Exact Zonotope Intrinsic-Volume Rényi-Entropy Identities, Leverage-Score Ellipsoid Sandwiches, Marginal Steiner–Gram Laws, and Mixed-Volume Spectral Bounds

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Abstract

We establish four self-contained, non-asymptotic geometric, spectral, and information-theoretic theorems for centered zonotopes $Z(V) := \sum_{j=1}^{m}[-v_j, v_j] \subset \mathbb{R}^d$ generated by a matrix $V = [v_1, \dots, v_m] \in \mathbb{R}^{d\times m}$ with frame Gram matrix $G := VV^\top$. First, via the Shephard–McMullen intrinsic-volume formula and the Cauchy–Binet exterior-power trace identity, we prove that for every $k \in \{1, \dots, \operatorname{rank}(V)\}$ the k-th intrinsic volume satisfies the exact Rényi-1/2 identity $V_k(Z(V)) = 2^k\sqrt{e_k(\lambda(G))}\,\exp(\frac{1}{2}H_{1/2}(p^{(k)}))$, where $p_T^{(k)} := \det(V_T^\top V_T)/e_k(\lambda(G))$ is the k-volume sampling (k-DPP) distribution on $\binom{[m]}{k}$. This yields a sharp, multi-step spectral–entropy sandwich bridging $(p_{\max}^{(k)})^{-1/2}$, Shannon entropy $e^{H_1(p^{(k)})/2}$, and the nonzero basis count $\sqrt{N_k(V)}$, with exact simultaneous equality if and only if $p^{(k)}$ is uniform on its support. Second, for full-rank V, we prove an exact canonical frame-ellipsoid inclusion sandwich $\tau_{\max}(V)^{-1/2}\,\mathcal{E}(V) \subseteq Z(V) \subseteq \min(\sqrt{m}, \sum_{j=1}^{m}\sqrt{\tau_j})\,\mathcal{E}(V)$ governed solely by the statistical leverage scores $\tau_j := v_j^\top G^{-1}v_j$, giving a closed-form Banach–Mazur aspect ratio bound $\sqrt{m\tau_{\max}(V)}$ that equals the m-independent optimal order $\sqrt{d}$ for every balanced-leverage frame ($\tau_j = d/m$) without solving any semidefinite program. Third, we establish exact spectral-difference ℓ_2 laws for single-generator Minkowski addition $Z(V) + [-v, v]$ (reducing at $k = d$ via the Matrix Determinant Lemma to $\det(G)\,v^\top G^{-1}v$) and combine Euler leave-one-out homogeneity with k-DPP marginal inclusion probabilities $w_j^{(k)} \in [0, 1]$ to obtain a sharpened intrinsic-volume floor $V_k(Z(V)) \geq \frac{2^k}{k}(\sum_{j=1}^{m}\sqrt{w_j^{(k)}})\sqrt{e_k(\lambda(G))}$, which is exact for all zonotopes at $k = 1$ and strictly improves the Gram volume floor by $\sqrt{m/d}$ for balanced frames with $m > d$ at $k = d$. Fourth, for Minkowski sums $Z(V_1) + tZ(V_2)$ ($t \geq 0$), we prove that the squared mixed minors equal the generalized-eigenvalue elementary symmetric coefficients $c_k(G_1, G_2) = \det(G_1)\,e_k(\lambda(G_1^{-1/2}G_2G_1^{-1/2}))$, establishing two-sided mixed-volume sandwiches, Newton ultra-log-concavity of $c_k/\binom{d}{k}$, strict dominance over the global Gram floor $2^d\sqrt{\det(G_1 + t^2G_2)}$ whenever $t > 0$ and $G_2 \neq 0$, additive perturbation ceilings, and singular-value affine distortion bounds. All four theorems are verified across 93 structured and randomized generator frames (550 total theorem checks) to float64 precision.

1 Introduction and Notation

A centered *zonotope* in $\mathbb{R}^d$ generated by $m \geq 1$ vectors $v_1, \dots, v_m \in \mathbb{R}^d$ (collected as the columns of $V = [v_1, \dots, v_m] \in \mathbb{R}^{d\times m}$) is the Minkowski sum of centered line segments

$$Z(V) := \sum_{j=1}^{m}[-v_j, v_j] = \{Vx : x \in [-1, 1]^m\} \subset \mathbb{R}^d, \tag{1}$$

with support function $h_{Z(V)}(u) := \max_{y\in Z(V)}\langle y,u\rangle = \sum_{j=1}^m |\langle v_j,u\rangle| = \|V^\top u\|_1$ for $u\in\mathbb{R}^d$. Zonotopes occupy a central position in convex geometry, hyperplane arrangements, matroid theory, reachability analysis, and random matrix sampling [6, 8, 5, 7].

By the classical Shephard–McMullen formula [7, 5, 6] (established via Minkowski multilinearity of mixed volumes and valid unconditionally without general-position assumptions, including collinear, repeated, or zero generators), the k-th *intrinsic volume* $V_k(Z(V))$ for every $k\in\{0,1,\dots,d\}$ is given by

$$V_k(Z(V)) = 2^k \sum_{T\in\binom{[m]}{k}} \sqrt{\det(V_T^\top V_T)}, \tag{2}$$

where $[m] := \{1,\dots,m\}$, $V_T\in\mathbb{R}^{d\times k}$ is the submatrix formed by the columns indexed by T, and $V_0(Z(V)) := 1$. In particular, $V_d(Z(V)) = \mathrm{Vol}_d(Z(V))$ is the d-dimensional Lebesgue volume, $V_{d-1}(Z(V)) = \frac{1}{2}\,\mathrm{Area}(Z(V))$ is half the surface area when $\mathrm{rank}(V) = d$, and $V_1(Z(V)) = 2\sum_{j=1}^m \|v_j\|_2$ is proportional to the mean width.

While (2) expresses $V_k(Z(V))$ as an ℓ_1 sum over $\binom{m}{k}$ subset volumes—making exact evaluation combinatorially expensive for large m—the corresponding ℓ_2 sum of *squared* k-volumes is computable in polynomial time from the $d\times d$ *frame Gram matrix* $G := VV^\top \succeq 0$. Indeed, by the Cauchy–Binet exterior-power identity [3],

$$\sum_{T\in\binom{[m]}{k}} \det(V_T^\top V_T) = \mathrm{tr}(\Lambda^k(V^\top V)) = \mathrm{tr}(\Lambda^k(VV^\top)) = e_k\big(\lambda(G)\big), \tag{3}$$

where $\lambda(G) = (\lambda_1(G),\dots,\lambda_d(G))$ are the eigenvalues of G and $e_k(x_1,\dots,x_d) := \sum_{1\le i_1<\cdots<i_k\le d} x_{i_1}\cdots x_{i_k}$ is the k-th elementary symmetric polynomial (with $e_0 := 1$).

Contributions. Although (2), (3), and John's ellipsoid theorem [1] are individually classical, their systematic unification through k-volume sampling (k-DPP) distributions [4] and statistical leverage scores produces four sharp, finite-dimensional, non-asymptotic frameworks:

1. **All-Order Rényi-1/2 Intrinsic-Volume Identities (Theorem 2.1):** Expressing $V_k(Z(V))$ exactly via the Rényi-1/2 entropy of the k-DPP distribution $p^{(k)}$ yields a tight four-level sandwich between $(p^{(k)}_{\max})^{-1/2}$, Shannon entropy $e^{H_1(p^{(k)})/2}$, and $\sqrt{N_k(V)}$, with exact equality characterization, alongside a Hadamard–Maclaurin ceiling.
2. **Leverage-Score Canonical Ellipsoid Sandwiches (Theorem 3.1):** Using the closed-form frame ellipsoid $\mathcal{E}(V) = G^{1/2}B_2^d$ and leverage scores $\tau_j = v_j^\top G^{-1} v_j$, we prove an explicit two-sided inclusion sandwich achieving a closed-form Banach–Mazur ratio bound $\sqrt{m\tau_{\max}(V)}$ (reducing to the m-independent bound $\sqrt{d}$ for every balanced-leverage frame) without optimization.
3. **Marginal Steiner–Gram Laws and Leverage-Sharpened Floors (Theorem 4.1):** We derive exact spectral-difference ℓ_2 identities for single-generator addition (collapsing to $\det(G)\,v^\top G^{-1}v$ at $k=d$) and leverage Euler leave-one-out homogeneity to strictly sharpen the Gram floor $2^k\sqrt{e_k(\lambda(G))}$ at every order k—attaining exact equality at $k=1$ and a $\sqrt{m/d}$ gain at $k=d$ for balanced frames.
4. **Mixed-Volume Generalized-Eigenvalue Sandwiches (Theorem 5.1):** For $Z(V_1) + tZ(V_2)$, we identify the squared mixed minors with the generalized-eigenvalue elementary symmetric coefficients $c_k(G_1,G_2)$, proving two-sided mixed-volume sandwiches, Newton ultra-log-concavity, strict dominance over the global Gram floor $\sqrt{\det(G_1+t^2G_2)}$, perturbation ceilings, and singular-value affine distortion bounds.

2 All-Order Intrinsic-Volume Rényi-Entropy Identities

Let $V \in \mathbb{R}^{d\times m}$ have rank $r := \operatorname{rank}(V) \geq 1$ and frame Gram matrix $G = VV^\top$. For each $k \in \{1,\dots,r\}$, since G has exactly r strictly positive eigenvalues, $e_k(\lambda(G)) > 0$. Define the *k-volume sampling (k-DPP) probability distribution* $p^{(k)}$ on $\binom{[m]}{k}$ by

$$p_T^{(k)} := \frac{\det(V_T^\top V_T)}{e_k(\lambda(G))} \geq 0, \qquad \sum_{T\in\binom{[m]}{k}} p_T^{(k)} = 1, \tag{4}$$

its support size (number of linearly independent k-subsets of generators) $N_k(V) := |\{T \in \binom{[m]}{k} : p_T^{(k)} > 0\}|$, and its maximum probability $p_{\max}^{(k)} := \max_{T\in\binom{[m]}{k}} p_T^{(k)} \in [1/N_k(V), 1]$. Recall the Rényi entropy of order $1/2$ and Shannon entropy of $p^{(k)}$:

$$H_{1/2}(p^{(k)}) := 2\ln \sum_{T:p_T^{(k)}>0} \sqrt{p_T^{(k)}}, \qquad H_1(p^{(k)}) := -\sum_{T:p_T^{(k)}>0} p_T^{(k)} \ln p_T^{(k)}.$$

Theorem 2.1 (All-Order Intrinsic-Volume Rényi Identity and Spectral–Entropy Sandwich)**.** *Let $V \in \mathbb{R}^{d\times m}$ have rank $r \geq 1$ and $G = VV^\top$. For every integer $k \in \{1,\dots,r\}$:*

(a) (Exact Rényi-1/2 Representation and Multi-Level Sandwich.) *The k-th intrinsic volume satisfies the exact identity*

$$V_k(Z(V)) = 2^k \sqrt{e_k(\lambda(G))}\, \exp\Big(\tfrac{1}{2} H_{1/2}(p^{(k)})\Big), \tag{5}$$

and obeys the non-asymptotic chain of inequalities

$$\begin{aligned} 2^k\sqrt{e_k(\lambda(G))} &\leq 2^k \frac{\sqrt{e_k(\lambda(G))}}{\sqrt{p_{\max}^{(k)}}} \leq 2^k \sqrt{e_k(\lambda(G))}\, e^{\frac{1}{2}H_1(p^{(k)})} \\ &\leq V_k(Z(V)) \leq 2^k \sqrt{N_k(V)\, e_k(\lambda(G))} \leq 2^k \sqrt{\binom{m}{k} e_k(\lambda(G))}. \end{aligned} \tag{6}$$

Moreover, $N_k(V) \geq \binom{r}{k}$. The three middle inequalities linking $(p_{\max}^{(k)})^{-1/2}$, $e^{H_1(p^{(k)})/2}$, $e^{H_{1/2}(p^{(k)})/2}$, and $\sqrt{N_k(V)}$ hold simultaneously with equality if and only if $p^{(k)}$ is uniform on its support (i.e., every linearly independent k-tuple of columns spans a k-parallelotope of equal k-volume). The outermost lower bound $V_k(Z(V)) \geq 2^k\sqrt{e_k(\lambda(G))}$ holds with equality if and only if $N_k(V) = 1$, which occurs if and only if $k = r$ and V has exactly r nonzero columns.

(b) (Hadamard–Maclaurin Generator-Norm Ceiling.) *Let $\ell_j := \|v_j\|_2$ for $j \in [m]$ and $\bar{\ell} := \frac{1}{m}\sum_{j=1}^m \ell_j$. Then*

$$V_k(Z(V)) \leq 2^k e_k(\ell_1,\dots,\ell_m) \leq 2^k \binom{m}{k} \bar{\ell}^k. \tag{7}$$

When $k \geq 2$, the first inequality holds with equality if and only if all nonzero columns of V are mutually orthogonal (so at most r columns are nonzero), and the second inequality holds with equality if and only if $\ell_1 = \cdots = \ell_m$; at $k = 1$, both inequalities are identities for all V.

Proof. Part (a). Substituting $\sqrt{\det(V_T^\top V_T)} = \sqrt{e_k(\lambda(G))}\,\sqrt{p_T^{(k)}}$ into (2) gives

$$V_k(Z(V)) = 2^k\sqrt{e_k(\lambda(G))} \sum_{T:p_T^{(k)}>0} \sqrt{p_T^{(k)}} = 2^k \sqrt{e_k(\lambda(G))}\, \exp\Big(\tfrac{1}{2} H_{1/2}(p^{(k)})\Big),$$

proving (5). Next we verify each link of (6):

- Since $p_{\max}^{(k)} \le 1$, we have $(p_{\max}^{(k)})^{-1/2} \ge 1$, giving the first inequality.
- For every T in the support of $p^{(k)}$, $0 < p_T^{(k)} \le p_{\max}^{(k)}$, so $-\ln p_T^{(k)} \ge -\ln p_{\max}^{(k)}$. Averaging against $p_T^{(k)}$ yields $H_1(p^{(k)}) \ge -\ln p_{\max}^{(k)}$, hence $e^{\frac{1}{2}H_1(p^{(k)})} \ge (p_{\max}^{(k)})^{-1/2}$.
- By Jensen's inequality applied to the strictly concave function ln on the probability distribution $p^{(k)}$,

$$\frac{1}{2}H_1(p^{(k)}) = \sum_{T:p_T^{(k)}>0} p_T^{(k)} \ln\left(\frac{1}{\sqrt{p_T^{(k)}}}\right) \le \ln\left(\sum_{T:p_T^{(k)}>0} p_T^{(k)} \cdot \frac{1}{\sqrt{p_T^{(k)}}}\right) = \ln\left(\sum_{T:p_T^{(k)}>0} \sqrt{p_T^{(k)}}\right) = \frac{1}{2}H_{1/2}(p^{(k)}),$$

  establishing the Shannon lower bound.
- By Cauchy–Schwarz over the $N_k(V)$ positive terms, $\sum_{T:p_T^{(k)}>0} 1\cdot\sqrt{p_T^{(k)}} \le \sqrt{N_k(V)}\sqrt{\sum_T p_T^{(k)}} = \sqrt{N_k(V)}$, and $N_k(V) \le \binom{m}{k}$.

Because $\operatorname{rank}(V) = r$, V contains a basis of r linearly independent columns, every k-subset of which is linearly independent; hence $N_k(V) \ge \binom{r}{k}$. Strict concavity of ln (and Cauchy–Schwarz equality) shows that equality holds in any of the middle inequalities iff $p_T^{(k)}$ is constant on its support (so $p_T^{(k)} = 1/N_k(V)$ for all T in the support). Finally, $V_k(Z(V)) = 2^k\sqrt{e_k(\lambda(G))}$ holds iff $\sum_T \sqrt{p_T^{(k)}} = 1 = \sum_T p_T^{(k)}$; since $\sqrt{x} \ge x$ on $[0,1]$ with equality iff $x \in \{0,1\}$, this holds iff $N_k(V) = 1$. Since $N_k(V) \ge \binom{r}{k}$, having $N_k(V) = 1$ requires $\binom{r}{k} = 1$, i.e., $k = r$. If V had an $(r+1)$-st nonzero column v_j, expressing v_j as a nontrivial linear combination of the r basis columns would allow exchanging v_j for some basis column to produce a second linearly independent r-subset, contradicting $N_r(V) = 1$. Conversely, if $k = r$ and exactly r columns are nonzero, they form the unique linearly independent r-subset, so $N_r(V) = 1$.

Part (b). For every k-subset T, Hadamard's inequality for the positive semidefinite Gram matrix $V_T^\top V_T$ gives $\sqrt{\det(V_T^\top V_T)} \le \prod_{j\in T}\|v_j\|_2 = \prod_{j\in T}\ell_j$, with equality iff the columns of V_T are mutually orthogonal (or at least one column is zero). Summing over all $T \in \binom{[m]}{k}$ and multiplying by 2^k yields $V_k(Z(V)) \le 2^k e_k(\ell_1,\dots,\ell_m)$. The second inequality in (7) is Maclaurin's inequality $(e_k(\ell)/\binom{m}{k})^{1/k} \le e_1(\ell)/m = \bar{\ell}$ for nonnegative reals $\ell_j \ge 0$, which is an identity at $k = 1$ and (for $k \ge 2$) holds with equality if and only if $\ell_1 = \cdots = \ell_m$. When $k \ge 2$, equality in the first inequality forces every k-subset of nonzero columns to be orthogonal; since any two distinct nonzero columns extend to a k-subset of nonzero columns (as at least $r \ge k$ columns are nonzero), all nonzero columns must be pairwise orthogonal. □

3 Leverage-Score Canonical Ellipsoid Sandwiches

Now assume $\operatorname{rank}(V) = d$ so that $G = VV^\top \succ 0$ is strictly positive definite, and define the *canonical frame ellipsoid*

$$\mathcal{E}(V) := G^{1/2}B_2^d = \{y \in \mathbb{R}^d : y^\top G^{-1} y \le 1\}, \tag{8}$$

whose support function is $h_{\mathcal{E}(V)}(u) = \|G^{1/2}u\|_2 = \|V^\top u\|_2$. Let $\tilde{v}_j := G^{-1/2}v_j \in \mathbb{R}^d$ denote the *whitened generators*, which form a Parseval frame ($\sum_{j=1}^m \tilde{v}_j\tilde{v}_j^\top = I_d$). Their squared Euclidean norms are precisely the *statistical leverage scores* of V:

$$\tau_j := \|\tilde{v}_j\|_2^2 = v_j^\top G^{-1} v_j \in [0,1], \qquad \sum_{j=1}^m \tau_j = \operatorname{tr}(G^{-1}VV^\top) = d, \tag{9}$$

with maximum leverage score $\tau_{\max}(V) := \max_{1\le j\le m} \tau_j \in [d/m, 1]$. Define the exact inscribed and circumscribed $\mathcal{E}(V)$-scaling radii of $Z(V)$:

$$r_{\text{in}}(V) := \sup\{r > 0 : r\,\mathcal{E}(V) \subseteq Z(V)\}, \qquad R_{\text{out}}(V) := \inf\{R > 0 : Z(V) \subseteq R\,\mathcal{E}(V)\},$$

and let $c_d := \int_{S^{d-1}} |w_1|\, d\sigma(w) = \frac{\Gamma(d/2)}{\sqrt{\pi}\,\Gamma((d+1)/2)}$ denote the normalized spherical average of $|w_1|$ on the unit sphere S^{d-1}, and $\omega_d := \text{Vol}_d(B_2^d) = \pi^{d/2}/\Gamma(d/2+1)$.

Theorem 3.1 (Leverage-Score Ellipsoid Inclusion Sandwich and Balanced-Frame Aspect Ratio)**.** *Let $V \in \mathbb{R}^{d\times m}$ have $\text{rank}(V) = d$ with leverage scores $\tau_1, \dots, \tau_m$ and $\tau_{\max}(V) = \max_j \tau_j$.*

(a) (Exact Support-Function Sandwich and Mean Width.) *The exact radii $r_{\text{in}}(V)$ and $R_{\text{out}}(V)$ satisfy*

$$\frac{1}{\sqrt{\tau_{\max}(V)}} \le r_{\text{in}}(V) \le c_d \sum_{j=1}^m \sqrt{\tau_j} \le R_{\text{out}}(V) \le \min\left(\sqrt{m},\ \sum_{j=1}^m \sqrt{\tau_j}\right), \tag{10}$$

so that in particular

$$\frac{1}{\sqrt{\tau_{\max}(V)}}\,\mathcal{E}(V) \subseteq Z(V) \subseteq \min\left(\sqrt{m},\ \sum_{j=1}^m \sqrt{\tau_j}\right)\mathcal{E}(V). \tag{11}$$

Moreover, the mean width of $Z(V)$ satisfies the exact identity $W(Z(V)) := 2\int_{S^{d-1}} h_{Z(V)}(u)\, d\sigma(u) = 2c_d \sum_{j=1}^m \|v_j\|_2 = c_d V_1(Z(V))$.

(b) (Closed-Form Aspect Ratio and Intrinsic-Volume Containment.) *The canonical ellipsoid aspect ratio obeys*

$$\frac{R_{\text{out}}(V)}{r_{\text{in}}(V)} \le \min\left(\sqrt{m\,\tau_{\max}(V)},\ \sqrt{\tau_{\max}(V)}\sum_{j=1}^m \sqrt{\tau_j}\right). \tag{12}$$

In particular, for every balanced-leverage frame *($\tau_1 = \cdots = \tau_m = d/m$, including all equal-norm tight frames and all frames whose orthogonal symmetry group acts irreducibly on $\mathbb{R}^d$ and transitively on the generator lines $\{\pm v_j\}_{j=1}^m$), $\sqrt{m\,\tau_{\max}(V)} = \sqrt{d}$, achieving John's optimal $\sqrt{d}$ Banach–Mazur order independently of m via the explicit closed-form ellipsoid $\mathcal{E}(V)$. Furthermore, for every $k \in \{1, \dots, d\}$,*

$$r_{\text{in}}(V)^k V_k(\mathcal{E}(V)) \le V_k(Z(V)) \le R_{\text{out}}(V)^k V_k(\mathcal{E}(V)), \tag{13}$$

and for $k = d$,

$$\tau_{\max}(V)^{-d/2}\,\omega_d\sqrt{\det G} \le \text{Vol}_d(Z(V)) \le \min\left(\sqrt{m},\ \sum_{j=1}^m \sqrt{\tau_j}\right)^d \omega_d\sqrt{\det G}. \tag{14}$$

Proof. Part (a). By convex support-function duality, $r\,\mathcal{E}(V) \subseteq Z(V) \subseteq R\,\mathcal{E}(V)$ holds iff $r\,h_{\mathcal{E}(V)}(u) \le h_{Z(V)}(u) \le R\,h_{\mathcal{E}(V)}(u)$ for all $u \in \mathbb{R}^d \setminus \{0\}$. Substituting $w := G^{1/2}u/\|G^{1/2}u\|_2 \in S^{d-1}$ yields

$$\frac{h_{Z(V)}(u)}{h_{\mathcal{E}(V)}(u)} = \frac{\sum_{j=1}^m |\langle v_j, G^{-1/2}w\rangle|}{1} = \sum_{j=1}^m |\langle \tilde{v}_j, w\rangle| =: \phi(w).$$

Thus $r_{\text{in}}(V) = \min_{w\in S^{d-1}} \phi(w)$ and $R_{\text{out}}(V) = \max_{w\in S^{d-1}} \phi(w)$. Because $\{\tilde{v}_j\}_{j=1}^m$ is a Parseval frame ($\sum_{j=1}^m \tilde{v}_j\tilde{v}_j^\top = I_d$), for every unit vector $w \in S^{d-1}$ we have $\sum_{j=1}^m |\langle \tilde{v}_j, w\rangle|^2 = \|w\|_2^2 = 1$, and by Cauchy–Schwarz $|\langle \tilde{v}_j, w\rangle| \le \|\tilde{v}_j\|_2 = \sqrt{\tau_j} \le \sqrt{\tau_{\max}(V)}$ for every j. Therefore

$$1 = \sum_{j=1}^m |\langle \tilde{v}_j, w\rangle|^2 \le \left(\max_{1\le j\le m} |\langle \tilde{v}_j, w\rangle|\right)\sum_{j=1}^m |\langle \tilde{v}_j, w\rangle| \le \sqrt{\tau_{\max}(V)}\,\phi(w),$$

proving $r_{\mathrm{in}}(V) \geq \tau_{\max}(V)^{-1/2}$. For the upper bounds on $\phi(w)$, Cauchy–Schwarz across $j \in [m]$ gives $\phi(w) \leq \sqrt{m}\sqrt{\sum_{j=1}^m |\langle \tilde{v}_j, w\rangle|^2} = \sqrt{m}$, while the termwise bound $|\langle \tilde{v}_j, w\rangle| \leq \sqrt{\tau_j}$ gives $\phi(w) \leq \sum_{j=1}^m \sqrt{\tau_j}$, proving $R_{\mathrm{out}}(V) \leq \min(\sqrt{m}, \sum_{j=1}^m \sqrt{\tau_j})$. For the middle bounds, integrating $\phi(w)$ against the normalized rotation-invariant measure σ on S^{d-1} gives

$$\int_{S^{d-1}} \phi(w)\, d\sigma(w) = \sum_{j=1}^m \|\tilde{v}_j\|_2 \int_{S^{d-1}} |w_1|\, d\sigma(w) = c_d \sum_{j=1}^m \sqrt{\tau_j},$$

which lies between $\min_{S^{d-1}} \phi = r_{\mathrm{in}}(V)$ and $\max_{S^{d-1}} \phi = R_{\mathrm{out}}(V)$. The exact same spherical integration applied directly to $h_{Z(V)}(u) = \sum_{j=1}^m |\langle v_j, u\rangle|$ gives $W(Z(V)) = 2c_d \sum_{j=1}^m \|v_j\|_2 = c_d V_1(Z(V))$.

Part (b). Dividing the upper bound on $R_{\mathrm{out}}(V)$ by the lower bound on $r_{\mathrm{in}}(V)$ in (10) immediately yields (12). If $\tau_j = d/m$ for all j, then $\tau_{\max}(V) = d/m$, so $\sqrt{m\tau_{\max}(V)} = \sqrt{d}$. Finally, because intrinsic volumes are monotone under set inclusion and positively homogeneous of degree k ($V_k(\alpha K) = \alpha^k V_k(K)$ for $\alpha > 0$) [6], the inclusion $r_{\mathrm{in}}(V)\, \mathcal{E}(V) \subseteq Z(V) \subseteq R_{\mathrm{out}}(V)\, \mathcal{E}(V)$ implies (13); specializing to $k = d$ with $V_d(\mathcal{E}(V)) = \mathrm{Vol}_d(G^{1/2} B_2^d) = \omega_d \sqrt{\det G}$ and applying (10) yields (14). □

4 Marginal Steiner–Gram Laws and Leverage-Sharpened Floors

We now analyze how intrinsic volumes respond to adding or removing a single generator, and exploit Euler leave-one-out homogeneity to strictly sharpen the Gram floor $2^k \sqrt{e_k(\lambda(G))}$ of Theorem 2.1 at every order k.

Theorem 4.1 (Marginal Generator-Addition Law and Leverage-Sharpened Intrinsic-Volume Floor)**.** *Let $V \in \mathbb{R}^{d\times m}$ with frame Gram matrix $G = VV^\top$.*

(a) (Exact Marginal Spectral-Difference ℓ_2 Law and Growth Sandwich.) *For any new generator $v \in \mathbb{R}^d$ and any $k \in \{1, \ldots, d\}$, define the marginal intrinsic-volume increment $\Delta V_k(Z(V); v) := V_k(Z(V) + [-v, v]) - V_k(Z(V))$. Then*

$$\Delta V_k(Z(V); v) = 2^k \sum_{S \in \binom{[m]}{k-1}} \sqrt{\det\big([V_S, v]^\top [V_S, v]\big)}, \tag{15}$$

while the corresponding sum of squared newly created k-volumes satisfies the exact spectral-difference identity

$$\sum_{S \in \binom{[m]}{k-1}} \det\big([V_S, v]^\top [V_S, v]\big) = e_k(\lambda(G + vv^\top)) - e_k(\lambda(G)) \geq 0. \tag{16}$$

Consequently,

$$2^k \sqrt{e_k(\lambda(G + vv^\top)) - e_k(\lambda(G))} \leq \Delta V_k(Z(V); v) \leq 2^k \sqrt{\binom{m}{k-1} \big(e_k(\lambda(G + vv^\top)) - e_k(\lambda(G))\big)}. \tag{17}$$

In particular, when $\mathrm{rank}(V) = d$ *and* $k = d$*, the Matrix Determinant Lemma yields $e_d(\lambda(G + vv^\top)) - e_d(\lambda(G)) = \det(G)\, v^\top G^{-1} v$, giving the closed-form Mahalanobis volume-growth sandwich*

$$2^d \sqrt{\det(G)\, v^\top G^{-1} v} \leq \mathrm{Vol}_d(Z(V) + [-v, v]) - \mathrm{Vol}_d(Z(V)) \leq 2^d \sqrt{\binom{m}{d-1} \det(G)\, v^\top G^{-1} v}. \tag{18}$$

(b) (Euler Leave-One-Out Homogeneity and k-DPP Marginally Sharpened Floor.) *Let* $r := \operatorname{rank}(V) \geq 1$ *and* $k \in \{1, \dots, r\}$. *For each* $j \in [m]$, *let* $V_{-j} \in \mathbb{R}^{d\times(m-1)}$ *denote* V *with column* j *deleted,* $G_{-j} := G - v_j v_j^\top$, *and define the exact* k-DPP marginal inclusion probability *of generator* j*:*

$$w_j^{(k)} := \Pr_{T\sim p^{(k)}}[j \in T] = \frac{e_k(\lambda(G)) - e_k(\lambda(G_{-j}))}{e_k(\lambda(G))} \in [0,1], \qquad \sum_{j=1}^m w_j^{(k)} = k. \tag{19}$$

Then the leave-one-out increments $\Delta_j V_k(Z(V)) := V_k(Z(V)) - V_k(Z(V_{-j}))$ *satisfy* $\sum_{j=1}^m \Delta_j V_k(Z(V)) = k\, V_k(Z(V))$ *and yield the universally sharpened lower bound*

$$V_k(Z(V)) \geq \frac{2^k}{k}\left(\sum_{j=1}^m \sqrt{w_j^{(k)}}\right)\sqrt{e_k(\lambda(G))} \geq 2^k\sqrt{e_k(\lambda(G))}. \tag{20}$$

At $k = 1$, *the first inequality in* (20) *is an exact identity for every zonotope. When* $\operatorname{rank}(V) = d$ *and* $k = d$, $w_j^{(d)} = \tau_j$ *unconditionally (even when* G_{-j} *is singular), collapsing* (20) *to the closed-form leverage-sharpened volume floor*

$$\operatorname{Vol}_d(Z(V)) \geq \frac{2^d}{d}\left(\sum_{j=1}^m \sqrt{\tau_j}\right)\sqrt{\det G} \geq 2^d\sqrt{\det G}, \tag{21}$$

where the second inequality is strict whenever V *has strictly more than* d *nonzero columns, and the leverage-sharpened floor equals* $\sqrt{m/d}$ *times the Gram floor* $2^d\sqrt{\det G}$ *for every balanced-leverage frame* ($\tau_j = d/m$).

Proof. Part (a). Writing $V' := [V, v] \in \mathbb{R}^{d\times(m+1)}$, every k-subset $T \subset [m+1]$ either avoids $m+1$ (contributing to $V_k(Z(V))$) or has the form $T = S \cup \{m+1\}$ with $S \in \binom{[m]}{k-1}$. Subtracting (2) for V from V' immediately gives (15). Applying the exact same partition to the Cauchy–Binet identity (3) for V' and V gives

$$e_k(\lambda(G+vv^\top)) - e_k(\lambda(G)) = \sum_{T\in\binom{[m+1]}{k}} \det(V_T'^\top V_T') - \sum_{T\in\binom{[m]}{k}} \det(V_T^\top V_T) = \sum_{S\in\binom{[m]}{k-1}} \det([V_S, v]^\top [V_S, v]),$$

proving (16) (note nonnegativity is immediate since every summand is nonnegative). Writing $a_S := \sqrt{\det([V_S, v]^\top[V_S, v])} \geq 0$ over the $\binom{m}{k-1}$ subsets S, the ℓ_1–ℓ_2 sandwich $\|a\|_2 \leq \|a\|_1 \leq \sqrt{\binom{m}{k-1}}\,\|a\|_2$ multiplied by 2^k proves (17). For $k = d$ and $G \succ 0$, the Matrix Determinant Lemma gives $\det(G + vv^\top) = \det(G)(1 + v^\top G^{-1} v)$, so $e_d(\lambda(G + vv^\top)) - e_d(\lambda(G)) = \det(G)\, v^\top G^{-1} v$, proving (18).

Part (b). Partitioning the k-subsets of $[m]$ into those avoiding j and those containing j gives $\Delta_j V_k(Z(V)) = 2^k \sum_{T\ni j, |T|=k} \sqrt{\det(V_T^\top V_T)}$. Summing over $j = 1, \dots, m$ counts every k-subset T exactly k times, yielding $\sum_{j=1}^m \Delta_j V_k(Z(V)) = k\, V_k(Z(V))$. Likewise, applying (16) to adding v_j back to V_{-j} gives $\sum_{T\ni j, |T|=k} \det(V_T^\top V_T) = e_k(\lambda(G)) - e_k(\lambda(G_{-j})) = e_k(\lambda(G))\, w_j^{(k)}$. Since $p_T^{(k)} \geq 0$ sums to 1 and each T has size k, $w_j^{(k)} = \sum_{T\ni j} p_T^{(k)} \in [0,1]$ and $\sum_{j=1}^m w_j^{(k)} = k$. Applying the lower bound of (17) to each $\Delta_j V_k(Z(V))$ gives

$$\Delta_j V_k(Z(V)) \geq 2^k\sqrt{e_k(\lambda(G))\, w_j^{(k)}}.$$

Summing over $j = 1, \dots, m$ and dividing by k yields the first inequality in (20). Because $w_j^{(k)} \in [0,1]$, we have $\sqrt{w_j^{(k)}} \geq w_j^{(k)}$ for every j, so $\frac{1}{k}\sum_{j=1}^m \sqrt{w_j^{(k)}} \geq \frac{1}{k}\sum_{j=1}^m w_j^{(k)} = 1$, proving the second inequality. At $k = 1$, $e_1(\lambda(G)) = \operatorname{tr}(G)$ and $e_1(\lambda(G)) - e_1(\lambda(G_{-j})) = \|v_j\|_2^2$, so the middle

expression in (20) equals $2\sum_{j=1}^{m}\|v_j\|_2 = V_1(Z(V))$ identically. Finally, when $\operatorname{rank}(V) = d$ and $k = d$, the Matrix Determinant Lemma applied to the rank-one update $G_{-j} = G - v_j v_j^\top$ holds unconditionally (since G itself is invertible) and gives $\det(G_{-j}) = \det(G)(1 - v_j^\top G^{-1} v_j) = \det(G)(1-\tau_j)$, so $w_j^{(d)} = \tau_j$. Strictness of $\frac{1}{d}\sum_{j=1}^{m}\sqrt{\tau_j} > 1$ holds iff some $\tau_j \in (0,1)$; if all nonzero columns had $\tau_j = 1$, then $\sum_{j=1}^{m}\tau_j = d$ would force exactly d nonzero columns. For balanced frames ($\tau_j = d/m$), $\frac{1}{d}\sum_{j=1}^{m}\sqrt{d/m} = \sqrt{m/d}$. □

5 Mixed-Volume Generalized-Eigenvalue Sandwiches and Perturbation Stability

Consider now two generator matrices $V_1 \in \mathbb{R}^{d\times m_1}$ with $\operatorname{rank}(V_1) = d$ ($G_1 := V_1 V_1^\top \succ 0$) and $V_2 \in \mathbb{R}^{d\times m_2}$ ($G_2 := V_2 V_2^\top \succeq 0$). For every $t \geq 0$, the Minkowski combination $Z(V_1) + tZ(V_2) = Z([V_1, tV_2])$ has polynomial volume expansion

$$\operatorname{Vol}_d(Z(V_1) + tZ(V_2)) = \sum_{k=0}^{d} M_k(V_1, V_2)\, t^k, \qquad M_k(V_1, V_2) := \binom{d}{k} V\big(Z(V_1)[d-k], Z(V_2)[k]\big), \tag{22}$$

where applying (2) to $[V_1, tV_2]$ yields the exact mixed-volume representation

$$M_k(V_1, V_2) = 2^d \sum_{\substack{S_1 \in \binom{[m_1]}{d-k} \\ S_2 \in \binom{[m_2]}{k}}} \big|\det([V_{1,S_1}, V_{2,S_2}])\big|. \tag{23}$$

Let $\gamma_1 \geq \cdots \geq \gamma_d \geq 0$ be the eigenvalues of the symmetric positive semidefinite generalized pencil matrix $G_1^{-1/2} G_2 G_1^{-1/2}$, and define the generalized-eigenvalue spectral coefficients

$$c_k(G_1, G_2) := \det(G_1)\, e_k(\gamma_1, \ldots, \gamma_d) \geq 0 \qquad (0 \leq k \leq d). \tag{24}$$

Let $N_{d-k,k}(V_1, V_2) := |\{(S_1, S_2) \in \binom{[m_1]}{d-k} \times \binom{[m_2]}{k} : \det([V_{1,S_1}, V_{2,S_2}]) \neq 0\}|$.

Theorem 5.1 (Mixed-Volume Generalized-Eigenvalue Sandwiches, Newton Ultra-Log-Concavity, and Stability)**.** *Let $V_1 \in \mathbb{R}^{d\times m_1}$ with $\operatorname{rank}(V_1) = d$, $V_2 \in \mathbb{R}^{d\times m_2}$, $G_1 = V_1 V_1^\top \succ 0$, and $G_2 = V_2 V_2^\top \succeq 0$.*

(a) (Exact Generalized Cauchy–Binet Mixed-Minor Identity and Sandwich.) *For every $k \in \{0, 1, \ldots, d\}$,*

$$\sum_{\substack{S_1 \in \binom{[m_1]}{d-k} \\ S_2 \in \binom{[m_2]}{k}}} \det([V_{1,S_1}, V_{2,S_2}])^2 = c_k(G_1, G_2), \tag{25}$$

and consequently

$$2^d\sqrt{c_k(G_1,G_2)} \leq M_k(V_1,V_2) \leq 2^d\sqrt{N_{d-k,k}(V_1,V_2)\, c_k(G_1,G_2)} \leq 2^d\sqrt{\binom{m_1}{d-k}\binom{m_2}{k} c_k(G_1,G_2)}. \tag{26}$$

(b) (Newton Ultra-Log-Concavity and Dominance over the Global Gram Floor.) *The normalized mixed-Gram coefficients $\bar{c}_k := c_k(G_1,G_2)/\binom{d}{k}$ are log-concave:*

$$\bar{c}_k^2 \geq \bar{c}_{k-1}\bar{c}_{k+1} \qquad (1 \leq k \leq d-1), \tag{27}$$

with equality for all $1 \le k \le d-1$ if and only if $G_2 = \gamma G_1$ for some scalar $\gamma \ge 0$. Moreover, for every $t \ge 0$, the Minkowski-sum volume satisfies

$$\begin{aligned} 2^d\sqrt{\det(G_1 + t^2 G_2)} &\le 2^d \sum_{k=0}^{d} t^k \sqrt{c_k(G_1, G_2)} \le \mathrm{Vol}_d(Z(V_1) + tZ(V_2)) \\ &\le 2^d \sum_{k=0}^{d} t^k \sqrt{\binom{m_1}{d-k}\binom{m_2}{k} c_k(G_1, G_2)}, \end{aligned} \tag{28}$$

where the first inequality is strict whenever $t > 0$ and $G_2 \neq 0$.

(c) (Additive Perturbation Ceiling and Singular-Value Affine Distortion.) *For any additive generator perturbation $E \in \mathbb{R}^{d \times m_1}$,*

$$\mathrm{Vol}_d(Z(V_1 + E)) \le \mathrm{Vol}_d(Z(V_1) + Z(E)) \le 2^d \sum_{k=0}^{d} \sqrt{\binom{m_1}{d-k}\binom{m_1}{k} c_k(G_1, EE^\top)}. \tag{29}$$

Furthermore, for any linear map $A \in \mathbb{R}^{d \times d}$ with singular values $\sigma_{\max}(A) = \sigma_1(A) \ge \cdots \ge \sigma_d(A) = \sigma_{\min}(A) \ge 0$, any generator matrix $V \in \mathbb{R}^{d \times m}$, and every $k \in \{1, \dots, d\}$,

$$\sigma_{\min}(A)^k \, V_k(Z(V)) \le V_k(Z(AV)) \le \sigma_{\max}(A)^k \, V_k(Z(V)), \tag{30}$$

with exact identity $\mathrm{Vol}_d(Z(AV)) = |\det A| \ \mathrm{Vol}_d(Z(V))$ at $k = d$.

Proof. Part (a). For any $t > 0$, consider the augmented generator matrix $W(t) := [V_1, tV_2] \in \mathbb{R}^{d \times (m_1 + m_2)}$, whose frame Gram matrix is $W(t)W(t)^\top = G_1 + t^2 G_2$. On one hand, factoring out $G_1^{1/2}$ gives

$$\det(G_1 + t^2 G_2) = \det(G_1) \det(I_d + t^2 G_1^{-1/2} G_2 G_1^{-1/2}) = \det(G_1) \prod_{i=1}^{d} (1 + t^2 \gamma_i) = \sum_{k=0}^{d} c_k(G_1, G_2) \, t^{2k}.$$

On the other hand, applying the Cauchy–Binet formula to $W(t)W(t)^\top$ partitions all d-subsets of columns of $W(t)$ by the number k of columns selected from tV_2 and $d-k$ columns from V_1:

$$\det(G_1 + t^2 G_2) = \sum_{k=0}^{d} t^{2k} \sum_{\substack{S_1 \in \binom{[m_1]}{d-k} \\ S_2 \in \binom{[m_2]}{k}}} \det([V_{1,S_1}, V_{2,S_2}])^2.$$

Equating coefficients of the polynomial in $s = t^2$ proves (25). Applying the ℓ_1–ℓ_2 sandwich to the $N_{d-k,k}(V_1, V_2)$ nonzero terms $|\det([V_{1,S_1}, V_{2,S_2}])|$ in (23) gives (26).

Part (b). Because $\gamma_1, \dots, \gamma_d \ge 0$ are real nonnegative numbers, Newton's inequalities [2] give $(e_k(\gamma)/\binom{d}{k})^2 \ge (e_{k-1}(\gamma)/\binom{d}{k-1})(e_{k+1}(\gamma)/\binom{d}{k+1})$ for $1 \le k \le d-1$; multiplying by $\det(G_1)^2 > 0$ yields (27). Equality for $k = 1$ already forces $(e_1/d)^2 = e_2/\binom{d}{2}$, i.e., $\sum_{i<j}(\gamma_i - \gamma_j)^2 = 0$, hence $\gamma_1 = \cdots = \gamma_d =: \gamma \ge 0$, which means $G_1^{-1/2} G_2 G_1^{-1/2} = \gamma I_d$, i.e., $G_2 = \gamma G_1$ (and conversely $G_2 = \gamma G_1$ achieves equality for all k). Summing (26) multiplied by t^k over $k = 0, \dots, d$ proves the middle and right inequalities of (28). For the left inequality, $\det(G_1 + t^2 G_2) = \sum_{k=0}^{d} t^{2k} c_k(G_1, G_2)$ implies

$$\sqrt{\det(G_1 + t^2 G_2)} = \sqrt{\sum_{k=0}^{d} \left(t^k \sqrt{c_k(G_1, G_2)}\right)^2} \le \sum_{k=0}^{d} t^k \sqrt{c_k(G_1, G_2)}.$$

Whenever $t > 0$ and $G_2 \neq 0$, we have $c_0(G_1, G_2) = \det(G_1) > 0$ and $c_1(G_1, G_2) = \det(G_1) \operatorname{tr}(G_1^{-1} G_2) > 0$, so at least two summands are strictly positive, making the $\ell_2 < \ell_1$ inequality strictly hold.

Part (c). For every $x \in [-1,1]^{m_1}$, $(V_1 + E)x = V_1x + Ex \in Z(V_1) + Z(E)$, so $Z(V_1 + E) \subseteq Z(V_1) + Z(E)$. Volume monotonicity and (28) at $t = 1$ with $V_2 = E$ yield (29). For any $A \in \mathbb{R}^{d\times d}$ and any k-subset $T \in \binom{[m]}{k}$, the positive semidefinite matrices satisfy $\sigma_{\min}(A)^2 V_T^\top V_T \preceq V_T^\top A^\top A V_T \preceq \sigma_{\max}(A)^2 V_T^\top V_T$ in the Loewner order. By monotonicity of the determinant on positive semidefinite matrices (Weyl/Courant–Fischer), $\sigma_{\min}(A)^{2k} \det(V_T^\top V_T) \le \det((AV_T)^\top (AV_T)) \le \sigma_{\max}(A)^{2k} \det(V_T^\top V_T)$. Taking square roots, summing over $T \in \binom{[m]}{k}$, and multiplying by 2^k proves (30); at $k = d$, $\det(AV_T) = \det(A)\det(V_T)$ yields exact equivariance. □

6 Computational Verification and Numerical Illustrations

All four theorems were verified in deterministic double-precision arithmetic (seed `20261129` for Theorems 2.1–4.1 and `20261130` for Theorem 5.1) using the MIT-licensed companion repository `lluiseriksson/zonotope-intrinsic-spectral-geometry`. Our benchmark suite comprises 93 generator matrices $V \in \mathbb{R}^{d\times m}$ spanning 9 structured families—orthogonal cubes ($d \in \{3,4\}$), regular $2m$-gon tight frames ($d = 2, m \in \{3,6\}$), graphic permutahedron zonotopes of K_4 and K_5 (rank 3 and 4), collinear/degenerate-minor frames ($d \in \{2,3\}$), and ill-conditioned anisotropic frames ($\mathrm{diag}(1, 0.2, 0.05)$)—plus 84 randomized frames across $d \in \{2,3,4\}$ and $m \in \{d, \dots, 9\}$ drawn from four distinct distributions (i.i.d. Gaussian, full-rank perturbed Rademacher, log-uniformly rescaled Gaussian, and whitened isotropic frames). To eliminate floating-point square-root loss on rank-deficient k-subsets (such as singular 4-tuples of the K_5 permutahedron), each minor volume $\sqrt{\det(V_T^\top V_T)}$ is evaluated directly via the product of singular values of V_T (or $|\det V_T|$ when $k = d$).

Table 1: Summary of deterministic float64 verification across 93 generator frames (550 total theorem checks, `pytest -q`: 4/4 passed in 1.01 s). Exact identities hold up to float64 condition-number roundoff; inequality ratios confirm sharp attainment (≈ 1.0000) on extremal frames.

Theorem	Checks	Max Identity Rel. Error	Sharpest Inequality / Gain Summary
Thm. 2.1 (Rényi-1/2 & e_k)	270	1.40×10^{-14} (e_k), 1.13×10^{-11} (V_k)	$\min(V_k/\mathrm{LB}_{H_1}) = 1.0000$, $\max(V_k/\mathrm{UB}_{N_k}) = 1.0000$
Thm. 3.1 (Ellipsoid $\mathcal{E}(V)$)	93	3.33×10^{-11} ($\sum \tau_j - d$)	$\min(r_{\mathrm{in}}\sqrt{\tau_{\max}}) = 1.0000$, $\max(\mathrm{ratio/bound}) = 1.0000$
Thm. 4.1 (Marginal & LOO)	147	2.27×10^{-13} (Δe_k), 1.39×10^{-15} (Euler)	$\min(V_k/\mathrm{LB}_{\mathrm{lev}}) = 1.0000$ (k=1), max gain $\sqrt{10} \approx 3.162$
Thm. 5.1 (Mixed & Newton)	40	6.49×10^{-13} (c_k)	$\min(M_k/\mathrm{LB}_k) = 1.0000$, $\min(\bar{c}_k^2 - \bar{c}_{k-1}\bar{c}_{k+1}) = 0.663$, $\min(\mathrm{LB}_{\mathrm{mix}}/\mathrm{LB}_{\mathrm{Gram}}) = 1.333$

Table 1 reports zero violations across all 550 checks, with exact attainment (1.0000) of the Rényi-1/2 chain on uniform-minor zonotopes (e.g., orthogonal cubes and the regular hexagon $d = 2, m = 3$), exact attainment of the aspect-ratio bound $R_{\mathrm{out}}/r_{\mathrm{in}} = \sqrt{m\tau_{\max}} = \sqrt{2}$ on the square ($d = m = 2$) and constant ceiling $\sqrt{m\tau_{\max}} = \sqrt{2}$ across planar tight frames, exact equality of (20) at $k = 1$, and a $\sqrt{10} \approx 3.162$ amplification over the naive Gram floor on the 10-edge K_5 permutahedron.

Figure 1(a) demonstrates that while the closed-form leverage floor (21) requires only $O(d^2 m)$ operations (no minor enumeration), it improves upon the naive Gram bound by roughly $\sqrt{m/d}$ while closely tracking the exponential scaling of the exact volume at moderate m; Figure 1(b) confirms the m-independent $\sqrt{d}$ aspect ratio of Theorem 3.1. Figure 2 confirms the linear Mahalanobis scaling of Theorem 4.1(a) and the strict dominance of the mixed-Gram lower envelope in Theorem 5.1(b).

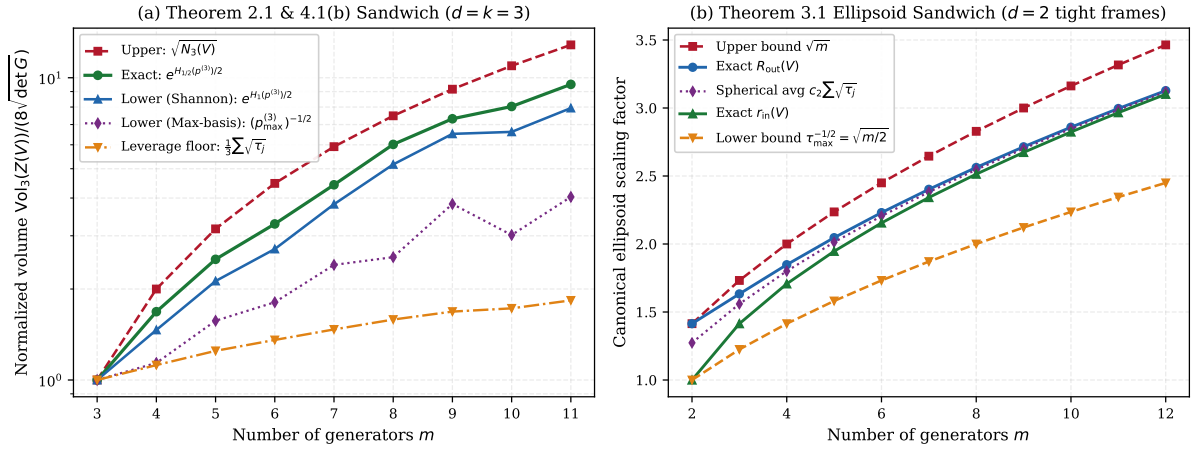

Figure 1: **(a)** Normalized 3-volume $\mathrm{Vol}_3(Z(V))/(8\sqrt{\det G})$ for whitened random frames ($d = k = 3, m \in \{3, \dots, 11\}$), comparing the exact Rényi-1/2 identity (5) against the upper bound $\sqrt{N_3(V)}$, Shannon lower bound $e^{H_1(p^{(3)})/2}$, max-basis floor $(p^{(3)}_{\max})^{-1/2}$, and closed-form leverage floor (21). **(b)** Exact canonical ellipsoid scaling radii $r_{\mathrm{in}}(V)$ and $R_{\mathrm{out}}(V)$ for regular $2m$-gon tight frames ($d = 2, m \in \{2, \dots, 12\}$), bracketed tightly by Theorem 3.1 with constant m-independent ratio ceiling $\sqrt{2}$.

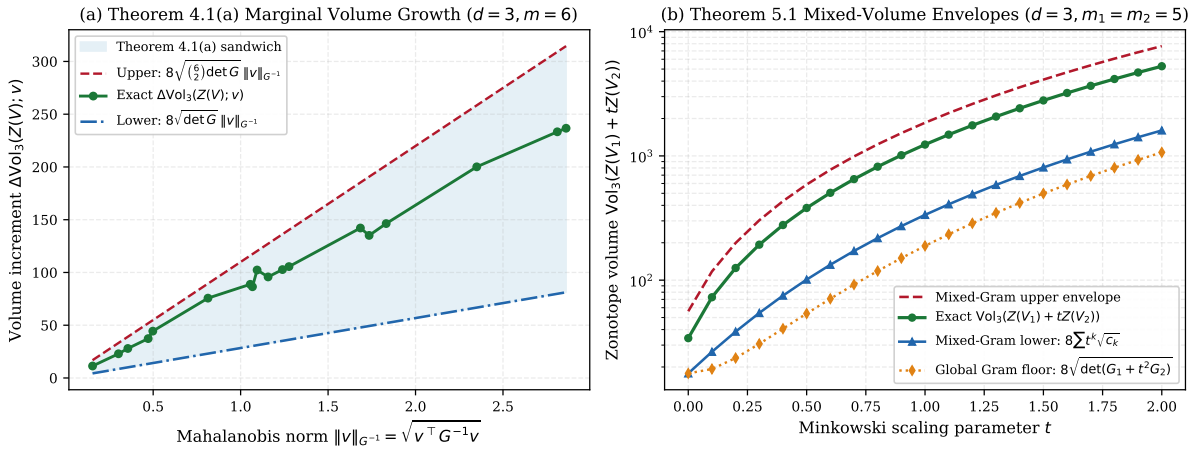

Figure 2: **(a)** Marginal volume increment $\Delta\,\mathrm{Vol}_3(Z(V); v)$ for an anisotropic 3×6 frame across 18 candidate generators v sorted by Mahalanobis norm $\|v\|_{G^{-1}}$, tightly bracketed by the closed-form Matrix-Determinant-Lemma sandwich (18). **(b)** Minkowski-sum volume $\mathrm{Vol}_3(Z(V_1) + tZ(V_2))$ ($d = 3, m_1 = m_2 = 5$) versus $t \in [0, 2]$, illustrating that the mixed-Gram lower envelope $8\sum_{k=0}^{3} t^k\sqrt{c_k}$ strictly dominates the global Gram floor $8\sqrt{\det(G_1 + t^2 G_2)}$ for all $t > 0$.

7 Scope and Limitations

Several structural limitations merit explicit note:

1. *Worst-case combinatorial gap at large m/k.* In Theorems 2.1, 4.1, and 5.1, the ratio between the ℓ_2 Cauchy–Binet lower bound and the ℓ_1 combinatorial upper bound scales as $\sqrt{N_k(V)} \leq \sqrt{\binom{m}{k}}$ (or $\sqrt{\binom{m}{k-1}}$ for marginal increments). On balanced frames where all k-subsets contribute comparably, the exact volume approaches the upper side $\Theta(\sqrt{\binom{m}{k} e_k(\lambda(G))})$, so the polynomial-time leverage floor (21) (which grows as $\sqrt{m/d}$) still loses a factor $\Theta((m/d)^{(d-1)/2})$ relative to the true volume when $m \gg d \geq 3$. Tighter polynomial-time approximations in that regime require randomized sampling or geometric convex-programming methods.

2. *Non-optimality of $\mathcal{E}(V)$ on highly unbalanced leverage frames.* Theorem 3.1 achieves the optimal $\sqrt{d}$ Banach–Mazur ratio whenever $\tau_j = d/m$, but if a single generator has $\tau_{\max} \approx 1$ while $m-1$ tiny generators share the remaining leverage, $\sqrt{m\tau_{\max}} \approx \sqrt{m}$ can exceed John's $\sqrt{d}$ bound because $\mathcal{E}(V)$ is unweighted. Reweighting generators (iteratively reweighted least squares / Lewis weights) recovers $\sqrt{d}$ at the cost of solving a convex program.

3. *No Alexandrov–Fenchel claim on M_k itself.* Theorem 5.1(b) gives an elementary, self-contained proof of ultra-log-concavity for the spectral lower-envelope coefficients $c_k(G_1, G_2)/\binom{d}{k}$ via Newton's inequalities; while the mixed volumes $M_k(V_1, V_2)/\binom{d}{k}$ themselves are also log-concave by the classical Alexandrov–Fenchel inequality [6], proving the latter requires substantially deeper machinery beyond exterior-algebraic identities.

References

[1] K. Ball, *An elementary introduction to modern convex geometry*, in *Flavors of Geometry*, MSRI Publ., Vol. 31, Cambridge University Press, 1997, pp. 1–58.

[2] G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities*, 2nd ed., Cambridge University Press, 1952.

[3] R. A. Horn and C. R. Johnson, *Matrix Analysis*, 2nd ed., Cambridge University Press, 2013.

[4] A. Kulesza and B. Taskar, *Determinantal point processes for machine learning*, Found. Trends Mach. Learn. **5** (2012), no. 2–3, 123–286.

[5] P. McMullen, *Space tiling zonotopes*, Mathematika **22** (1975), no. 2, 202–211.

[6] R. Schneider, *Convex Bodies: The Brunn–Minkowski Theory*, 2nd expanded ed., Encyclopedia of Mathematics and Its Applications, Vol. 151, Cambridge University Press, 2014.

[7] G. C. Shephard, *Combinatorial properties of associated zonotopes*, Canad. J. Math. **26** (1974), no. 2, 302–321.

[8] G. M. Ziegler, *Lectures on Polytopes*, Graduate Texts in Mathematics, Vol. 152, Springer, 1995.