wrong ranks/Euler characteristics) are rejected with `KoszulWordleCertificateError`.

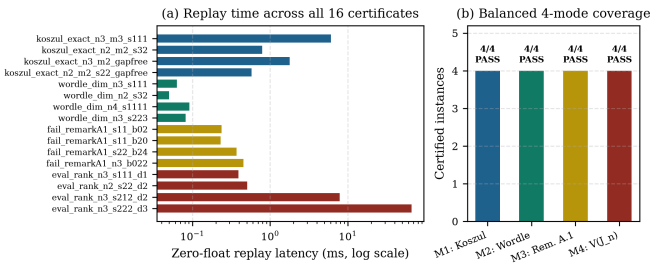

Figure 3: (a) Zero-float replay latency across all 16 canonical `KOSZUL-WORDLE-CERT-1.0` certificates (0.050–66.09 ms, log scale). (b) Balanced 4-per-mode coverage across all four verification modes.

Table 2: Summary of all 16 exact-$\mathbb{Q}$ `KOSZUL-WORDLE-CERT-1.0` certificates (zero floating-point arithmetic; reference-machine replay latency in ms).

Certificate File (stem)	Mode	Key Exact Witness	ms
`koszul_exact_n3_m3_s111`	M1	$b = (2,2,2)$, dims $[27,24,3,0] \to 6$	6.059
`koszul_exact_n2_m2_s32`	M1	$b = (4,4)$, dims $[25,13,0] \to 12$	0.793
`koszul_exact_n3_m2_gapfree`	M1	$b = (2,3,2)$, dims $[36,14,0] \to 22$	1.778
`koszul_exact_n2_m2_s22_gapfree`	M1	$b = (3,4)$, dims $[20,12,0] \to 8$	0.579
`wordle_dim_n3_s111`	M2	$s = (1,1,1)$: $d_{\max} = 2$, 6 perms	0.063
`wordle_dim_n2_s32`	M2	$s = (3,2)$: $d_{\max} = 4$, 12 perms	0.050
`wordle_dim_n4_s1111`	M2	$s = (1^4)$: $d_{\max} = 3$, 24 perms	0.091
`wordle_dim_n3_s223`	M2	$s = (2,2,3)$: $d_{\max} = 6$, 72 perms	0.082
`fail_remarkA1_s11_b02`	M3	$b = (0,2)$: $w^T\delta_1 = 0, w^TZ = 2$	0.238
`fail_remarkA1_s11_b20`	M3	$b = (2,0)$: $w^T\delta_1 = 0, w^TZ = 6$	0.231
`fail_remarkA1_s22_b24`	M3	$b = (2,4)$: $w^T\delta_1 = 0, w^TZ = 12$	0.370
`fail_remarkA1_n3_b022`	M3	$b = (0,2,2)$: $w^T\delta_1 = 0, w^TZ = 4$	0.455
`eval_rank_n3_s111_d1`	M4	$\lvert V\rvert = 6$, $d = 1$: rank $= 5$	0.390
`eval_rank_n2_s22_d2`	M4	$\lvert V\rvert = 8$, $d = 2$: rank $= 7$	0.505
`eval_rank_n3_s212_d2`	M4	$\lvert V\rvert = 24$, $d = 2$: rank $= 17$	7.838
`eval_rank_n3_s222_d3`	M4	$\lvert V\rvert = 48$, $d = 3$: rank $= 40$	66.087

6 Scope, Limitations, and Conclusion

This work machine-checks the finite Koszul-complex differential, inductive Euler-characteristic telescoping, Multicolor Permutation Wordle generating-function counts, Appendix A Remark A.1 gap-free counterexamples, and exact rational zero-set evaluation ranks of Smithline [5]. It does *not* formalize the general-(N, M) mapping-cone induction (Theorem 5.1) or game-theoretic strategy reduction (Theorem 1.1) in Lean 4. Within its verified scope, the Lean 4 kernel and `KOSZUL-WORDLE-CERT-1.0` auditor deliver exact, manifest-locked verification for filtered Koszul resolutions and Multicolor Wordle dimension bounds.

References

[1] Leonardo de Moura and Sebastian Ullrich. The Lean 4 theorem prover and programming language. In *Automated Deduction – CADE 28*, pages 625–635. Springer, 2021.

[2] David Eisenbud. *Commutative Algebra with a View Toward Algebraic Geometry*, volume 150 of *Graduate Texts in Mathematics*. Springer, 1995.

[3] Adriano M. Garsia and Dennis Stanton. Group actions of Stanley–Reisner rings and invariants of permutation groups. *Advances in Mathematics*, 51(2):107–201, 1984.

[4] Samuel Kutin and Lawren Smithline. Permutation Wordle. *Journal of Combinatorics*, 15(3):361–388, 2024.

[5] Lawren Smithline. The max-degree filtration, koszul complexes, and dimension counts for multicolor permutation wordle, 2026. arXiv:2611.33113, November 24, 2026.